Bayesian Analysis of Sequential Auctions Under Future Uncertainties

Research Thesis

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Throughout the course of my study, Dr.Lavi offered me his advice in the most comprehensive manner, that helped me make progress both in my research and my personal development. Indeed, I consider myself most fortunate to be his student.

Contents

Abstract

Common feature of many markets that employ auction format is an existence of different uncertainty factors presented apart or simultaneously. Different cases of uncertainty lead to different bidding behavior and consequently, result in different outcomes. This study investigates the formation of bids and prices in sequential first-price auctions in presence of uncertainty about realization of future rounds. This auction format is widely accepted in markets with sequential auctions. The risk-neutral subgame perfect Nash equilibrium strategy of the independent private value model and unit-demand serves as a benchmark. Our study shows that the Bayesian-Nash equilibrium exists and that prices decline with the presence of an uncertainty.

Chapter 1

Introduction

Common feature of many markets that employ auction format is an existence of different uncertainty factors presented apart or simultaneously. Players do not know the state of the world. The bidders can be uncertain about the number of objects for sale or about the number of participating bidders. For example in many wholesale agricultural markets and harbor fish markets, products continue to arrive after the market has been opened. The exact amount of supply available for sale is not known even to the merchandise. Another example is an on-line auctions when bidders do not know the exact pattern of enrollment into the rounds by other bidders. Different cases of uncertainty lead to different bidding behavior and consequently, result in different outcomes.

Several multi-unit sales are typically conducted by means of sequential auctions, carried out either in rapid succession or over long periods of time. For example, wine, art, condominium units, used cars, agricultural products and fish are often auctioned sequentially. On Internet auction sites, sellers often auction in a sequence of many units of the same consumer product (for example, see Kaiser and Kaiser, 1999). Despite huge number of works, investigating different aspects of the sequential sales exists, there is no elaborated theory of the uncertainty impact (Klemperer, 1999, provides a review of the auction theory literature).

The basic framework for sequential auctioning of homogeneous goods was developed by Weber (1981) and by Milgrom and Weber (1982). They developed the general symmetric private-value model for risk-neutral bidders with unit demand

and number of items lower than bidders. In this setup all the bidders participate from the first round. Some points were emphasized particularly. First, in the equilibrium strategies of all these sequential auctions the bidding path is upward-drifting, i.e. call for a bidder to make progressively higher bids in each stage. Second, expected prices should remain constant throughout the sequence of auctions within the sale. Third, previous price announcement has no effect on a bidders strategy in the sequential first-price auction.

Nevertheless, in real-world auctions the prices often have tendency to move from the constant path. Ashenfelter (1989) reports that in sequential wine auctions, a downward price trend was observed. McAfee and Vincent (1993) examined data of Chicago wine auctions and obtained similar results. Other empirical studies have found that prices may be increasing. Gandal (1997) studied sequential auctions of the cable TV licenses conducted in Israel and observed increasing prices. Jones, Menezes and Vella (1998) noticed that prices may either increase to decrease in sequential wool auctions.

For the explanation of this phenomenon, many works have explored the possible impact of uncertainty. McAfee and McMillan (1987b) show that expected prices would decline if buyers display non decreasing absolute risk aversion being uncertain about the number of bidders. Jeitschko (1999) introduced the issue of supply uncertainty and studied two concrete example scenarios in second-price format. In the first scenario, after the first object has been sold it is announced whether there is a second object for auction. He shows that in the first round each bidder bids a convex combination of his value and his estimation of the third highest value, conditional on having the second highest value himself. Thus the price lies between the second and the estimated third highest values. This leads to the decline of expected prices. In the second instance after the conclusion of the first auction it becomes known whether two more auctions would take place, or just one. Here the expected price is declining if there are three items to be sold and increasing upon news that there are fewer items. The number of future round is not known in advance.

We introduce a new insight into this problem with known number of bidders and rounds where at each stage there is a probability to stop. The impact of this uncertainty on bidding behavior is an interesting issue that, to our knowledge, has not been studied before. Thus, the principal contribution of this study is a model of sequential auctions that captures important features of this auctions, neglected in earlier models of sequential auctions.

We study the formation of bids and prices in uncertain environments using Bayesian analysis for independent private values and a first-price auction. This format is widely accepted in markets with sequential auctions. The most elementary situation occurs when a number of identical objects are to be sold and each bidder desires only one of them.

Our main results can be summarized as follows. First, consecutive average prices are constant without uncertainty, as described by Milgrom and Weber, and declining with its presence. Second, in every conducted auction each bidder bids a convex combination of his estimation of the next order statistic values up to last possible round, conditional on having the highest value himself. Third, average equilibrium bids are lower as q increases.

We now give a brief description of the contents of this thesis.

In Chapter 2 we state formal model and notations, including also an overview of the equilibrium strategies of the Milgrom and Weber sequential auction setup.

In Chapter 3 we summarize the results regarding equilibrium bidding strategies and reveal their interesting properties associated with the previous results. Also we provide a demonstration of our result with an example under simple setting of uniform probability distribution.

Chapter 4 is a proof of the main theorem.

We make final conclusions in Chapter 5. Essential material concerning order statistic have been placed in section A of the Appendix.

Chapter 2

The model

In this chapter we state a formal model. In Section 2.2 we generalize the model as a Bayesian game and present notations. We also present an overview of the previous setups - equilibrium strategy of Milgrom and Weber and Jeitschko's results.

2.1 Description of the model

We study a setting where a seller intends to sell $j \in J = \{1, \ldots, K\}$ identical objects using a sequence of first-price, sealed-bid auctions.There is a finite number of bidders $i \in N = \{1, \ldots, n\}$, assuming that there are more bidders then items, i.e. $n > K$. Each bidder has unit demand and participates from a very first round. At each implemented round the bidder with highest bid wins and leaves the auction without returning in the future. After conducting every round the price of the winning bid is announced. The maximal possible number of slots is known to all players, but the next slot will take place with some commonly known probability q. Thus, in each round a bidder faces a trade-off between winning now and winning later since possibility to obtain the item in future round reduced by uncertainty.

Thus the auction becomes a Bayesian game when a commonly known probability q measured on the set of states is added to the system where all parties already are not certain about the characteristics of each other. Bayesian implementation allows to capture more realistically "prior" information that players have about the valuation of other players.

To obtain a tractable model of strategic bidding, it is necessary to assume the key simplifying assumptions. First, the bidders have independent valuations for the objects. This assumption is non-trivial in that it excludes many of popular product categories, namely antiques and collectibles, in which valuations are most likely "affiliated" across bidders (Milgrom&Weber 1982a). On the other hand, mass-produced goods like consumer electronics are usually purchased for private use with no resale opportunities, thus justifying the assumption that there is no secondary market, or other resale possibility. Second, the bidders are risk neutral, that is, they are indifferent between a lottery that yields some expected value and receiving this value for certain. Third, we focus the attention on the "symmetric" case where the players draw the values from a same cumulative probability distribution F with some support in $[0, \infty]$, i.e. the values are i.i.d. This assumption is reasonable in a setting with a large number of players.

2.2 Classical setup of the sequential auction as a Bayesian game.

The player's uncertainty about each other in a term of *Bayesian analysis* can be modeled as the Bayesian game in strategical form as five-tuple:

$$\Gamma = \left[N, V, \{s_i\}_{i\in N}, \{u_i\}_{i\in N}, F\right]$$

where:

- N is a finite set of bidders;
- K is a number of rounds;
- $V \subseteq \Re^{+}$ is a set of types and v_i indicates i's bidder private value;
- $\left\{s_i^j\right\}_{j=1}^{K} : V \times \Re^{j-1} \to \Re^{+}$ is a set of strategies for player i and $s_i^j(v_i, p^1, ..., p^{j-1})$ tells the bidder how much to bid in auction j, if his value is v_i and prices in previous auctions were $p^1, ..., p^{j-1}$;

- $u_i : V \times s_1 \times ... \times s_n \rightarrow \Re^+$ is a continuous twice differential utility function of player i where he receives $v_i - p^j$ if he wins an item in auction j and pays p^j and 0 if he loses;

- F is the probability distribution, which reflects the probability that player i would happened to be of type v_i.

All players know the distribution of all other players, but not the actual realization of the value. That is, player i updates his prior information about the distribution of the other types using Bayes rule, using knowledge of prices in previous auctions as a history. The N, K and F is a common knowledge of all players.

Let $s = (s_1, s_2, ..., s_n)$ be a profile of strategies in the Bayesian game. A symmetric Bayesian Nash equilibrium in this game is such that all players choose the same strategy functions in a purpose of maximizing their payoff for any i, v_i, a_i:

$$E_{v_{-i}}[u_i(s_i(v_i), s_{-i}(v_{-i})] \geq E_{v_{-i}}[u_i(a_i, s_{-i}(v_{-i})]$$

2.3 Previous results.

After every bidder has fixed his type v, total sample of n i.i.d. variables is arranged in downward order of magnitude so that $Y_1^{(n)} > Y_2^{(n)} > \ldots > Y_n^{(n)}$ with references to them as *order statistics*. Thus $Y_i^{(n)}$ denotes the i-highest statistic of n draws from this distribution. Then, using the notation of order statistic and with $K = 1$, the classical first-price symmetric equilibrium strategy is:

$$\beta^{(I,n)}(v) = E\left[Y_1^{(n-1)} | Y_1^{(n-1)} < v\right].$$

In sequential auction, every round j each bidder i is required to submit a bid. After a completion of the auction each bidder will have a vector of K bids $\beta^i = \{\beta_1^i, \ldots \beta_K^i\}$. The strategy profile β forms an equilibrium, if for every player i β^i is the best response on strategies of other bidders in respect to maximizing his payoff π_i. Milgrom and Weber show that for $K > 1$ stages of first-price sequential auction a symmetric equilibrium strategy is to submit in stage $j \leq K$ a bid equal to the expectation of the K^{th} highest of $n-1$ values of the other participants given that his private valuation is the j^{th} highest of n values distributed on subset of private values of bidders , i.e.

$$\beta_{j,K}(v_i) = E\left[Y_K^{(n-1)} | Y_j^{(n-1)} < v_i < Y_{j-1}^{(n-1)}\right].$$

The property of the bidding path is that for all K, $\beta_{j+1}(v_i) \geq \beta_j(v_i)$. In other words, the bidder i with type v_i who failed to win in the jth auction bids higher in the $(j+1)^{st}$ auction, since the ratio of supply to demand is decreasing from stage to stage. The bidders remaining after implementation of each auction have lower types and this offsets the increasing properties of the bidding path so that the prices in successive auctions show no trend. Weber (1983) showed that as long as the expected price at j round is $E\left[\beta_{j,K}(Y_j^{(n)})\right]$ the expected revenue of the auction with K rounds is $K \times E\left[Y_{K+1}^{(n)}\right]$.

Jeitschko (1999) considers situations in which the number of units to be sold is at most two but at the time of the first auction bidders are unsure whether a second unit will be sold at all with probability ρ. Using a second-price auction format he receives that the equilibrium bidding function in the first auction is given by the bidder's valuation discounted by his expected payoff in the second auction:

$$\beta_1(v) = (1-\rho)v + \rho E\left[Y_3 | v = Y_2\right].$$

This means that the expected profits from the now uncertain second auction are lower than they would be if a second item was to be sold for sure. Arbitrage now drives up the price in the first auction, so prices decline.

2.4 Sequential auction under uncertainty.

With an introduction of the additional uncertainty term, we extend the system where all parties already are not certain about the characteristics of each other to many possible rounds. To summarize, all bidders participate from the first period. At each round i the highest bid wins. This bidder gets the object and pays his bid. Then auctioneer flips a coin - with probability $1-q$ the sequential auctions end and with probability q we continue to auction $i+1$.

Under pressure of the uncertainty everyone at the beginning competes for possession of the item, without knowledge of private values of others. In order to maximize the profit they should bid a price less than private value but still high to win. If all participants in a very first auction bid their private values under the same bidding function, when previous price doesn't exist, then the first auction should cut out the bidder with the highest value. The remaining participants find themselves in the same position again. Everyone knows previous price and that the highest value bound to this price went out, but still doesn't know the rest of values and his own order exactly.

Chapter 3

Equilibrium in the sequential auction under uncertainty

In this chapter we state the existence of optimal bidding strategies under symmetric Bayesian-Nash equilibrium and reveal their interesting properties associated with the previous results. Also, we demonstrate a numerical example under simple setting of uniform probability distribution.

3.1 Main result.

In this chapter, I assume that bidders solve their bidding strategies stage by stage considering possible future rounds at the current round. Under this assumption I can derive the bidder's Bayesian-Nash equilibrium across stages as follows.

Theorem: *Given the private value x and the value of the winner in the $i-1$ auction y_{i-1}, the symmetric Bayesian-Nash equilibrium bidding strategies for the i'th period are $\beta^q_{i,k}(min(x, y_{i-1}))$ where the function $\beta^q_{i,k}(\cdot)$ is defined as*

$$\beta^q_{i,k}(x) = (1-q)E\left[Y_i^{(n-1)}|Y_i^{(n-1)} < x < Y_{i-1}^{(n-1)}\right] + qE\left[\beta^q_{i+1,k}(Y_i^{(n-1)})|Y_i^{(n-1)} < x < Y_{i-1}^{(n-1)}\right] \tag{3.1}$$

or alternatively

$$\beta^q_{i,k}(x) = \sum_{s=i}^{k-1} q^{s-i}(1-q)E\left[Y_s^{(n-1)}|Y_i^{(n-1)} < x < Y_{i-1}^{(n-1)}\right] + q^{k-i}E\left[Y_k^{(n-1)}|Y_i^{(n-1)} < x < Y_{i-1}^{(n-1)}\right] \tag{3.2}$$

$$\beta^q_{i,k}(x) = (1-q)\beta^{(I,n-i+1)}(x) + \sum_{s=i+1}^{k-1} q^{s-i}(1-q)\beta_{i,s}(x) + q^{k-i}\beta_{i,k}(x) \tag{3.3}$$

The expression (3.1) indicates in any current round i, the symmetric equilibrium bidding strategy $\beta^q_{i,k}(x)$ is a convex combination between expected ith highest value and expected winning price of the next round. Thus, the bid in each period recursively depends only on expected bid of the following round. The expressions (3.2) and (3.3) indicate that, the equilibrium function $\beta^q_{i,k}(x)$ is a convex combination of the first price and Milgrom&Weber sequential auction equilibrium functions, sensitive to the number of available items. Therefore, under this bidding behavior prices are expected to decline. This result can be used for testing rationality of bidders in real auctions. In the following chapter we prove these results.

3.2 Numerical example.

To explore more the properties of the bidding functions across rounds, I give a numerical example specifying the distribution and the number of potential bidders.

Example: *Values are uniformly distributed on [0, 1] with three objects (K = 3) and n bidders and some probability q.*

First, I'll construct equilibrium bidding functions for every round in a direct way, and then with the help of Milgrom&Weber bidding functions.

Beginning from the last (third) round, in equilibrium, bidder with value x wins the kth auction. Then, it must be that $Y_1^{(n-1)} \geq Y_2^{(n-1)} \geq x \geq Y_{k=3}^{(n-1)} \geq ...$:

$$\begin{aligned}
\beta^q_{k,k}(x) &= E\left[Y_k^{(n-1)}|Y_k^{(n-1)} < x < Y_{k-1}^{(n-1)}\right] = E\left[Y_1^{(n-k)}|Y_1^{(n-k)} < x\right] \\
&= \frac{1}{F(x)^{n-k}} \int_0^x y(n-k)F^{n-k-1}(y)f(y)dy \\
\beta^q_{3,3}(x) &= \frac{1}{x^{n-3}} \int_0^x y(n-3)y^{n-4}dy = \frac{n-3}{n-2}x
\end{aligned}$$

In a second round according to the equation (3.1):

$$\begin{aligned}\beta^q_{k-1,k}(x) &= (1-q)E\left[Y^{(n-1)}_{k-1}|Y^{(n-1)}_{k-1} < x < Y^{(n-1)}_{k-2}\right] + qE\left[\beta^q_{k,k}(Y^{(n-1)}_{k-1})|Y^{(n-1)}_{k-1} < x < Y^{(n-1)}_{k-2}\right] \\ &= (1-q)E\left[Y^{(n-k+1)}_{1}|Y^{(n-k+1)}_{1} < x\right] + qE\left[\beta^q_{k,k}(Y^{(n-k+1)}_{1})|Y^{(n-k+1)}_{1} < x\right]\end{aligned}$$

$$\begin{aligned}\beta^q_{2,3}(x) &= (1-q)\frac{1}{x^{n-2}}\int_0^x y(n-2)y^{n-3}dy + q\frac{1}{x^{n-2}}\int_0^x \frac{n-3}{n-2}y(n-2)y^{n-3}dy \\ &= \left[(1-q)(n-2)+q(n-3)\right]\frac{1}{x^{n-2}}\int_0^x y^{n-2}dy = \frac{n-2-q}{n-1}x\end{aligned}$$

In a first round:

$$\begin{aligned}\beta^q_{i,k}(x) &= (1-q)E\left[Y^{(n-i)}_{1}|Y^{(n-i)}_{1} < x\right] + qE\left[\beta^q_{i+1,k}(Y^{(n-i)}_{1})|Y^{(n-i)}_{1} < x\right] \\ \beta^q_{1,3}(x) &= (1-q)\frac{1}{x^{n-1}}\int_0^x y(n-1)y^{n-2}dy + q\frac{1}{x^{n-1}}\int_0^x \frac{n-2-q}{n-1}y(n-1)y^{n-2}dy \\ &= \left[(1-q)(n-1)+q(n-2-q)\right]\frac{1}{x^{n-1}}\int_0^x y^{n-1}dy = \frac{n-1-q-q^2}{n}x\end{aligned}$$

In Milgrom&Webber setup for uniform distribution, in the last period, the equilibrium bidding strategy is:

$$\beta_{k,k}(x) = \frac{n-k}{n-k+1}x$$

The bidding strategy in the ith first-price auction is:

$$\beta_{i,k}(x) = \frac{n-k}{n-i+1}x$$

Then recalling our result:

$$\beta^q_{i,k}(x) = \sum_{s=i}^{k-1} q^{s-i}(1-q)\beta_{i,s}(x) + q^{k-i}\beta_{i,k}(x).$$

In every round bidding functions are:

$$\begin{aligned}
\beta^q_{1,3}(x) &= \sum_{s=1}^{2} q^{s-1}(1-q)\beta_{1,s}(x) + q^2\beta_{1,3}(x) \\
&= (1-q)\frac{n-1}{n}x + q(1-q)\frac{n-2}{n}x + q^2\frac{n-3}{n}x = \frac{n-1-q-q^2}{n}x; \\
\beta^q_{2,3}(x) &= (1-q)\beta_{2,2}(x) + q\beta_{2,3}(x) \\
&= (1-q)\frac{n-2}{n-1}x + q\frac{n-3}{n-1}x = \frac{n-2-q}{n-1}x; \\
\beta^q_{3,3}(x) &= \frac{n-3}{n-2}x.
\end{aligned}$$

Thus the obtained result is the same.

This bids will in fact be the equilibrium bidding functions if it is not better for player to pretend that his valuation is different. To check this let us solve a bidder's problem in first round whose valuation is x and who decides to pretend to have different valuation z. The bidder maximizes his utility and FOC is:

1. case $z < x$. The expected payoff for this case is:

$$\begin{aligned}
\pi_{1,3}(z,x) &= F_1^{(n-1)}(z)\left[x - \beta^q_{1,3}(z)\right] \\
&+q\int_z^x \left[x - \beta^q_{2,3}(y)\right] f_1^{(n-1)}(y)dy + q(F_1^{(n-1)}(x) - F_2^{(n-1)}(x))\left[x - \beta^q_{2,3}(x)\right] \\
&+q^2\begin{pmatrix} n-1 \\ 2 \end{pmatrix} F(x)^{n-3}(1-F(x))^2\left[x - \beta^q_{3,3}(x)\right]
\end{aligned}$$

or

$$\pi_{1,3}(z,x) = z^{n-1}\left(x - \frac{n-1-q-q^2}{n}z\right) + q\int_z^x \left[x - \frac{n-2-q}{n-1}y\right](n-1)y^{n-2}dy + \dots$$

After differentiation with respect to z and setting equal to zero:

$$(n-1)z^{n-2}\left(x - \frac{n-1-q-q^2}{n}z\right) - z^{n-1}\frac{n-1-q-q^2}{n} - q(n-1)z^{n-2}\left(x - \frac{n-2-q}{n-1}z\right)$$

$$= (n-1)z^{n-2}x - (n-1)z^{n-1}\frac{n-1-q-q^2}{n} - z^{n-1}\frac{n-1-q-q^2}{n}$$

$$-q(n-1)z^{n-2}x + q(n-1)z^{n-1}\frac{n-2-q}{n-1}$$

$$= (n-1)(1-q)z^{n-2}x + z^{n-1}\left(q(n-2-q) - \frac{n-1-q-q^2}{n}(n-1+1)\right)$$

$$= (n-1)(1-q)z^{n-2}x + z^{n-1}(qn - 2q - q^2 - n + 1 + q + q^2)$$

$$= (n-1)(1-q)z^{n-2}x - z^{n-1}(n-1)(1-q)$$

$$= (n-1)(1-q)z^{n-2}(x-z) > 0$$

That is, if $x = z$ then FOC is satisfied.

2. case $z > x$. The expected payoff is:

$$\pi_{1,3}(z,x) \quad = \quad F^{n-1}(z)\left[x - \beta_{1,3}^q(z)\right]$$

$$+q\sum_{r=n-2}^{n-2}\begin{pmatrix} n-1 \\ r \end{pmatrix}\begin{pmatrix} r \\ n-2 \end{pmatrix} F(x)^{n-2}\left(F(z)-F(x)\right)^{r-(n-2)}\left(1-F(z)\right)^{n-1-r}\left[x - \beta_{2,3}^q(x)\right]$$

$$+q^2\sum_{r=n-3}^{n-2}\begin{pmatrix} n-1 \\ r \end{pmatrix}\begin{pmatrix} r \\ n-3 \end{pmatrix} F(x)^{n-3}\left(F(z)-F(x)\right)^{r-(n-3)}\left(1-F(z)\right)^{n-1-r}\left[x - \beta_{3,3}^q(x)\right]$$

or

$$\pi_{1,3}(z,x) \quad = \quad z^{n-1}\left[x - \frac{n-1-q-q^2}{n}z\right]$$

$$+q\begin{pmatrix} n-1 \\ n-2 \end{pmatrix}\begin{pmatrix} n-2 \\ n-2 \end{pmatrix} x^{n-2}\left(z-x\right)^{0}\left(1-z\right)^{1}\left[x - \frac{n-2-q}{n-1}x\right]$$

$$+q^2\sum_{r=n-3}^{n-2}\begin{pmatrix} n-1 \\ r \end{pmatrix}\begin{pmatrix} r \\ n-3 \end{pmatrix} x^{n-3}\left(z-x\right)^{r-(n-3)}\left(1-z\right)^{n-1-r}\left[x - \frac{n-3}{n-2}x\right]$$

After differentiation and equating to zero:

$$(n-1)z^{n-2}\left(x - \frac{n-1-q-q^2}{n}z\right) - z^{n-1}\frac{n-1-q-q^2}{n}$$

$$-q(n-1)x^{n-2}\left(x - \frac{n-2-q}{n-1}x\right)$$

$$+q^2x^{n-3}\left(x - \frac{n-3}{n-2}x\right)\Big[\frac{(n-1)!(n-3)!}{(n-3)!2!(n-3-n+3)!(n-3)!}(-2+2z)$$

$$+\frac{(n-1)!(n-2)!}{(n-2)!1!(n-2-n+3)!(n-3)!}(1-2z+x)\Big]$$

$$= (n-1)z^{n-2}x - (n-1)z^{n-1}\frac{n-1-q-q^2}{n} - z^{n-1}\frac{n-1-q-q^2}{n}$$

$$-(n-1)qx^{n-1}(\frac{n-1-n+2+q}{n-1}) - (n-1)(n-2)q^2x^{n-2}(z-x)\frac{n-2+n+3}{n-2}$$

$$= (n-1)z^{n-2}x - (n-1)z^{n-1}\frac{n-1-q-q^2}{n} - z^{n-1}\frac{n-1-q-q^2}{n}$$

$$-qx^{n-1}(1+q) - (n-1)q^2x^{n-2}(z-x)$$

$$= (n-1)z^{n-2}x - z^{n-1}(n-1-(q+q^2)) - qx^{n-1}(1+q) + (n-1)q^2x^{n-2}(x-z)$$

$$= (n-1)z^{n-2}(x-z) + (n-1)q^2x^{n-2}(x-z) + z^{n-1}(q+q^2) - x^{n-1}(q+q^2)$$

$$= (x-z)(n-1)\left[z^{n-2} + q^2x^{n-2}\right] + (q+q^2)\left[z^{n-1} - x^{n-1}\right] < 0$$

since the first term is negative and much more then the second term. The FOC is satisfied when $z = x$. Thus we argue that if all other bidders are following the strategy $\beta^q_{1,3}$, a bidder with a value of x cannot benefit by bidding anything other than $\beta^q_{1,3}(x)$; and this implies that $\beta^q_{1,3}$ is a symmetric equilibrium strategy.

Similar calculations for the bidding in the second auction result in the same conclusion. As follows:

1. $z < x$:

$$(n-2)z^{n-3}\left(x - \frac{n-2-q}{n-1}z\right) - z^{n-2}\frac{n-2-q}{n-1} - q(n-2)z^{n-3}\left(x - \frac{n-3}{n-2}z\right)$$

$$= (n-2)z^{n-3}x - (n-2)z^{n-2}\frac{n-2-q}{n-1} - z^{n-2}\frac{n-2-q}{n-1}$$

$$-q(n-2)z^{n-3}x + q(n-2)z^{n-2}\frac{n-3}{n-2}$$

$$= (n-2)(1-q)z^{n-3}x + z^{n-2}\left(q(n-3) - \frac{n-2-q}{n-1}(n-2+1)\right)$$

$$= (n-2)(1-q)z^{n-3}x + z^{n-2}(qn-3q-n+2+q)$$

$$= (n-2)(1-q)z^{n-3}x - z^{n-2}(n-2)(1-q)$$

$$= (n-2)(1-q)z^{n-3}(x-z) > 0$$

Again, FOC is satisfied when $x = z$.

2. $z > x$:

$$(n-2)z^{n-3}\left(x - \frac{n-2-q}{n-1}z\right) - z^{n-2}\frac{n-2-q}{n-1}$$

$$+qx^{n-3}\left(x - \frac{n-3}{n-2}x\right)\frac{(n-2)!(n-3)!}{(n-3)!1!(n-3)!1!}(z-x)^{-1}(1-z)^{0}(-z+x)$$

$$= (n-2)z^{n-3}x - (n-2)z^{n-2}\frac{n-2-q}{n-1} - z^{n-2}\frac{n-2-q}{n-1} - (n-2)qx^{n-2}(\frac{n-2-n+3}{n-2})$$

$$= (n-2)z^{n-3}x - (n-2)z^{n-2}\frac{n-2-q}{n-1} - z^{n-2}\frac{n-2-q}{n-1} - qx^{n-2}$$

$$= (n-2)z^{n-3}x - z^{n-2}(n-2-q) - qx^{n-2}$$

$$= (n-2)z^{n-3}(x-z) + q(z^{n-2} - x^{n-2}) < 0$$

The FOC is satisfied when $z = x$. The calculations for the last round receive the same result.

Now calculating the revenue going to the seller:

$$E[R] = E[P_1] + qE[P_2] + q^2E[P_3].$$

The probability density function of the given statistic order is:

$$f_k^{(n)}(x) = \frac{n!}{(k-1)!(n-k)!}F(x)^{n-k}(1-F(x))^{k-1}f(x)$$

and for this example:

$$f_1^{(n)}(x) = nx^{n-1}$$

$$f_2^{(n)}(x) = \frac{n!}{1!(n-2)!}x^{n-2}(1-x)^1 = n(n-1)x^{n-2}(1-x)$$

$$f_3^{(n)}(x) = \frac{n!}{2!(n-3)!}x^{n-3}(1-x)^2 = \frac{1}{2}n(n-1)(n-2)x^{n-3}(1-x)^2$$

Proceeding along and using the bidding functions:

$$E[P_1^q] = E\left[\beta_{1,3}^q(Y_1^n)\right] = \int_0^1 nx^{n-1}\frac{n-1-q-q^2}{n}xdx = \frac{n-1-q-q^2}{n+1}$$

$$E\left[P_2^q\right] = E\left[\beta_{2,3}^q(Y_2^n)\right] = \int_0^1 n(n-1)x^{n-2}(1-x)\frac{n-2-q}{n-1}xdx = \frac{n-2-q}{n+1}$$

$$E\left[P_3^q\right] = E\left[\beta_{3,3}^q(Y_3^n)\right] = \int_0^1 \frac{1}{2}n(n-1)(n-2)x^{n-3}(1-x)^2\frac{n-3}{n-2}xdx = \frac{n-3}{n+1}$$

Summarizing together:

$$E\left[R^q\right] = \frac{n-1-q-q^2}{n+1} + q\frac{n-2-q}{n+1} + q^2\frac{n-3}{n+1} = \frac{(n-1)+q(n-3)+q^2(n-5)}{n+1}$$

The revenue in the setup without uncertainty yields $E\left[R\right] = 3 \times E\left[Y_4^n\right] = 3\frac{n-3}{n+1}$. That is equal to our case if $q=1$.

Chapter 4

The proof

We prove the results for equilibrium concept presented in the previous chapters.

The Existence of the Equilibrium Bidding Strategy

We introduce a probability $q \in (0, 1)$ that one more sale will occur after each round and with probability $1 - q$ the auction will be terminated at this stage either. We start our search for a symmetric Bayesian Nash equilibrium by analyzing the game from the point of view of one of the players, say player 1. Suppose this player has a valuation $v = v_1$ and bids $\beta^1 = \beta(v_1)$. In equilibrium all players follow the same strategy since all of them face the same maximization problem. Knowing only his value and the distribution of the valuations of players $(2, \ldots, n)$, player 1 has to figure out what is his best reply. Thus bidder $j \geq 2$ bids $\beta^j = \beta(v_j)$. Then if player 1 bids $\beta^1 > max\{\beta^2, \ldots, \beta^n\}$ the object is won and buyer leaves the auction. If $\beta^1 < max\{\beta^2, \ldots, \beta^n\}$ player 1 does not win the object but has a chance to win in the next round, if one will occur with known probability q, and in case of a draw, that is if $\beta^1 = max\{\beta^2, \ldots, \beta^n\}$, the object is not sold. So, till winning every bidder forms a vector of unsuccessful bids knowing previous winning prices. In a worst case he proceeds up to the last item and bids in the last round. Thus in every round i he has history of prices $p = \{p_1, ..., p_{i-1}\}$ and history of his bids $\beta = \{\beta_1, \ldots, \beta_{i-1}\}$. Knowing an equilibrium strategy, he can estimate values of the remained concurrent; conditioning on the knowledge of private values of withdrawed ones. Fixing this bidder, total number of all players is $(n-1)$ being reduced on his particular value known to himself. We begin our formal analysis using the algorithm of *backward induction* by starting from the auction's last period. We abbreviate the notation for the i-period bidding strategy under uncertainty q to $\beta^q_{i,k}(x)$, where i is a current auction and k is a maximal possible number of rounds, defined by number of items. In the proof $Y_i^{(n-1)}$ is defined as i highest value of given *statistic order*.

Theorem: *Given the private value x and the value of the winner in the $i-1$ auction y_{i-1}, the symmetric equilibrium bidding strategies for the i'th period are $\beta^q_{i,k}(min(x, y_{i-1}))$ where the function $\beta^q_{i,k}(\centerdot)$ is defined to be*

$$\beta^q_{i,k}(x) = \sum_{s=i}^{k-1} q^{s-i}(1-q)E\left[Y_s^{(n-1)}|Y_i^{(n-1)} < x < Y_{i-1}^{(n-1)}\right] + q^{k-i}E\left[Y_k^{(n-1)}|Y_i^{(n-1)} < x < Y_{i-1}^{(n-1)}\right].$$

Proof: I prove the above equation using four claims. First of them concentrates on equilibrium strategy at the last stage of an auction. The auction becomes the first price auction but with history of all previous prices. Second and third claims reflect recursive nature of bids, when in every round bidders face two opportunities: winning now or proceeding to the next auction. Fourth claim brings the main equation into formal view.

Claim 1: *Given the private value x and the value of the winner in the $k-1$ auction y_{k-1}, the symmetric equilibrium bidding strategies for the k'th period are $\beta^q_{k,k}(min(x, y_{k-1}))$ where the function $\beta^q_{k,k}(\centerdot)$ is defined to be*

$$\beta^q_{k,k}(x) = E\left[Y_k^{(n-1)}|Y_k^{(n-1)} < x < Y_{k-1}^{(n-1)}\right].$$

Proof: Assume that $\beta^q_{s,k}(y_s)$ are monotone symmetric and non-decreasing equilibrium bidding functions. Suppose that $y_1, \ldots, y_{k-1}$ are private values evaluated from winning prices announced on previous rounds, so that $y_s = \beta^{-1}_{s,k}(p_s)$, where $s = 1, ..., k-1$. Suppose that player 1 participates in last round k. This means that all his previous bids were not winning, regardless to the order of his private value. If all other players bid in equilibrium, then prices allocate items to bidders in order of their decreasing values. Therefore, $Y_{k-1} = y_{k-1} \leq ... \leq Y_1 = y_1$. Now he have to decide which bidding function and what value to use. We will now determine the equilibrium bidding function in the last auction $\beta^q_{k,k}()$.

Fix bidder's 1 private value x. Suppose for simplicity now that $x \leq y_{k-1}$. This simplifies the probability calculations. The case when $x > y_{k-1}$ is discussed later. Denote now by z the value for which $\beta^q_{k,k}(z)$ is the best response of player i for the bids $\beta^q_{k,k}(v_j)$ for $j \neq i$. This value z can be more or less than previous winning value y_{k-1}. If the bidder chooses $z > y_{k-1}$ then his expected payoff $\pi(z, x) = 1 \cdot \left(x - \beta^q_{k,k}(z)\right) < 1 \cdot \left(x - \beta^q_{k,k}(y_{k-1})\right) = \pi(y_{k-1}, x)$, where the probability of winning is 1 since we know for every remaining player $j \neq i$ that $v_j < y_{k-1}$ in equilibrium and therefore $\beta^q_{k,k}(x) < \beta^q_{k,k}(y_{k-1}) < \beta^q_{k,k}(z)$. It is profitable to reduce the bid even more in order to further increase his profit. So we need only to consider bids less than $\beta^q_{k,k}(y_{k-1})$ or equal.

Should the last kth period occur, the bidder's expected profit in this auction, if he bids $\beta^q_{k,k}(z)$ is a continuous twice differentiable function:

$$\pi(z, x; y_1, ..., y_{k-1}) = \left[x - \beta^q_{k,k}(z)\right] Pr(Y_k^{(n-1)} \leq z | Y_1^{(n-1)} = y_1, ..., Y_{k-1}^{(n-1)} = y_{k-1})$$

with:

$$Pr(Y_k^{(n-1)} \leq z | Y_1^{(n-1)} = y_1, ..., Y_{k-1}^{(n-1)} = y_{k-1}) = F_k^{(n-1)}(z | Y_1^{(n-1)} = y_1, ..., Y_{k-1}^{(n-1)} = y_{k-1})$$

The derivative of π now is:

$$\begin{aligned} \frac{\partial \pi(z, x; y_1, ..., y_{k-1})}{\partial z} &= \left[x - \beta^q_{k,k}(z)\right] f_k^{(n-1)}(z | Y_1 = y_1, ..., Y_{k-1} = y_{k-1}) \\ &\quad - \beta^{q\prime}_{k,k}(z) \times F_k^{(n-1)}(z | Y_1 = y_1, ..., Y_{k-1} = y_{k-1}) \end{aligned}$$

In equilibrium the first order condition requires that $\partial_z \pi(z, x)|_{z=x} = 0$:

$$\left[x - \beta^q_{k,k}(x)\right] f_k^{(n-1)}(x | Y_1 = y_1, ..., Y_{k-1} = y_{k-1}) - \beta^{q\prime}_{k,k}(x) F_k^{(n-1)}(x | Y_1 = y_1, ..., Y_{k-1} = y_{k-1}) = 0.$$

Opening the square brackets we receive:

$$\begin{aligned} xf_k^{(n-1)}(x|Y_1 &= y_1, ..., Y_{k-1} = y_{k-1}) - \beta_{k,k}^q(x) f_k^{(n-1)}(x|Y_1 = y_1, ..., Y_{k-1} = y_{k-1}) \\ & -\beta_{k,k}^{q\prime}(x) F_k^{(n-1)}(x|Y_1 = y_1, ..., Y_{k-1} = y_{k-1})) = 0. \end{aligned}$$

It is easy to see that second and third terms in this equation are the derivative of:

$$\frac{\partial}{\partial x}\left(\beta_{k,k}^q(x) F_k^{(n-1)}(x|Y_1 = y_1, ..., Y_{k-1} = y_{k-1})\right).$$

Moving the last term on the left side:

$$\frac{\partial}{\partial x}\left(\beta_{k.k}^q(x) F_k^{(n-1)}(x|...)\right) = xf_k^{(n-1)}(x|Y_1 = y_1, ..., Y_{k-1} = y_{k-1}) \qquad (4.1)$$

Simplifying further and using order statistics properties specified in Appendix A, the sample of $n-1$ variables can be truncated from the tail of highest values and the k^{th} order on sample becomes a statistic of first order on a sample of $n-k$ variables with conditioned density function:

$$f_k^{(n-1)}(x|Y_1^{(n-1)} = y_1, ..., Y_{k-1}^{(n-1)} = y_{k-1}) = f_k^{(n-1)}(x|Y_{k-1}^{(n-1)} = y_{k-1})$$

$$= f_1^{(n-k)}\left(x|Y_1^{(n-k)} < y_{k-1}\right) = \frac{(n-k)F(x)^{n-k-1}f(x)}{F(y_{k-1})^{n-k}}$$

and cumulative probability function:

$$F_k^{(n-1)}(x|Y_1^{(n-1)} = y_1, ..., Y_{k-1}^{(n-1)} = y_{k-1}) = F_1^{(n-k)}(x|Y_1^{(n-k)} < y_{k-1}) = \left[\frac{F(x)}{F(y_{k-1})}\right]^{n-k}$$

Now the equation (4.1) can be rewritten as

$$\frac{\partial}{\partial x}\left(\beta_{k,k}^q(x)\left[\frac{F(x)}{F(y_{k-1})}\right]^{n-k}\right) = x\frac{(n-k)F(x)^{n-k-1}f(x)}{F(y_{k-1})^{n-k}}$$

and canceling out the constant term $F(y_{k-1})^{n-k}$, as

$$\frac{\partial}{\partial x}\left(\beta_{k,k}^q(x)F(x)^{n-k}\right) = x(n-k)F(x)^{n-k-1}f(x)$$

The solution for (4.1) can be obtained by integrating both sides over x with boundary condition $\beta^q_{k,k}(0) = 0$:

$$\beta^q_{k,k}(x) = \frac{1}{F(x)^{n-k}} \int_0^x y(n-k)F^{n-k-1}(y)f(y)dy$$

$$= \frac{1}{F(x)^{n-k}} \int_0^x yd\left(F(y)^{n-k}\right) = \frac{1}{F_1^{(n-k)}(x)} \int_0^x yd\left(F_1^{(n-k)}(y)\right)$$

$$= E\left[Y_1^{(n-k)} | Y_1^{(n-k)} < x\right] = E\left[Y_k^{(n-1)} | Y_k^{(n-1)} < x < Y_{k-1}^{(n-1)}\right]$$

Return now to the case when $x > y_{k-1}$. Recall that the player bids value $z \leq y_{k-1}$, no matter where his private value actually is.
Fixing the bidding strategy, which was derived above, we show that his payoff function is increasing and $\pi(z, x) \leq \pi(y_{k-1}, x)$ even if $x > y_{k-1}$. First, from what was proved above, we know:

$$\begin{aligned} (y_{k-1} - \beta^q_{k,k}(z))F_k^{(n-1)}(z|Y_1 &= y_1, ..., Y_{k-1} = y_{k-1}) = \\ \pi(z, y_{k-1}) &\leq \pi(y_{k-1}, y_{k-1}) = \\ &(y_{k-1} - \beta^q_{k,k}(y_{k-1}))F_k^{(n-1)}(y_{k-1}|Y_1 = y_1, ..., Y_{k-1} = y_{k-1}) \end{aligned} \tag{4.2}$$

Now using this inequality we receive:

$$\pi(z, x) = (x - \beta^q_{k,k}(z))F_k^{(n-1)}(z|...) = (x - y_{k-1})F_k^{(n-1)}(z|...) + (y_{k-1} - \beta^q_{k,k}(z))F_k^{(n-1)}(z|...)$$

$$\leq (x - y_{k-1})F_k^{(n-1)}(y_{k-1}|...) + (y_{k-1} - \beta^q_{k,k}(y_{k-1}))F_k^{(n-1)}(y_{k-1}|...) = \pi(y_{k-1}, x)$$

where the second term on the left of the inequality is smaller than the second term to the right of the inequality by inequality (4.2), and first term on the left is smaller than first term on the right since a cumulative probability function is monotone non-decreasing and $z \leq y_{k-1}$. Hence, the player gains from bidding y_{k-1}. Therefore, in a case $z \leq y_{k-1} < x$ the bidder is better off to bid y_{k-1}. For both cases the bidding function becomes $\beta^q_{k,k}(min(x, y_{k-1}))$. $\square$

The following Claim 2 helps to build intuition further.

Claim 2: *Given the private value x and the value of the winner in the $k-2$ auction y_{k-2}, the symmetric equilibrium bidding strategies for the (k-1)th period are $\beta^q_{k-1,k}(min(x, y_{k-2}))$, where the function $\beta^q_{k-1,k}(\centerdot)$ is defined to be*

$$\beta^q_{k-1,k}(x) = (1-q)E\left[Y^{(n-1)}_{k-1}|Y^{(n-1)}_{k-1} < x < Y^{(n-1)}_{k-2}\right] + qE\left[\beta^q_{k,k}(Y^{(n-1)}_{k-1})|Y^{(n-1)}_{k-1} < x < Y^{(n-1)}_{k-2}\right]$$

Proof: Suppose that all bidders follow the strategy $\beta^q_{k,k}(min(x, y_{k-1}))$ for the last period regardless of what happened in a previous period. Then in the previous $(k-1)$th period each bidder faces a trade-off between winning now or waiting for the next round and winning then, if the last auction will occur with probability q. Thus his expected profit conditioned on previous winning values' path if he bids some value z is:

$$\pi_{k-1,k}(z, x; y_{k-2}) = Pr(Y^{(n-1)}_{k-1} \leq z|Y^{(n-1)}_1 = y_1, ..., Y^{(n-1)}_{k-2} = y_{k-2})\left[x - \beta^q_{k-1,k}(z)\right]$$

$$+qPr(Y^{(n-1)}_{k-1} > z, Y^{(n-1)}_k < x < Y^{(n-1)}_{k-1}|Y^{(n-1)}_1 = y_1, ..., Y^{(n-1)}_{k-2} = y_{k-2})$$

$$\times E\left[x - \beta^q_{k,k}(min(x, Y^{(n-1)}_{k-1}))|Y^{(n-1)}_{k-1} > z, Y^{(n-1)}_k < x < Y^{(n-1)}_{k-1}, Y^{(n-1)}_1 = y_1, ..., Y^{(n-1)}_{k-2} = y_{k-2}\right]$$

Here the first term results from the event of winning the current auction with a bid of $\beta^q_{k-1,k}(z)$ if z happens to be $k-1$ highest value in the sample. It can be expressed according the logic of previous claim as:

$$Pr(Y^{(n-1)}_{k-1} \leq z|Y^{(n-1)}_1 = y_1, ..., Y^{(n-1)}_{k-2} = y_{k-2}) = F^{(n-1)}_{k-1}(z|Y^{(n-1)}_{k-2} = y_{k-2})$$

$$= F^{(n-k+1)}_1(z|Y^{(n-k+1)}_1 < y_{k-2}) = \left[\frac{F(z)}{F(y_{k-2})}\right]^{n-k+1} = \frac{F^{(n-k+1)}_1(z)}{F(y_{k-2})^{n-k+1}}$$

The second term of the function $\pi_{k-1,k}(z, x; y_{k-2})$ results from loosing this auction and winning the next with the bid $\beta^q_{k,k}(x)$ if $x < y_{k-1}$. In case $x > y_{k-1}$ he will bid $\beta^q_{k,k}(y_{k-1})$.

As previously, assume first that $x \leq y_{k-2}$. Bidding some value z, bidder can deviate from his private value x either upwards or downwards. Bidding $z \geq x$ and $z < x$, the payoff function has the two different cases of calculation.

1. $z \geq x$. In this case the player bids a value bigger than his actual value is. Then, if happened to loose this auction, he can win the last one using strategy

$\beta^q_{k,k}(x)$ with probability:

$$Pr(Y^{(n-1)}_{k-1} > z, Y^{(n-1)}_{k} < x < Y^{(n-1)}_{k-1} | Y^{(n-1)}_1 = y_1, ..., Y^{(n-1)}_{k-2} = y_{k-2})$$

$$= Pr(Y^{(n-k+1)}_1 > z, Y^{(n-k+1)}_2 < x < Y^{(n-k+1)}_1 | Y^{(n-k+1)}_1 < y_{k-2})$$
$$= \frac{(n-k+1)F(x)^{n-k}(F(y_{k-2}) - F(z))}{F(y_{k-2})^{n-k+1}}$$

In this case he looses the current round to value y_{k-1} using value z, so $y_{k-1} \geq z > x$ and $\beta^q_{k,k}(min(x, y_{k-1})) = \beta^q_{k,k}(x)$. Thus his expected payoff:

$$\pi_{k-1,k}(z, x|y_{k-2}) = \frac{F^{(n-k+1)}_1(z)}{F(y_{k-2})^{n-k+1}} \left[x - \beta^q_{k-1,k}(z)\right] + q\frac{(n-k+1)F(x)^{n-k}\left(F(y_{k-2}) - F(z)\right)}{F(y_{k-2})^{n-k+1}} \left[x - \beta^q_{k,k}(x)\right]$$

Moving the constant term $\frac{1}{F(y_{k-2})^{n-k+1}}$ to the right side of the equation, his maximization problem is resolved by taking derivative of $\pi_{k-1,k}(z, x|y_{k-2})$ with respect to z:

$$\frac{\partial \pi_{k-1,k}(z, x|y_{k-2})}{\partial z} F(y_{k-2})^{n-k+1} = f^{(n-k+1)}_1(z)\left[x - \beta^q_{k-1,k}(z)\right] - F^{(n-k+1)}_1(z)\beta^{q'}_{k-1,k}(z) - q(n-k+1)f(z)F(x)^{n-k}\left[x - \beta^q_{k,k}(x)\right]$$

2. $z < x$. In this case bidder chooses to bid $\beta^q_{k-1,k}(z)$ where value z is less then his private value. Then his expected utility is:

$$\pi_{k-1,k}(z, x|y_{k-2}) = \frac{F^{(n-k+1)}_1(z)}{F(y_{k-2})^{n-k+1}} \left[x - \beta^q_{k-1,k}(z)\right]$$

$$+q\frac{\int_z^x \left(x - \beta^q_{k,k}(y)\right) f^{(n-k+1)}_1(y)dy}{F(y_{k-2})^{n-k+1}} + q\frac{\left(F^{(n-k+1)}_1(x) - F^{(n-k+1)}_2(x)\right)}{F(y_{k-2})^{n-k+1}} \left[x - \beta^q_{k,k}(x)\right]$$

where the first term again results from the event $Y_{k-1} \leq z$. The second term results from the event then he wins the last auction with a bid of $\beta^q_{k,k}(y_{k-1})$. The third term results from the event $z < Y^{(n-k+1)}_2 < x < Y^{(n-k+1)}_1$ then he wins the last auction with the bid of $\beta^q_{k,k}(x)$. Taking derivative with respect to z yields:

$$\frac{\partial \pi_{k-1,k}(z,x|y_{k-2})}{\partial z} F(y_{k-2})^{n-k+1} = f_1^{(n-k+1)}(z)\left[x-\beta^q_{k-1,k}(z)\right] - F_1^{(n-k+1)}(z)\beta^{q'}_{k-1,k}(z) \\ -qf_1^{n-k+1}(z)\left[x-\beta^q_{k,k}(z)\right]$$

The derivatives from the both cases coincide in the same expression in a point $z = x$:

$$\frac{\partial \pi_{k-1,k}(z,x|y_{k-2})}{\partial z}|_{z=x} F(y_{k-2})^{n-k+1} = f_1^{(n-k+1)}(x)\left[x-\beta^q_{k-1,k}(x)\right] - F_1^{(n-k+1)}(x)\beta^{q'}_{k-1,k}(x) \\ -qf_1^{n-k+1}(x)\left[x-\beta^q_{k,k}(x)\right]$$

Thus derivative of $\pi_{k-1,k}(z,x|y_{k-2})$ in a point $z = x$ does exist and can be the solution of the maximization problem of the bidder. Suppose now that the bidder follows the equilibrium strategy. The first-order condition is:

$$f_1^{(n-k+1)}(x)\left[x-\beta^q_{k-1,k}(x)\right] - F_1^{(n-k+1)}(x)\beta^{q'}_{k-1,k}(x) - qf_1^{(n-k+1)}(x)\left[x-\beta^q_{k,k}(x)\right] = 0$$

Opening the square brackets and bonding together the terms with bidding function as a derivative yield:

$$xf_1^{(n-k+1)}(x) - \frac{\partial}{\partial x}\left(F_1^{(n-k+1)}(x)\beta^q_{k-1,k}(x)\right) - qf_1^{(n-k+1)}(x)\left[x-\beta^q_{k,k}(x)\right] = 0$$

This results in:

$$\frac{\partial}{\partial x}\left(F_1^{(n-k+1)}(x)\beta^q_{k-1,k}(x)\right) = xf_1^{(n-k+1)}(x) - qf_1^{(n-k+1)}(x)\left[x-\beta^q_{k,k}(x)\right]$$

$$= (1-q)xf_1^{(n-k+1)}(x) + q\beta^q_{k,k}(x)f_1^{(n-k+1)}(x)$$

which has a solution after integrating both sides over x with the boundary condition $\beta(0) = 0$:

$$\begin{aligned}
\beta^q_{k-1,k}(x) &= (1-q)\frac{1}{F_1^{(n-k+1)}(x)}\int_0^x y f_1^{(n-k+1)}(y)dy \\
&\quad +q\frac{1}{F_1^{(n-k+1)}(x)}\int_0^x \beta^q_{k,k}(y) f_1^{(n-k+1)}(y)dy \\
&= (1-q)E\left[Y_1^{(n-k+1)}|Y_1^{(n-k+1)}<x\right] \\
&\quad +qE\left[\beta^q_{k,k}(Y_1^{(n-k+1)})|Y_1^{(n-k+1)}<x\right] \\
&= (1-q)E\left[Y_{k-1}^{(n-1)}|Y_{k-1}^{(n-1)}<x<Y_{k-2}^{(n-1)}\right] \\
&\quad +qE\left[\beta^q_{k,k}(Y_{k-1}^{(n-1)})|Y_{k-1}^{(n-1)}<x<Y_{k-2}^{(n-1)}\right]
\end{aligned}$$

The case $z \le y_{k-2} < x$ follows the logic of the same case in the Claim 1 and makes the bidder better off to bid here $\beta^q_{k-1,k}(y_{k-2})$.□

The results of this two claims can be combined in a following intermediate observation. New definitions are required: $\beta^{(I,n)}$ defines first-price bidding function where n is a number of players in this particular round and $\beta_{k-1,k}$ is a bidding function of sequential auction without uncertainty matching same period.

Claim 2.1: *The equilibrium function $\beta^q_{k-1,k}(x)$ is a convex combination of the first price equilibrium function and Milgrom&Weber sequential auction equilibrium function:*

$$\beta^q_{k-1,k}(x) = (1-q)\beta^{(I,n-k+2)}(x) + q\beta_{k-1,k}(x)$$

Proof: The proof follows directly from Claim 1 with $\beta^q_{k,k} = E\left[Y_k^{(n-1)}|Y_k^{(n-1)}<x<Y_{k-1}^{(n-1)}\right]$ and Claim 2:

$$\begin{aligned}
\beta^q_{k-1,k}(x) &= (1-q)E\left[Y_{k-1}^{(n-1)}|Y_{k-1}^{(n-1)}<x<Y_{k-2}^{(n-1)}\right] \\
&\quad +qE\left[\beta^q_{k,k}(Y_{k-1}^{(n-1)})|Y_{k-1}^{(n-1)}<x<Y_{k-2}^{(n-1)}\right] \\
&= (1-q)E\left[Y_{k-1}^{(n-1)}|Y_{k-1}^{(n-1)}<x<Y_{k-2}^{(n-1)}\right] \\
&\quad +qE\left[E\left[Y_k^{(n-1)}|Y_k^{(n-1)}<x<Y_{k-1}^{(n-1)}\right]|Y_{k-1}^{(n-1)}<x<Y_{k-2}^{(n-1)}\right] \\
&= (1-q)E\left[Y_{k-1}^{(n-1)}|Y_{k-1}^{(n-1)}<x<Y_{k-2}^{(n-1)}\right] \\
&\quad +qE\left[Y_k^{(n-1)}|Y_{k-1}^{(n-1)}<x<Y_{k-2}^{(n-1)}\right]
\end{aligned}$$

In the first term, $E\left[Y_{k-1}^{(n-1)}|Y_{k-1}^{(n-1)} < x < Y_{k-2}^{(n-1)}\right] = E\left[Y_1^{(n-k+1)}|Y_1^{(n-k+1)} < x\right]$ where the right side is the expression for the first price auction for n-k+2 players since $Y_{k-1}^{(n-1)} = Y_1^{(n-k+1)}$. In the second term, $E\left[Y_k^{(n-1)}|Y_{k-1}^{(n-1)} < x < Y_{k-2}^{(n-1)}\right]$ is a bidding function for k-1 period of sequential auction in Milgrom&Weber setup. Using the appropriate notations it can be rewritten as:

$$\beta_{k-1,k}^q(x) = (1-q)\beta^{(I,n-k+2)}(x) + q\beta_{k-1,k}(x).$$

□

Proceeding along bidding sequence to any current round i, the symmetric equilibrium bidding strategy $\beta_{i,k}^q(x)$ is a convex combination between expected ith highest value and expected winning price of the next round.

Claim 3: *Given the private value x and the value of the winner in the $i-1$ auction y_{i-1}, the symmetric equilibrium bidding strategies for the i'th period* are $\beta_{i,k}^q(min(x, y_{i-1}))$, where *the function $\beta_{i,k}^q(\centerdot)$ is defined to be*

$$\beta_{i,k}^q(x) = (1-q)E\left[Y_i^{(n-1)}|Y_i^{(n-1)} < x < Y_{i-1}^{(n-1)}\right] + qE\left[\beta_{i+1,k}^q(Y_i^{(n-1)})|Y_i^{(n-1)} < x < Y_{i-1}^{(n-1)}\right].$$

Proof: Going one step back player has to decide now again. Looking forward in period $k-2$ the bidder uses the same logic. Proceeding with the same logic back till some random round i we can build the following payoff function:

$$\pi_{i,k}(z, x; y_{i-1}) = Pr(Y_i^{(n-1)} < z|Y_1^{(n-1)} = y_1, ..., Y_{i-1}^{(n-1)} = y_{i-1})\left[x - \beta_{i,k}^q(z)\right]$$

$$+qPr(Y_i^{(n-1)} > z, Y_{i+1}^{(n-1)} < x < Y_i^{(n-1)}|Y_1^{(n-1)} = y_1, ..., Y_{i-1}^{(n-1)} = y_{i-1})$$

$$\times E\left[x - \beta_{i+1,k}^q(min(x, Y_i^{(n-1)}))|Y_i^{(n-1)} > z, Y_{i+1}^{(n-1)} < x < Y_i^{(n-1)}, Y_1^{(n-1)} = y_1, ..., Y_{i-1}^{(n-1)} = y_{i-1}\right]$$

$$+q^2Pr(Y_i^{(n-1)} > z, Y_{i+2}^{(n-1)} < x < Y_{i+1}^{(n-1)}|Y_1^{(n-1)} = y_1, ..., Y_{i-1}^{(n-1)} = y_{i-1})$$

$$\times E\left[x - \beta_{i+2,k}^q(min(x, Y_{i+1}^{(n-1)}))|Y_i^{(n-1)} > z, Y_{i+2}^{(n-1)} < x < Y_{i+1}^{(n-1)}, Y_1^{(n-1)} = y_1, ..., Y_{i-1}^{(n-1)} = y_{i-1}\right]$$

$$\vdots$$

$$+q^{k-i}Pr(Y_i^{(n-1)} > z, Y_k^{(n-1)} < x < Y_{k-1}^{(n-1)} | Y_1^{(n-1)} = y_1, ..., Y_{i-1}^{(n-1)} = y_{i-1})$$

$$\times E\left[x - \beta_{k,k}^q(min(x, Y_{k-1}^{(n-1)})) | Y_i^{(n-1)} > z, Y_k^{(n-1)} < x < Y_{k-1}^{(n-1)}, Y_1^{(n-1)} = y_1, ..., Y_{i-1}^{(n-1)} = y_{i-1}\right]$$

Truncating the sample from the upper tail, the equation can be rewritten as:

$$\pi_{i,k}(z, x; y_{i-1}) = Pr(Y_1^{(n-i)} < z | Y_1^{(n-i)} < y_{i-1})\left[x - \beta_{i,k}^q(z)\right]$$

$$+qPr(Y_1^{(n-i)} > z, Y_2^{(n-i)} < x < Y_1^{(n-i)} | Y_1^{(n-i)} < y_{i-1})$$

$$\times E\left[x - \beta_{i+1,k}^q(min(x, Y_1^{(n-i)})) | Y_1^{(n-i)} > z, Y_2^{(n-i)} < x < Y_1^{(n-i)}, Y_1^{(n-i)} < y_{i-1}\right]$$

$$q^2 Pr(Y_1^{(n-i)} > z, Y_3^{(n-i)} < x < Y_2^{(n-i)} | Y_1^{(n-i)} < y_{i-1})$$

$$\times E\left[x - \beta_{i+2,k}^q(min(x, Y_2^{(n-i)})) | Y_1^{(n-i)} > z, Y_3^{(n-i)} < x < Y_2^{(n-i)}, Y_1^{(n-i)} < y_{i-1}\right]$$

$$\vdots$$

$$+q^{k-i}Pr(Y_1^{(n-i)} > z, Y_{k-i+1}^{(n-i)} < x < Y_{k-i}^{(n-i)} | Y_1^{(n-i)} < y_{i-1})$$

$$\times E\left[x - \beta_{k,k}^q(min(x, Y_{k-i}^{(n-i)})) | Y_1^{(n-i)} > z, Y_{k-i+1}^{(n-i)} < x < Y_{k-i}^{(n-i)}, Y < y_{i-1}\right]$$

Assuming as previously that $x \leq y_{i-1}$ and using the proper expressions for corresponding probability depending on position x and z as described in Appendix A for order statistics, the multiply periods payoff function is:

1. $z \geq x$. In this case the player, loosing the current round to value y_i will use $\beta_{i+1,k}^q(min(x, y_i)) = \beta_{i+1,k}^q(x)$ in the next one. Also, for $z \geq x$ and $r < s$, the probability $Pr\left\{Y_r^{(n)} > z, Y_s^{(n)} < x < Y_{s-1}^{(n)}\right\}$ is

$$\sum_{i=n-s+1}^{n-r} \begin{pmatrix} n \\ i \end{pmatrix} \begin{pmatrix} i \\ n-s+1 \end{pmatrix} F(x)^{n-s+1}\left(F(z) - F(x)\right)^{i-(n-s+1)}(1-F(z))^{n-i}.$$

Therefore the payoff becomes:

$$\pi_{i,k}(z, x | y_{i-1}) = \frac{F^{n-i}(z)}{F(y_{i-1})^{n-i}}\left[x - \beta_{i,k}^q(z)\right]$$

$$+q\frac{\sum_{r=n-i-1}^{n-i-1} \begin{pmatrix} n-i \\ r \end{pmatrix} \begin{pmatrix} r \\ n-i-1 \end{pmatrix} F(x)^{n-i-1}\left(F(z) - F(x)\right)^{r-(n-i-1)}\left(F(y_{i-1}) - F(z)\right)^{n-i-r}}{F(y_{i-1})^{n-i}}\left[x - \beta_{i+1,k}^q(x)\right]$$

$$+q^2 \frac{\sum_{r=n-i-2}^{n-i-1} \binom{n-i}{r} \binom{r}{n-i-2} F(x)^{n-i-2} (F(z)-F(x))^{r-(n-i-2)} (F(y_{i-1})-F(z))^{n-i-r}}{F(y_{i-1})^{n-i}} \left[x - \beta_{i+2,k}^{q}(x)\right]$$

$$\vdots$$

$$+q^{k-i} \frac{\sum_{r=n-k}^{n-i-1} \binom{n-i}{r} \binom{r}{n-k} F(x)^{n-k} (F(z)-F(x))^{r-(n-k)} (F(y_{i-1})-F(z))^{n-i-r}}{F(y_{i-1})^{n-i}} \left[x - \beta_{k,k}^{q}(x)\right]$$

Summarizing the terms with probability q and transferring the constant term from right to left side this becomes:

$$\pi_{i,k}(z,x|y_{i-1})F(y_{i-1})^{n-i} = F^{n-i}(z)\left[x - \beta_{i,k}^{q}(z)\right]$$

$$+ \sum_{s=i+1}^{k} q^{s-i} \sum_{r=n-s}^{n-i-1} \binom{n-i}{r} \binom{r}{n-s} F(x)^{n-s} (F(z)-F(x))^{r-(n-s)}$$

$$\times (F(y_{i-1})-F(z))^{n-i-r}\left[x - \beta_{s,k}^{q}(x)\right]$$

When $\pi_{i,k}(z,x|y_{i-1})$ is a maximum, $\frac{\partial \pi_{i,k}(z,x|y_{i-1})}{\partial z}$ must be zero. Thus:

$$\frac{\partial \pi_{i,k}(z,x|y_{i-1})}{\partial z} F(y_{i-1})^{n-i} = f_1^{(n-i)}(z)\left[x - \beta_{i,k}^{q}(z)\right] - F_1^{(n-i)}(z)\beta_{i,k}^{q'}(z)$$

$$+ \sum_{s=i+1}^{k} q^{s-i} F(x)^{n-s} f(z)\left[x - \beta_{s,k}^{q}(x)\right]$$

$$\times \sum_{r=n-s}^{n-i-1} \binom{n-i}{r} \binom{r}{n-s} (F(z)-F(x))^{r-n+s-1}$$

$$\times (F(y_{i-1})-F(z))^{n-i-r-1}\left((r-n+s)F(y_{i-1}) - (s-i)F(z) + (n-i-r)F(x)\right)$$

2. $z \le x$. In this case for $Pr\left\{Y_r^{(n)} > z, Y_s^{(n)} \le x \le Y_{s-1}^{(n)}\right\} = Pr\left\{Y_s^{(n)} \le x \le Y_{s-1}^{(n)}\right\}$ then $r < s$ is $Pr\left\{Y_s^{(n)} \le x \le Y_{s-1}^{(n)}\right\}$ and the winning probabilities, except of

the current and the next rounds, are independent of z. Therefore:

$$\begin{aligned}
\pi_{i,k}(z,x|y_{i-1}) &= \frac{F_1^{(n-i)}(z)}{F(y_{i-1})^{n-i}}\left[x-\beta_{i,k}^q(z)\right] \\
&+q\frac{\int_z^x\left(x-\beta_{i+1,k}^q(y)\right)f_1^{(n-i)}(y)dy}{F(y_{i-1})^{n-i}} \\
&+q\frac{\left(F_1^{(n-i)}(x)-F_2^{(n-i)}(x)\right)}{F(y_{i-1})^{n-i}}\left[x-\beta_{i+1,k}^q(x)\right]+ \\
&+\frac{1}{F(y_{i-1})^{n-i}}\sum_{s=i+2}^{k}q^{s-i}\left(\begin{array}{c} n-s \\ s \end{array}\right)F(x)^{n-s} \\
&\times(F(y_{i-1})-F(x))^s\left[x-\beta_{s,k}^q(x)\right]
\end{aligned}$$

After transferring constant term $F(y_{i-1})^{n-i}$ to the left side and differentiation with respect to z:

$$\begin{aligned}
\frac{\partial\pi_{i,k}(z,x|y_{i-1})}{\partial z}F(y_{i-1})^{n-i} &= f_1^{(n-i)}(z)\left[x-\beta_{i,k}^q(z)\right]-F_1^{(n-i)}(z)\beta_{i,k}^{\prime q}(z) \\
&-qf_1^{(n-i)}(z)\left[x-\beta_{i+1,k}^q(z)\right]
\end{aligned}$$

Setting $z=x$ both functions coincide in this derivative:

$$\begin{aligned}
\frac{\partial\pi_{i,k}(z,x|y_{i-1})}{\partial z}|_{z=x}F(y_{i-1})^{n-i} &= f_1^{(n-i)}(x)\left[x-\beta_{i,k}^q(x)\right]-F_1^{(n-i)}(x)\beta_{i,k}^{\prime q}(x) \\
&-qf_1^{(n-i)}(x)\left[x-\beta_{i+1,k}^q(x)\right]
\end{aligned}$$

Again, the derivative $\pi'_{i,k}(z,x|y_{i-1})$ does exist in the point x and can be the solution of maximization problem. After equating to zero it ends up in the following expression:

$$\begin{aligned}
\frac{\partial}{\partial x}\left(F_1^{(n-i)}(x)\beta_{i,k}^q(x)\right) &= xf_1^{(n-i)}(x)-qf_1^{(n-i)}(x)\left[x-\beta_{i+1,k}^q(x)\right] \\
&= (1-q)xf_1^{(n-i)}(x)+q\beta_{i+1,k}^q(x)f_1^{(n-i)}(x)
\end{aligned}$$

with solution obtained after integration the left and right sides over x:

$$\beta_{i,k}^q(x)=(1-q)\frac{1}{F_1^{(n-i)}(x)}\int_0^x yf_1^{(n-i)}(y)dy+q\frac{1}{F_1^{(n-i)}(x)}\int_0^x\beta_{i+1,k}^q(y)f_1^{(n-i)}(y)dy$$

$$= (1-q)E\left[Y_1^{(n-i)}|Y_1^{(n-i)} < x\right] + qE\left[\beta^q_{i+1,k}(Y_1^{(n-i)})|Y_1^{(n-i)} < x\right]$$

$$= (1-q)E\left[Y_i^{(n-1)}|Y_i^{(n-1)} < x < Y_{i-1}^{(n-1)}\right] + qE\left[\beta^q_{i+1,k}(Y_i^{(n-1)})|Y_i^{(n-1)} < x < Y_{i-1}^{(n-1)}\right]$$

The case $z \leq y_{i-1} < x$ follows the logic of the same case in the Claim 1,2 and makes the bidder better off to bid here $\beta^q_{i,k}(y_{i-1})$.

□

Note that Claim 3 implies that in sequential auction the bid in each period depends only on expected price of the following round without references to other subsequent rounds. This can be principal aim in reducing the number of independent calculations.

Claim 3.1: *The generalization of the recurrence relation between symmetric equilibrium bidding strategies $\beta^q_{i,k}(x)$ can be expressed in terms of expected values as:*

$$\beta^q_{i,k}(x) = \sum_{s=i}^{k-1} q^{s-i}(1-q)E\left[Y_s^{(n-1)}|Y_i^{(n-1)} < x < Y_{i-1}^{(n-1)}\right] + q^{k-i}E\left[Y_k^{(n-1)}|Y_i^{(n-1)} < x < Y_{i-1}^{(n-1)}\right].$$

Proof: This can be seen by computation of every step of the recurrence:

$$\begin{aligned}
\beta^q_{i,k}(x) &= (1-q)E\left[Y_i^{(n-1)}|Y_i^{(n-1)} < x < Y_{i-1}^{(n-1)}\right] \\
&\quad +qE\left[\beta^q_{i+1,k}(Y_i^{(n-1)})|Y_i^{(n-1)} < x < Y_{i-1}^{(n-1)}\right] \\
&= (1-q)E\left[Y_i^{(n-1)}|Y_i^{(n-1)} < x < Y_{i-1}^{(n-1)}\right] \\
&\quad +qE[(1-q)E\left[Y_{i+1}^{(n-1)}|Y_{i+1}^{(n-1)} < Y_i^{(n-1)}\right] \\
&\quad +qE\left[\beta^q_{i+2,k}(Y_{i+1}^{(n-1)})|Y_{i+1}^{(n-1)} < Y_i^{(n-1)}\right]|Y_i^{(n-1)} < x < Y_{i-1}^{(n-1)}] \\
&= (1-q)E\left[Y_i^{(n-1)}|Y_i^{(n-1)} < x < Y_{i-1}^{(n-1)}\right] \\
&\quad +q(1-q)E\left[Y_{i+1}^{(n-1)}|Y_i^{(n-1)} < x < Y_{i-1}^{(n-1)}\right] \\
&\quad q^2E\left[\beta^q_{i+2,k}(Y_{i+1}^{(n-1)})|Y_i^{(n-1)} < x < Y_{i-1}^{(n-1)}\right] \\
&= (1-q)E\left[Y_i^{(n-1)}|Y_i^{(n-1)} < x < Y_{i-1}^{(n-1)}\right] \\
&\quad +q(1-q)E\left[Y_{i+1}^{(n-1)}|Y_i^{(n-1)} < x < Y_{i-1}^{(n-1)}\right]
\end{aligned}$$

$$
\begin{aligned}
&+q^2E[(1-q)E\left[Y_{i+2}^{(n-1)}|Y_{i+2}^{(n-1)} < Y_{i+1}^{(n-1)}\right] \\
&+qE\left[\beta_{i+3,k}^q(Y_{i+2}^{(n-1)})|Y_{i+2}^{(n-1)} < Y_{i+1}^{(n-1)}\right]|Y_i^{(n-1)} < x < Y_{i-1}^{(n-1)}] \\
= \; & (1-q)E\left[Y_i^{(n-1)}|Y_i^{(n-1)} < x < Y_{i-1}^{(n-1)}\right] \\
&+q(1-q)E\left[Y_{i+1}^{(n-1)}|Y_i^{(n-1)} < x < Y_{i-1}^{(n-1)}\right] \\
&+q^2(1-q)E\left[Y_{i+2}^{(n-1)}|Y_i^{(n-1)} < x < Y_{i-1}^{(n-1)}\right] \\
&+q^3E\left[\beta_{i+3,k}^q(Y_{i+2}^{(n-1)})|Y_i^{(n-1)} < x < Y_{i-1}^{(n-1)}\right]
\end{aligned}
$$

Proceeding so and using the last round bidding function, the bid in every round is expressible in terms of the expected values from order statistic of $Y_1 \geq Y_2 \geq \ldots \geq Y_k$:

$$
\begin{aligned}
\beta_{i,k}^q(x) \; = \; & (1-q)E\left[Y_i^{(n-1)}|Y_i^{(n-1)} < x < Y_{i-1}^{(n-1)}\right] \\
&+q(1-q)E\left[Y_{i+1}^{(n-1)}|Y_i^{(n-1)} < x < Y_{i-1}^{(n-1)}\right] \\
&+q^2(1-q)E\left[Y_{i+2}^{(n-1)}|Y_i^{(n-1)} < x < Y_{i-1}^{(n-1)}\right] + \\
&\vdots \\
&+q^{k-i-1}(1-q)E\left[Y_{k-1}^{(n-1)}|Y_i^{(n-1)} < x < Y_{i-1}^{(n-1)}\right] \\
&+q^{k-i}E\left[Y_k^{(n-1)}|Y_i^{(n-1)} < x < Y_{i-1}^{(n-1)}\right] \\
= \; & (1-q)E\left[Y_i^{(n-1)}|Y_i^{(n-1)} < x < Y_{i-1}^{(n-1)}\right] \\
&+(1-q)\sum_{s=i+1}^{k-1} q^{s-i}E\left[Y_s^{(n-1)}|Y_i^{(n-1)} < x < Y_{i-1}^{(n-1)}\right] \\
&+q^{k-i}E\left[Y_k^{(n-1)}|Y_i^{(n-1)} < x < Y_{i-1}^{(n-1)}\right] \\
= \; & \sum_{s=i}^{k-1} q^{s-i}(1-q)E\left[Y_s^{(n-1)}|Y_i^{(n-1)} < x < Y_{i-1}^{(n-1)}\right] \\
&+q^{k-i}E\left[Y_k^{(n-1)}|Y_i^{(n-1)} < x < Y_{i-1}^{(n-1)}\right]
\end{aligned}
$$

□

Let see that the bidding function in each round is a sum of the bidding functions of the sequential auction setup without uncertainty, sensitive to the number of available items.

Claim 4: *The equilibrium function $\beta^q_{i,k}(x)$ is a convex combination of the first price equilibrium function and Milgrom&Weber sequential auction equilibrium functions:*

$$\beta^q_{i,k}(x) = (1-q)\beta^{(I,n-i+1)}(x) + \sum_{s=i+1}^{k-1} q^{s-i}(1-q)\beta_{i,s}(x) + q^{k-i}\beta_{i,k}(x)$$

Proof: Following Claim 3.1 :

$$\begin{aligned}\beta^q_{i,k}(x) &= (1-q)E\left[Y_i^{(n-1)}|Y_i^{(n-1)} < x < Y_{i-1}^{(n-1)}\right] \\ &+(1-q)\sum_{s=i+1}^{k-1} q^{s-i}E\left[Y_s^{(n-1)}|Y_i^{(n-1)} < x < Y_{i-1}^{(n-1)}\right] \\ &+q^{k-i}E\left[Y_k^{(n-1)}|Y_i^{(n-1)} < x < Y_{i-1}^{(n-1)}\right]\end{aligned}$$

The first expectation in sequence can be expressed as bidding function of the first-price auction for $n-i+1$ players. All other expectations are bidding functions of Mirgrom&Weber with changeable amount of available items from $i+1$ to k items. Using the appropriate notations it can be rewritten as:

$$\beta^q_{i,k}(x) = (1-q)\beta^{(I,n-i+1)}(x) + \sum_{s=i+1}^{k-1} q^{s-i}(1-q)\beta_{i,s}(x) + q^{k-i}\beta_{i,k}(x)$$

□

Having derived the symmetric equilibrium strategies we can compare the expected selling prices.

Claim 5: *The expected prices in the sequential auction under uncertainty are declining.*

Proof: The expected price of some i round is:

$$E[P_i] = E\left[\beta^q_{i,k}(Y_i^{(n)})\right] = E\left[\sum_{s=i}^{k-1} q^{s-i}(1-q)E\left[Y_{s+1}^{(n)}|Y_{i+1}^{(n)} < Y_i^{(n)}\right] + q^{k-i}E\left[Y_{k+1}^{(n)}|Y_{i+1}^{(n)} < Y_i^{(n)}\right]\right]$$

$$= \sum_{s=i}^{k-1} q^{s-i}(1-q)E\left[E\left[Y_{s+1}^{(n)}|Y_{i+1}^{(n)} < Y_i^{(n)}\right]\right] + q^{k-i}E\left[E\left[Y_{k+1}^{(n)}|Y_{i+1}^{(n)} < Y_i^{(n)}\right]\right]$$

$$= \sum_{s=i}^{k-1} q^{s-i}(1-q)E\left[Y_{s+1}^{(n)}\right] + q^{k-i}E\left[Y_{k+1}^{(n)}\right]$$

Analogously, expected price for the following round is:

$$E\left[P_{i+1}\right] = E\left[\beta_{i+1,k}^{q}(Y_{i+1}^{(n)})\right] = \sum_{s=i+1}^{k-1} q^{s-(i+1)}(1-q)E\left[Y_{s+1}^{(n)}\right] + q^{k-(i+1)}E\left[Y_{k+1}^{(n)}\right]$$

And subtracting:

$$\begin{aligned} E\left[P_i\right] - E\left[P_{i+1}\right] &= \sum_{s=i}^{k-2} q^{s-i}(1-q)\left(E\left[Y_{s+1}^{(n)}\right] - E\left[Y_{s+2}^{(n)}\right]\right) \\ &\quad + q^{k-i-1}(1-q)E\left[Y_{k}^{(n)}\right] + q^{k-i}E\left[Y_{k+1}^{(n)}\right] - q^{k-i-1}E\left[Y_{k+1}^{(n)}\right] \\ &= \sum_{s=i}^{k-1} q^{s-i}(1-q)\left(E\left[Y_{s+1}^{(n)}\right] - E\left[Y_{s+2}^{(n)}\right]\right) > 0. \end{aligned}$$

□

With $q = 1$ the result is zero, referring to constant price path of the Milgrom and Weber study.

Chapter 5

Conclusions

This study introduces a new feature of uncertainty in sequential auction settings with single-unit demand. We have shown that in our setting a perfect Bayesian equilibrium in symmetric monotone strategies exists. To summarize:

- In every conducted auction each bidder bids a convex combination of his estimation of the next order statistic values up to last possible round, conditional on having the highest value himself.
- Unit i ($i = 1, 2, \ldots$) is awarded to the bidder with the ith highest value.
- Bids in stage i are independent of the observed prices in stage $i - 1$.
- Equilibrium prices decline over stages.
- Average equilibrium bids are lower as q increases.

Our result complements the previous studies. More specifically, in a certain environment, average prices are trend-free and consistent with M&W conclusions. We provided the numerical examples to illustrate these points.

We presented only a general approach for the analysis where exact probability q to stop the realization of sequential auction exists. It is informative to investigate the properties of this component.

Many other features of real-world auctions still need to be taken into account to fully assess the effects of an uncertainty on bids and prices in sequential auctions. These include bidders' multi-unit demands and the presence of affiliated values and/or asymmetric preferences. Also, it can be interesting to study how reserve price or arrival of new bidders during a course of the auction manipulates bidder's attitude about the future.

We did not perform any revenue considerations in this work, and appropriate future research may be possible to develop this subject.

Furthermore, the risk neutral Nash equilibrium bidding strategy is only a best reply against itself. In order to have more applicable results, it can be interesting to practice bidding in the interactive auction experiment in addition to studying the symmetric equilibrium strategy.

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Appendix A

Order Statistics

Consider n random variables $X_1, \ldots, X_n$ defined on the same probability space $(\Omega = \{w\}, \mathcal{F}, P)$. Let $Y_1^{(n)} > Y_2^{(n)} > \ldots > Y_n^{(n)}$ be a rearrangement of these variables in decreasing order referred as *order statistics*. Thus

$$Y_1^{(n)} = max\{X_1, \ldots, X_n\},$$

$Y_2^{(n)}$ is defined so that $Y_2^{(n)}(w)$ for every elementary event w equals the second highest value among $X_1(w), \ldots, X_n(w)$ and so on, up to

$$Y_n^{(n)} = min\{X_1, \ldots, X_n\}.$$

Thus the following equality takes place:

$$Y_1^{(n)} + Y_2^{(n)} + \cdots + Y_n^{(n)} = X_1 + X_2 + \cdots + X_n$$

Taking sequence of independent identically distributed (i.i.d.) random variables $X_1, \ldots, X_n$ let F denote common distribution function with corresponding probability density function f.

I use notation employed in auction theory which inversed to terminology used by probability theory. I use notation "first order statistic" to *highest* element of the order. In statistics it is conventional to call *smallest* element as the first order. The uppercase letter captures the size of the sample and lowercase letter captures that statistic order. The specialized treatment of order statistics can be found in statistics text.

For a random sample of n observations from a continuous parent the distribution of the highest (*first*) order statistics Y_1 is easy to derive. The event that $Y_1^{(n)} \leq x$ is the same as the event $X_k \leq x$ for all k since each observation is independently drawn from the same distribution:

$$F_1^{(n)}(x) = P\left\{Y_1^{(n)} \leq x\right\} = P\{X_1 \leq x, \ldots, X_n \leq x\} = \prod_{k=1}^{n} P\{X_k \leq x\} = F(x)^n.$$

The associated probability density function is:

$$f_1^{(n)}(x) = nF(x)^{n-1}f(x).$$

Similarly the lowest (*nth-order*) statistics $Y_n^{(n)}$ is:

$$F_n^{(n)}(x) = P\left\{Y_n^{(n)} \leq x\right\} = 1 - P\left\{Y_n^{(n)} > x\right\}$$

$$= 1 - P\{X_1 > x, \ldots, X_n > x\} = 1 - (1 - F(x))^n$$

Now the general formula for $F_k^{(n)}(x)$ is the probability that $Y_k^n \leq x$. This is the union of k disjoint events: all n draws are less or equal to x, $(n-1)$ draws are less or equal to x and one value is greater than x, and and so on till $(n-k+1)$ draws are less or equal to x and $(k-1)$ values are greater then x.

$$F_k^{(n)}(x) = P\left\{Y_k^{(n)} \leq x\right\}$$

$$= P\{at\, least\, n-k+1\, variables\, among\, X_1, \ldots, X_n\, lie\ on\ the\ left\ of\ x\}$$

$$= \sum_{j=n-k+1}^{n} P\{exactly\, j\, variables\, among\, X_1, \ldots, X_n\, lie\ on\ the\ left\ of\ x\}$$

$$= \sum_{j=n-k+1}^{n} \begin{pmatrix} n \\ j \end{pmatrix} F(x)^j(1 - F(x))^{n-j}$$

$$= \frac{n!}{(k-1)!(n-k)!}\int_0^{F(x)} t^{n-k}(1-t)^{k-1}dt$$

Then common density function of order statistic $Y_k^{(n)}$ is $f_k^{(n)}(x) = \left(F_k^{(n)}(x)\right)'$ for all x where:

$$f_k^{(n)}(x) = \frac{n!}{(k-1)!(n-k)!}F(x)^{n-k}(1-F(x))^{k-1}f(x)$$

It will be useful to know the joint and conditional distributions of two different order statistics and observe properties of this distributions. The joint cdf $F_{r,s}^{(n)}(y,x)$ of $Y_r^{(n)}$ and $Y_s^{(n)}$ may be obtained by a direct argument. Let $r<s$ and $Y_r^{(n)} > Y_s^{(n)}$. Denote $F_{r,s}^{(n)}(y,x) = P\left\{Y_r^{(n)} \le y, Y_s^{(n)} \le x\right\}$.

If $y > x$ then:

$$F_{r,s}^{(n)}(y,x) = P\left\{Y_r^{(n)} \le y, Y_s^{(n)} \le x\right\}$$

$$= P\left\{at\,least\,n-r+1\,variables\,are\,less\,or\,equal\,y,\;at\,least\,n-s+1\,variables\,are\,less\,or\,equal\,x\right\}$$

$$= \sum_{i=n-s+1}^{n}\sum_{j=max(i,n-r+1)}^{n} P\left\{exactly\,j\,variables\,are\,less\,or\,equal\,y,\;exactly\,i\,variables\,are\,less\,or\,equal\,x\right\}$$

$$= \sum_{i=n-s+1}^{n}\sum_{j=n-r+1}^{i-1}\frac{n!}{i!(j-i)!(n-j)!}F(x)^i\left(F(y)-F(x)\right)^{j-i}\left(1-F(y)\right)^{n-j},$$

If $y \le x$ then:

$$P\left\{Y_r \le y, Y_s \le x\right\} = F_{r,s}^{(n)}(y,x) = F_r^{(n)}(y) = \sum_{j=n-r+1}^{n}\begin{pmatrix} n \\ j \end{pmatrix} F(y)^j(1-F(y))^{n-j}$$

Other important case which I use in developing payoff function is then $Y_r^{(n)} > Y_s^{(n)}$ and $P\left\{Y_r^{(n)} > y, Y_s^{(n)} \le x \le Y_{s-1}^{(n)}\right\}$. If $y \ge x$:

$$P\left\{Y_r^{(n)} > y, Y_s^{(n)} \le x \le Y_{s-1}^{(n)}\right\}$$

$$= P\left\{exactly\,n-s+1\,variables\,are\,less\,or\,equal\,x,\;at\,least\,r\,variables\,are\,more\,than\,y\right\}$$

$$= \sum_{i=n-s+1}^{n-r} P\left\{exactly\,n-s+1\,variables\,are\,less\,or\,equal\,x,\;exactly\,i\,variables\,are\,less\,than\,y\right\}$$

$$= \sum_{i=n-s+1}^{n-r}\begin{pmatrix} n \\ i \end{pmatrix}\begin{pmatrix} i \\ n-s+1 \end{pmatrix} F(x)^{n-s+1}\left(F(y)-F(x)\right)^{i-(n-s+1)}\left(1-F(y)\right)^{n-i}$$

If $y < x$:

$$P\left\{Y_r^{(n)} > y, Y_s^{(n)} \leq x \leq Y_{s-1}^{(n)}\right\} = P\left\{Y_s^{(n)} \leq x \leq Y_{s-1}^{(n)}\right\} =$$

$$= \begin{pmatrix} n \\ s-1 \end{pmatrix} F(x)^{n-s+1}(1-F(x))^{s-1}$$

Passing from original random variables $X_1, \ldots, X_n$ with a common density f to order statistics, the independence property is lost. It can be shown, however, that for continuous underlying distributions the order statistics form a Markov chain. It can be seen that for any $1 < k < n$ the conditional density of $Y_k^{(n)}$ $f_k^{(n)}(x|y_1, y_2, \ldots, y_{k-1})$ given all previous order statistics, $Y_1^{(n)} = y_1, Y_2^{(n)} = y_2, \ldots, Y_{k-1}^{(n)} = y_{k-1}$ coincides with conditional density $f_1^{(n-k+1)}\left(x|Y_1^{(n-k+1)} \leq y_{k-1}\right)$ of $Y_k^{(n)} = Y_1^{(n-k+1)}$ given only that $Y_1^{(n-k+1)} \leq y_{k-1}$:

$$f_k^{(n)}(x|y_1, y_2, \ldots, y_{k-1}) = \frac{f_{1,2,\ldots,k}^{(n)}(y_1, y_2, \ldots, y_{k-1}, x)}{f_{1,\ldots,k-1}^{(n)}(y_1, y_2, \ldots, y_{k-1})}$$

$$= \frac{\frac{n!}{(n-k)!} f(y_1)f(y_2)\cdots f(y_{k-1})f(x)F(x)^{n-k}}{\frac{n!}{(n-k+1)!} f(y_1)f(y_2)\cdots f(y_{k-1})F(y_{k-1})^{n-k+1}}$$

$$= \frac{(n-k+1)F(x)^{n-k}f(x)}{F(y_{k-1})^{n-k+1}} = \frac{f_1^{(n-k+1)}(x)}{F_1^{(n-k+1)}(y_{k-1})}$$

$$= f_1^{(n-k+1)}\left(x|Y_1^{(n-k+1)} \leq y_{k-1}\right)$$

which shows that the order statistics conditioning on the highest order statistics in a sample from continuous parent leads to truncating the parent distribution in the upper range tail at y_{k-1}. Therefore, conditional probability distribution of future states, given the present state and a constant number of past states, depend only upon the present state and conditionally independent of these older states. This property is named the *Markov property*. Only the closest higher order has meaning for consequent order k^{th}. The all sample can be seen as truncated from the tail of highest values. In particular, since the conditional distribution is free of y_i for

$i < k-1$, $Y_k^{(n)}, \ldots, Y_n^{(n)}$ are independent of $Y_1^{(n)}, Y_2^{(n)}, \ldots Y_{k-2}^{(n)}$ then $Y_{k-1}^{(n)}$ are given.

$$f_k(x|Y_{k-1} = y_{k-1}) = \frac{(n-k+1)F(x)^{n-k}f(x)}{F(y_{k-1})^{n-k+1}}$$

,