The Incompatibility of Strategy-proofness and Pareto-optimality in Quasi-linear Settings with Public Budget Constraints

Research Thesis

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Abstract

Almost all of the studies regarding auction theory focus on the player's value of getting one specific alternative or another and completely ignore the existence of players' maximum upper bound on their possible payment to the auction - their budgets. This fact often causes a mismatch between practice and theory while attempting actual implementation of last. The existing literature studying budgets indicates that addressing them properly to the model breaks down the usual quasi-linear setting and causes significant changes in it's conception and techniques.

We study the problem of allocating multiple identical items that may be complements to budget-constrained bidders with private values. We show that there does not exist a deterministic mechanism that is individually rational, strategy-proof, Pareto-efficient, and that does not make positive transfers. This is true even if there are only two players, two items, and the budgets are common knowledge. The same impossibility naturally extends to more abstract social choice settings with an arbitrary outcome set, assuming players with quasi-linear utilities and public budget limits. Thus, the case of infinite budgets (in which the VCG mechanism satisfies all these properties) is really the exception.

Chapter 1

Introduction

Almost all of the studies regarding auction theory focus on the player's value of getting one specific alternative or another (in case of auctions selling goods or services, such an alternative is simply getting or not that good/service or another) and completely ignore the existence of players' maximum upper bound on their possible payment to the auction - their budgets. This fact often causes a mismatch between practice and theory while attempting actual implementation of last. Indeed, budgets should be definitely taken into consideration as they constitute an important role in the strategic consideration of the bidders. The existing literature studying budgets indicates that addressing them properly to the model breaks down the usual quasi-linear setting and causes significant changes in it's conception and techniques.

It is well known that the possibility of designing strategy-proof and Pareto-efficient mechanisms depends on the structure of players' utilities. Most remarkably, when utilities are quasi-linear in money, the VCG mechanism (Groves (1973)) is strategy-proof and Pareto-efficient. In recent years, several works study, in the context of auctions, whether this possibility holds if the quasi-linearity assumption is slightly relaxed, to allow for hard budget constraints. For example, Aggarwal, Muthukrishnan, Pal and Pal (2009) and Ashlagi,Braverman, Hassidim, Lavi and Tennenholtz (2010) construct strategy-proof and Pareto-efficient deterministic auctions for unit-demand bidders with private values and budget constraints. In addition, naturally, these auctions are individually rational, and never make positive transfers. In contrast, with multi-unit demand, Fiat, Leonardi, Saia and Sankowski (2011) show that there does not exist such auction, even with only two items, two bidders with multi-unit demand, and public (commonly known) budgets.

For *identical* items, the situation with multi-unit demand is better: Dobzinski, Lavi and Nissan (2008) show that Ausubel's clinching auction (uniquely) satisfies all above-mentioned properties, assuming public budgets and additive valuations. In this paper we show that when the identical items are complements, the impossibility returns. Specifically, even with two identical items, two bidders with valuations that are not additive, and even if budgets are public (i.e., commonly known), there still does not exist individually rational, strategy-proof, and Pareto-efficient auction that makes no positive transfers.

The no-positive-transfers requirement that we use is quite weak, as we only require the *sum* of players' payments to be non-negative. This is clearly an important design criterion in most auction settings. In fact, with an unlimited amount of positive transfers, one can simply increase the players' budgets to be ineffective, and then use the VCG mechanism (and indeed the Groves mechanism can be tuned to make only positive transfers).

Our main proof is quite simple, and does not rely on previous impossibilities. In comparison, the proof of Fiat et al. (2011) for non-identical items relies on the uniqueness result of Dobzinski et al. (2008). Briefly, the proof is composed of two claims that contradict each other, but must be satisfied by any mechanism with the above four properties. Let b_i denote the budget of player i, and assume $b_2 > b_1$. We show that if $v_2(2) > b_1$ (where $v_i(q)$ denotes player i's value for q items), player 2 must pay zero when she receives one item. On the other hand, if $v_2(2) < b_1$, prices must be regular VCG prices. These two claims contradict, since when player 2 has low values she prefers to falsely declare high values.

The additional proof (in some different setting variation) we present relies on the uniqueness result of Dobzinski et al. (2008) and is based on the contradiction between the prices paid in general mechanism and in the Adaptive Clinching Auction of Dobzinski et al. (2008).

There are many classic as well as recent results on impossibilities in mechanism design without the quasi-linearity assumption, starting from Gibbard (1973) and Satterthwaite (1975), where the typical result is that only dictatorship (or sequential dictatorship) is strategy-proof and Pareto-efficient. Interestingly, here, even dictatorship is not a candidate, as dictatorship is simply not efficient when utilities are quasi-linear. Our result implies that there really is *no* mechanism that satisfies the above desired properties, for almost all structures of valuations over some abstract set of outcomes. The two extremes of zero budgets, and of infinite budgets, are just two rare exceptions.

A possible way to advance, given these impossibilities, is to study the "constrained efficiency" problem of maximizing efficiency subject to Bayesian incentive-compatibility constraints, as initiated in Maskin (2000). It is also important to understand the powers and limitations of randomized mechanisms, as initiated by Bhattacharya, Conitzer, Munagala and Xiax (2010) for the case of a divisible good.

Literature survey is given in chapter 2. In chapter 3, we formally define our model and state our result. The first (and the main) proof is given in chapter 4, the second version of the proof in some different setting is given in chapter 5.

Chapter 2

Literature Survey

The existing literature studying budgets indicates that adressing them properly to the model breaks down the usual quasi-linear setting and causes significant changes in it's conception and techniques.

When utilities are quasi-linear in money, the VCG mechanism (Groves (1973)) is srategy-proof and Pareto-efficient. In recent years, in auctions context, several works study whether this possibility holds if the quasi-linearity assumption is slightly relaxed, to allow for hard budget constraints. For example, Aggarwal, Muthukrishnan, Pal and Pal (2009) and Ashlagi,Braverman, Hassidim, Lavi and Tennenholtz (2010) construct strategy-proof and Pareto-efficient deterministic auctions for unit-demand bidders with private values and budget constraints.

Aggarwal et al. (2009) modeled advertising auctions in terms of an assignment model with linear utilities, extended with bidder and item specific maximum and minimum prices. The model of bidder preferences allows for a wider range of bidder behaviors than just profit maximization (i.e. it is not assumed that the bidder's payoff is quasi-linear in payment). They proved that in their model the existence of a stable matching is guaranteed, and under a non-degeneracy assumption a bidder-optimal stable matching exists as well. The work gives an algorithm to find such matching in polynomial time, and uses it to design truthful mechanism that generalizes GSP (Generalized Second Price), is truthful for profitmaximizing bidders, correctly implements features like bidder-specific minimum prices and position-specific bids, and works for rich mixtures of bidders and preferences. The authors' main technical contributions are the existence of bidder-optimal matchings and strategyproofness of the resulting mechanism.

Ashlagi et al. (2010) designed a Generalized Position Auction for players with private values and private budgets and proved the existence and uniqueness of the outcome. Their modification of the Generalized English Auction of Edelman, Ostrovsky and Schwarz (2007) retrieves all its positive properties, while taking budgets into account; it has a unique ex-post equilibrium which outcome is envy-free and Pareto-efficient. Ashlagi et al. (2010) showed that any other mechanism that satisfies the properties of Pareto-efficiency and envy-freeness obtains in ex-post equilibrium the same outcome as their Generalized Position Auction, at least whenever all true budgets are distinct.

Both auctions mentioned above are individually rational and never make positive transfers.

In contrast, with multi-unit demand, Fiat, Leonardi, Saia and Sankowski (2011) show that there does not exist strategy-proof and Pareto-efficient auction, even with only two items, two bidders with multi-unit demand, and public (commonly known) budgets. In their work, they give a combinatorial auction that is incentive compatible with respect to private valuations, individually rational, and the auctioneer makes no positive transfers. Their result is an extension of Dobzinski et al. (2008) from the case of multiple identical items to a new combinatorial setting where items are distinct and different agents may be interested in different items. They also show that in their setting, with public budgets, private valuations and private sets of interest, there can be no truthful Pareto-optimal auction (the proof for non-identical items relies on the uniqueness result of Dobzinski et al. (2008), mentioned below).

For identical items, the situation with multi-unit demand is better: Dobzinski, Lavi and Nissan (2008) show that Ausubel's clinching auction (uniquely) satisfies all above-mentioned properties, assuming public budgets and additive valuations. Their work studies multi-unit auctions where the bidders have a budget constraint and provides the result of impossibility of incentive-compatible auction that always produces a Pareto-optimal outcome, for any finite number $m > 1$ of units of an indivisible good and any $n \geq 2$ number of players. The authors show that when budgets are public knowledge there exists a unique auction that is truthful and Pareto-optimal - The Adaptive Clinching Auction (the uniqueness holds when there are exactly two bidders). The Adaptive Clinching Auction satisfies Individual Rationality, Incentive Compatibility, optimality and always allocates all items.

There are many classic as well as recent results on impossibilities in mechanism design without the quasi-linearity assumption, where the typical result is that only dictatorship (or sequential dictatorship) is strategy-proof and Pareto-efficient.

Gibbard (1973) proved in his work that any non-dictatorial voting scheme (any scheme which makes a community's choice depend entirely on individuals' professed preferences among the alternatives) with at least three possible outcomes is subject to individual manipulation (is not strategy-proof).

Satterthwaite (1975) considers in his work a committee which must select one alternative from a set of three or more alternatives and proves that if a committee is choosing among at least three alternatives, then every strategy-proof voting procedure vests in one committee member absolute power over the committee's choice. In other words, every strategy-proof voting procedure is dictatorial.

Hatfield (2009) considered the problem of allocating a fixed number of available items to agents in an incentive compatible and Pareto optimal way, when each agent has a quota that must be filled exactly. He showed that the set of mechanisms that are strategy-proof, Pareto-optimal, and nonbossy is the set of sequential dictatorships, even when the preferences of the agents are restricted to be responsive (the change in utility from substituting one item for another depends only on those two items, nor the other items the agent obtains).

The work by Budish and Cantillon (2010) uses data consisting of students' strategically reported preferences and their underlying true preferences to study the course allocation mechanism used at Harvard Business School (HBS) since the mid-1990s to allocate roughly 9000 elective course seats to about 900 second-year MBA students every year. The study finds out that the HBS mechanism is simple to manipulate in theory, is heavily manipulated by students in practice, and these manipulations cause congestion and substantial inefficiency, assesed either ax-ante (i.e. before priority orders drawn) or ex-post. Yet, ex-ante welfare is higher under the HBS mechanism than under the strategy-proof and ex-post efficient alternative, the Random Serial Dictatorship.

There are some possible ways to advance, given the discussed above impossibilities. One of them is to study the "constrained efficiency" problem of maximizing efficiency subject to Bayesian incentive-compatibility constraints, as initiated in Maskin (2000). He studies in his work the model that consists of one (indivisible) unit of a capital asset to be privatized and n potential (risk neutral) buyers, and exhibits an efficient auction (an auction for which the winner is the buyer with the highest valuation) subject to the constraint that buyers may be budget or liquidity constrained.

Another direction is to understand the powers and limitations of randomized mechanisms. The work by Bhattacharya, Conitzer, Munagala and Xiax (2010) following the Dobzinski et al. (2008) study, tries to improve the results reached by Dobzinski et al. (2008). It claims that a small randomized modification to the Adaptive Clincing Auction makes it Pareto-optimal and incentive compatible with private budgets (it is shown that in case of one infinitely divisible good a bidder cannot improve his utility by reporting budget smaller than the truth, which implies that the adaptive clinching auction is truthfull only when over-reporting the budget is impossible).

Chapter 3

Problem Statement and Main Result

We study the following very simple setting. The simplicity of the setting only strengthens our result, since it is an impossibility. There are two identical items and two players $1, 2$. Each player i has a private valuation function $v_i(q)$ that denotes i's non-negative real value for $q = 1, 2$ items, where $v_i(2) \geq v_i(1)$ (free disposal), and for notational simplicity we also use $v_i(0) = 0$. Each player additionally has a commonly known budget constraint $b_i > 0$. The utility of player i from obtaining $q = 1, 2$ items for a price p_i is quasi-linear up to the budget constraint, i.e., it is $v_i(q) - p_i$ if $p_i \leq b_i$, and it is some arbitrary negative number if $p_i > b_i$. An *outcome* is a tuple (q_1, q_2, p_1, p_2) where $q_1, q_2 \in \{0, 1, 2\}$, $q_1 + q_2 \leq 2$, and $p_i \leq b_i$ for $i = 1, 2$. (since budgets are public it simplifies notation and it is without loss of generality to disallow outcomes in which a player pays more than her budget).

Relying on the revelation principle, we consider only direct mechanisms. In a direct deterministic mechanism, each player i reports some type $\tilde{v}_i(\cdot)$, and, given these reports and the knowledge of b_1, b_2, the mechanism decides on an outcome $(q_i(\tilde{v}_1(\cdot), \tilde{v}_2(\cdot)), p_i(\tilde{v}_1(\cdot), \tilde{v}_2(\cdot)))_{i=1,2}$. Having fixed the parameters b_1, b_2, and some direct mechanism, we denote by $u_i((\tilde{v}_1(\cdot), \tilde{v}_2(\cdot)), v_i(\cdot))$ player i's resulting utility from the outcome of the mechanism when her private type is $v_i(\cdot)$ and the declarations are $(\tilde{v}_1(\cdot), \tilde{v}_2(\cdot))$.

We consider the following four standard and desirable properties. The first two properties address strategic issues, and need to be satisfied for any player i, any true valuation $v_i(\cdot)$ of i, any possible declaration $\tilde{v}_i(\cdot)$ of player i, and any possible declaration of the other player (say j), $\tilde{v}_j(\cdot)$.

Individual rationality (IR). A mechanism is ex-post individually rational if player i can always obtain a non-negative utility by truth-telling, i.e., $u_i((v_i(\cdot), \tilde{v}_j(\cdot)), v_i(\cdot)) \geq 0$.

Strategy proofness (SP). A mechanism is strategy-proof if truth-telling is a dominant strategy, i.e., $u_i((v_i(\cdot), \tilde{v}_j(\cdot)), v_i(\cdot)) \geq u_i((\tilde{v}_i(\cdot), \tilde{v}_j)(\cdot), v_i(\cdot))$.

The last two properties address design issues, and need to be satisfied for any tuple of valuations/declarations $(v_1(\cdot), v_2(\cdot))$.

No Positive Transfers (NPT). A mechanism satisfies no positive transfers if $p_1(v_1(\cdot), v_2(\cdot)) + p_2(v_1(\cdot), v_2(\cdot)) \geq 0$.

Pareto optimality (PO). A mechanism is Pareto-optimal if its outcome is Pareto optimal. Formally, there does not exist another outcome $\tilde{o} = (\tilde{q}_1, \tilde{q}_2, \tilde{p}_1, \tilde{p}_2)$ that is preferred by both bidders, i.e., $v_i(\tilde{q}_i) - \tilde{p}_i \geq v_i(q_i(v_1(\cdot), v_2(\cdot))) - p_i(v_1(\cdot), v_2(\cdot))$ for $i = 1, 2$, and by the seller, i.e., $\tilde{p}_1 + \tilde{p}_2 \geq p_1(v_1(\cdot), v_2(\cdot)) + p_2(v_1(\cdot), v_2(\cdot))$, with at least one strict inequality.[1]

If valuations were additive, Ausubel's clinching auction would satisfy all these properties, as Dobzinski et.al (2008) show. Even in our setting, there do exist mechanisms that satisfy any three of these four properties. Without IR, for example, one can charge each player her full budget, and choose some arbitrary allocation independently of the declarations. Without NPT, as remarked earlier, there exists a possible Groves mechanism. This paper shows that combining the four properties together is unfortunately impossible.

Theorem 1. *In our setting, if $b_1 \neq b_2$, there does not exist any mechanism that satisfies IR, SP, NPT, and PO.*

This theorem implies the same impossibility for all settings that generalize ours, since any mechanism for a more general setting can be used to construct a mechanism with the same properties for our setting. In particular, this gives an impossibility for any number of identical or non-identical items and any number of players with general valuations over the set of items. Similarly, it implies the same impossibility for the abstract social choice setting with at least three alternatives and an unrestricted domain of players' valuations.

[1]Note that $\tilde{o}$ must satisfy $\tilde{p}_i \leq b_i$ by definition. The seller must be included in this definition to preclude the trivial improvement of bidders' utilities by reducing prices (e.g., to zero).

More formaly, the general setting is defined as follows: there are two players 1,2 and 3 general alternatives a,b,c (set x - some alternative chosen). Each player i has a private valuation function $v_i(x)$ that denotes i 's non-negative real value for some alternative x. Each player also has a commonly known budget constraint $b_i > 0$. The utility of player i from the chosen alternative x is $v_i(x) - p_i(x)$ if $p_i(x) \leq b_i$, and is some arbitrary negative number if $p_i(x) > b_i$. An outcome is a triple $(x, p_1(x), p_2(x))$ where $x \in (a, b, c)$ and $p_i(x) \leq b_i$ for $i = 1, 2$.

The input of the mechanism would be then as follows:

player	a	b	c	b_i
1	$v_1(a)$	$v_1(b)$	$v_1(c)$	b_1
2	$v_2(a)$	$v_2(b)$	$v_1(c)$	b_2

Table 3.1: values and budgets declared by the players

Theorem 2. *In general setting, if $b_1 \neq b_2$, there does not exist any mechanism that satisfies IR, SP, NPT, and PO.*

Theorem 2 derives from theorem 1 (if there was a mechanism for a general setting it could've been used to construct a mechanism for a more spesific setting of theorem 1), however we will give a direct proof of theorem 2 for the case when $b_1 > 2b_2$ in chapter 5.

Chapter 4

The Proof (main version)

The main proof is composed of four components, that together yield a contradiction to the existence of a mechanism that satisfies the four mentioned properties. Without loss of generality we assume throughout that $b_2 > b_1$.

First component: $q_i = 0$ implies $p_i = 0$. The first component shows that if player i receives no items, her price is exactly zero. A very similar argument appears in Dobzinski et al. (2008) and in Fiat et al. (2011), we include it here mainly for completeness, but also since the exact technical connection is vague (as these papers study a different setting). We should also note that this claim does not immediately follow from IR and NPT – these requirements only imply that $p_i \leq 0$, and a-priori it may well be that $p_i < 0$ (i.e., a positive transfer to i) if the other player who receives both items has a positive payment that can balance the transfer. We start with a simple case.

Claim 1. *If $v_i(2) = v_i(1) = 0$ then $p_i = 0$.*

Proof : Let j denote the other player, let $v_j(\cdot)$ be her declaration, and let (q_1, q_2, p_1, p_2) be the resulting outcome. If $v_j(2) = 0$, IR implies $p_i, p_j \leq 0$ and NPT then implies $p_i = p_j = 0$. Thus, assume $v_j(2) > v_j(1) > 0$. In this case PO implies $q_j = 2$, otherwise the outcome that keeps the same prices and sets $\tilde{q}_j = 2, \tilde{q}_i = 0$ Pareto improves the previous outcome (player j's utility strictly increases since $v_j(2) > v_j(1) > 0$ and player i's utility does not change since $v_i(2) = v_i(1) = 0$). Now, IR implies $p_j \leq v_j(2)$. Since for *any* $v_j(2) > v_j(1) > 0$

(having fixed $v_i(\cdot) = 0$), player j receives both items, strategy-proofness implies that $p_j \leq 0$, otherwise player j can report some positive $\tilde{v}_j(2) < p_j$, still receive both items, and pay strictly less, thus strictly improving her utility and contradicting SP.

Since $p_j \leq 0$, and IR requires $p_i \leq 0$, NPT now implies $p_j = p_i = 0$, and the claim follows. □

Claim 2. *Whenever $q_i = 0$, $p_i = 0$.*

Proof : Suppose by contradiction that there exist $v_1(\cdot), v_2(\cdot)$ such that $q_i(v_1(\cdot), v_2(\cdot)) = 0$ but $p_i(v_1(\cdot), v_2(\cdot)) \neq 0$. By IR, $p_i(v_1(\cdot), v_2(\cdot)) < 0$, hence i's utility is strictly positive. Then, in case i's true value is $v_i'(\cdot) = 0$, i can increase her utility by misreporting her type to be $v_i(\cdot)$ (if she reports her true value $v_i'(\cdot)$, Claim 1 implies that her utility will be zero). This contradicts SP. Thus, $p_i(v_1(\cdot), v_2(\cdot)) = 0$, as claimed. □

Second component: the case where $v_2(1) = 0$. (Recall that we assume $b_2 > b_1$.)

Claim 3. *Suppose $\min\{b_2, v_2(2)\} > v_1(2) > b_1$, and $v_2(1) = 0$. Then $q_2 = 2$, and $q_1 = p_1 = 0$.*

Proof : Suppose that the outcome of the mechanism for this tuple of valuations is q_1, q_2, p_1, p_2. We show that the claim directly follows from PO. Suppose by contradiction that $q_2 < 2$. Then, we argue that the following outcome is a Pareto improvement: $\tilde{q}_1 = 0, \tilde{p}_1 = p_1 - \min\{b_2, v_2(2)\}, \tilde{q}_2 = 2, \tilde{p}_2 = p_2 + \min\{b_2, v_2(2)\}$. To verify this, note that $\tilde{p}_2 \leq b_2$ since $p_2 \leq 0$ (by IR, since $q_2 \leq 1$ and $v_2(1) = 0$). Also, by definition, $\tilde{p}_1 + \tilde{p}_2 = p_1 + p_2$. Player 2's utility does not decrease since the added value is $v_2(2)$ and the added price is at most that. Player 1's utility strictly increases, since the decrease in her value is at most $v_1(2)$ and the decrease in her price is $\min\{b_2, v_2(2)\} > v_1(2)$. This shows that we have indeed constructed a Pareto improvement, which is a contradiction. We conclude that it must be that $q_2 = 2$ and $q_1 = 0$. Claim 2 now implies $p_1 = 0$, and the claim follows. □

While the last claim requires $v_2(2) > v_1(2) > b_1$, the next claim allows very large or very small values, without a connection between the values of the two players.

Claim 4. *Suppose $v_2(1) = 0$ and $v_1(2) > v_1(1)$. Then either $q_1 = 2$ or $q_2 = 2$.*

Proof :This again follows directly from PO. Since $v_i(2) > v_i(1)$ for $i = 1, 2$, PO implies $q_1 + q_2 = 2$. Thus, we only need to rule out the case that $q_1 = q_2 = 1$. This outcome is Pareto dominated by the following outcome: $\tilde{q}_1 = 2, \tilde{p}_1 = p_1, \tilde{q}_2 = 0, \tilde{p}_2 = p_2$. (Player 1's utility strictly increases since $v_1(2) > v_1(1)$, player 2's utility is the same in both cases as $v_2(1) = 0$, and prices are the same.) □

Third component: some bounds on prices when $v_2(2) > b_1$. We will show a case where player 2's price for one item is at most zero. This makes the contradiction very close, as clearly an efficient mechanism cannot give an item "for free". We start with 2's price for two items.

Claim 5. *Suppose $v_2(2) > b_1$, $v_2(1) = 0$, and $v_1(2) > v_1(1)$. Then $q_2 = 2$, and $p_2 \leq b_1$.*

Proof :This now follows from SP. By Claim 4, either $q_1 = 2$, or $q_2 = 2$. Suppose by contradiction that $q_1 = 2$, and consider the case where player 1's true valuation is $\tilde{v}_1(\cdot)$, where $\min\{b_2, v_2(2)\} > \tilde{v}_1(2) > b_1$. By Claim 3, if player 1 truthfully reports $\tilde{v}_1(\cdot)$ as her valuation, her resulting utility will be exactly zero. If, however, player 1 misreports her valuation to be $v_1(\cdot)$, she will receive two items, and by IR will pay at most b_1. Since $\tilde{v}_1(2) > b_1$, misreporting her value in this case strictly increases player 1's utility, contradicting SP. Thus, $q_2 = 2$. Since this is true for *every* $v_2(2) > b_1$, IR and SP imply $p_2 \leq b_1$. □

Claim 6. *Suppose $v_2(2) > v_2(1) > b_1$, $v_2(2) - v_2(1) \leq b_1 < v_1(1) < v_1(2)$. Then $q_2 = 1$ and $p_2 \leq 0$.*

Proof :We first show that $q_2 = 1$. If $q_2 = 0$, player 2's resulting utility is exactly zero by Claim 2. However, if player 2 will declare a false valuation $\tilde{v}_2(\cdot)$ such that $\tilde{v}_2(2) = v_2(2)$ and $\tilde{v}_2(1) = 0$ she will receive two items and will pay at most b_1 (by Claim 5), hence will obtain a strictly positive utility, contradicting SP. If $q_2 = 2$, the following outcome is a Pareto improvement: $\tilde{q}_1 = \tilde{q}_2 = 1, \tilde{p}_1 = p_1 + \Delta, \tilde{p}_2 = p_2 - \Delta$, where $\Delta = v_2(2) - v_2(1)$, and this contradicts PO. Thus, $q_2 = 1$. Since player 2 can receive two items and pay at most b_1 by declaring $\tilde{v}_2(1) = 0$, SP implies $v_2(1) - p_2 \geq v_2(2) - b_1$, i.e. $p_2 \leq b_1 - (v_2(2) - v_2(1))$. Since this is true also when $v_2(2) - v_2(1) = b_1$, SP implies that $p_2 \leq 0$. □

Forth component: allocation and prices when $v_2(2) < b_1$. We now study the complementary case where $v_2(2)$ is relatively small. We show that in this case the mechanism must choose the VCG outcome. In particular, in some cases player 2 receives one item and pays a strictly positive price. This will imply the theorem by contradicting SP, as if player 2 misreports her value to be larger than b_1 she can receive one item "for free". We start with a standard claim.

Claim 7. *Let i, j be two distinct players, and fix some valuation $\tilde{v}_j$ for player j. If $q_i(v_i(\cdot), v_j(\cdot)) = q_i(\tilde{v}_i(\cdot), v_j(\cdot))$ for two valuations $v_i(\cdot), \tilde{v}_i(\cdot)$ of player i, then it must be that $p_i(v_i(\cdot), v_j(\cdot)) = p_i(\tilde{v}_i(\cdot), v_j(\cdot))$ as well.*

Proof : Otherwise, if w.l.o.g. $p_i(v_i(\cdot), v_j(\cdot)) < p_i(\tilde{v}_i(\cdot), v_j(\cdot))$, when i's true valuation is $\tilde{v}_i(\cdot)$ she can increase her utility by declaring $v_i(\cdot)$ (this way, her price will decrease while the allocation remains the same), contradicting SP. □

We continue by analyzing the case where all values are smaller than b_1. One should not be surprised that in this case the mechanism must be VCG, as this case is similar to the case when there are no budgets at all, and for this case it is well known that the unique strategy-proof and efficient mechanism is VCG. In this sense, the proof of the following claim is quite standard, and we provide it mainly for completeness.

Claim 8. *Suppose $\min_{i=1,2} b_i > \max_{i=1,2} v_i(2)$. Then the allocation (q_1, q_2) maximizes the welfare $v_1(q_1) + v_2(q_2)$, if $q_i = 2$ then $p_i = v_j(2)$, and if $q_i = 1$ then $p_i = v_j(2) - v_j(1)$, where j is the other player.*

Proof :

Allocation. First suppose $v_i(2) > \max\{v_j(2), v_1(1) + v_2(1)\}$. In this case, if $q_i < 2$, the following outcome is a Pareto improvement: $\tilde{q}_i = 2, \tilde{p}_i = p_i + (v_i(2) - v_i(q_i)), \tilde{q}_j = 0, \tilde{p}_j = p_j - (v_i(2) - v_i(q_i))$. By IR, $p_i \leq v_i(q_i)$. Thus, $\tilde{p}_i \leq v_i(2) < b_i$. Since $v_i(2) \geq v_i(q_i)$, $\tilde{p}_j \leq p_j \leq b_j$. Clearly, $\tilde{p}_1 + \tilde{p}_2 = p_1 + p_2$, and player i's utility is exactly the same in both outcomes, as the added value $(v_i(2) - v_i(q_i))$ is exactly balanced by the increase in price. Finally, since $v_i(2) - v_i(q_i) > v_j(q_j)$ (whether $q_i \leq 1$ and $q_j \leq 1$, or $q_j = 2$ and $q_i = 0$), player j's utility strictly increases, as her decrease in value is $v_j(q_j)$ and her decrease in price is $v_i(2) - v_i(q_i)$. This shows that we have indeed constructed a Pareto improvement, contradicting PO. We conclude that in this case $q_i = 2$.

Now suppose $v_1(1) + v_2(1) > \max_{i=1,2} v_i(2)$. We need to show that $q_1 = q_2 = 1$. Suppose by contradiction that there exists a player i with $q_i = 0$, and let j be the other player. Then, the following outcome is a Pareto improvement: $\tilde{q}_1 = \tilde{q}_2 = 1, \tilde{p}_i = v_i(1), \tilde{p}_j = p_j - v_i(1)$. We have $\tilde{p}_i < b_i$ since $v_i(1) < b_i$ by assumption, and $\tilde{p}_j < b_j$ since $p_j < b_j$ by IR. By Claim 2, $p_i = 0$, hence $p_1 + p_2 = p_j = \tilde{p}_1 + \tilde{p}_2$. Player i's utility is zero in both outcomes. The utility of player j strictly increases: $v_j(\tilde{q}_j) - \tilde{p}_j - (v_j(q_j) - p_j) = v_j(\tilde{q}_j) - v_j(q_j) + v_i(1) \geq v_j(1) - v_j(2) + v_i(1) > 0$. We have therefore showed a Pareto improvement, contradicting PO. Hence, $q_1 = q_2 = 1$.

Payments. We first show that $q_i = 2$ implies $p_i = v_j(2)$. By the allocation part of the proof $q_i = 2$ for every $\tilde{v}_i(\cdot)$ such that $\min(b_1, b_2) > \tilde{v}_i(2) > v_j(2)$ and $\tilde{v}_i(1) = 0$. Thus, SP and IR imply $p_i \leq v_j(2)$, otherwise player i can declare $p_i > \tilde{v}_i(2) > v_j(2)$ and $\tilde{v}_i(1) = 0$, still win two items, and pay at most $\tilde{v}_i(2)$ which is strictly less than p_i. Similarly, again by the allocation part of the proof, $q_i = 0$ for every $\tilde{v}_i(\cdot)$ such that $v_j(2) > \tilde{v}_i(2)$ and $\tilde{v}_i(1) = 0$. By Claim 2, if player i has true value $v_j(2) > \tilde{v}_i(2)$ and $\tilde{v}_i(1) = 0$ her utility is exactly zero. Thus, SP implies that $p_i \geq v_j(2)$, otherwise player i with true value $v_j(2) > \tilde{v}_i(2) > p_i$ and $\tilde{v}_i(1) = 0$ can falsely declare v_i and obtain strictly positive utility. As a conclusion, $p_i = v_j(2)$.

Second, we argue that $q_i = 1$ implies $p_i = v_j(2) - v_j(1)$. Assume $v_j(1) > 0$, otherwise $v_i(2) \geq v_i(1) + v_j(1)$ and we can assume $q_i = 2$. By the allocation part of the proof $q_i = 1$ for every $\tilde{v}_i(\cdot)$ such that $\min(b_1, b_2) > \tilde{v}_i(2) = \tilde{v}_i(1) > v_j(2) - v_j(1)$, and $q_i = 0$ for every $\tilde{v}_i(\cdot)$ such that $\tilde{v}_i(2) = \tilde{v}_i(1) < v_j(2) - v_j(1)$.

Similarly to the previous paragraph, by SP this implies $p_i = v_j(2) - v_j(1)$ (if $p_i > v_j(2) - v_j(1)$, i can increase utility by declaring $p_i > \tilde{v}_i(2) = \tilde{v}_i(1) > v_j(2) - v_j(1)$, and if $p_i < v_j(2) - v_j(1)$, when i's true utility is $p_i < \tilde{v}_i(2) = \tilde{v}_i(1) < v_j(2) - v_j(1)$ she can increase utility by declaring v_i). This concludes the proof of Claim 8.

□

Concluding the proof of Theorem 1. We show how all the above implies that, in our setting, there does not exist a mechanism that satisfies IR, SP, NPT, and PO. Assume by contradiction the existence of such a mechanism. Fix valuations $v_1(\cdot)$ and $v_2(\cdot)$ such that $v_1(2) > b_1 + v_1(1)$ and $v_1(1) > b_1 > v_2(2) > v_2(1) > 0$. For these valuations, it cannot be that $q_1 = 0$, otherwise a Pareto improvement is $\tilde{q}_1 = 2, \tilde{q}_2 = 0, \tilde{p}_1 = p_1 + v_2(2), \tilde{p}_2 = p_2 - v_2(2)$ (in fact $p_1 = 0$ by Claim 2). We next show $q_1 \neq 1$ and $q_1 \neq 2$, achieving a contradiction.

Suppose $q_1 = 1$. If $b_1 - p_1 \geq v_2(1)$, a Pareto improvement is $\tilde{q}_1 = 2, \tilde{q}_2 = 0, \tilde{p}_1 = p_1 + v_2(1), \tilde{p}_2 = p_2 - v_2(1)$, which is a contradiction. If $b_1 - p_1 < v_2(1)$, player 1 can increase her utility by declaring a valuation $\tilde{v}_1(\cdot)$ such that $v_2(2) - v_2(1) < \tilde{v}_1(2) = \tilde{v}_1(1) < b_1$. By Claim 8, in this case player 1 will still receive one item, and will pay $v_2(2) - v_2(1) < b_1 - v_2(1) < p_1$. Thus, player 1 is able to strictly increase her utility, contradicting SP. We conclude that $q_1 \neq 1$.

Finally, suppose $q_1 = 2$. By Claim 2 the utility of player 2 in this case is exactly zero. However, by Claim 6, if player 2 declares a valuation $\tilde{v}_2(\cdot)$ such that $\tilde{v}_2(2) > \tilde{v}_2(1) > b_1$ and $\tilde{v}_2(2) - \tilde{v}_2(1) \leq b_1$, she will receive one item and will pay a non-positive price, resulting in a positive utility. That is, player 2 can increase her utility to be strictly positive instead of zero by declaring $\tilde{v}_2(\cdot)$, a contradiction to SP. We conclude that $q_1 \neq 2$ as well, and therefore we have contradicted the existence of a mechanism that satisfies IR, SP, NPT, and PO. This concludes the proof of Theorem 1.

Chapter 5

Additional Proof for the General Setting

In this chapter, we give the proof of theorem 2 for the case when $b_1 > 2b_2$. We first solve the problem for our general setting and then show that the solution contradicts the unique Clinching Auction of Dobzinski et al. (2008).

Let us remember that in our general setting the input of the mechanism is as follows:

player	a	b	c	b_i
1	$v_1(a)$	$v_1(b)$	$v_1(c)$	b_1
2	$v_2(a)$	$v_2(b)$	$v_1(c)$	b_2

Table 5.1: values and budgets declared by the players

If it is possible to design the desired mechanism for general valuations then it is also possible to design such mechanism for any restricted domain. Thus, we consider the restricted domain in which $v_1(a) > v_1(c) > v_1(b)$ and $v_2(a) < v_2(c) < v_2(b)$ and show that for this domain there does not exist a mechanism that satisfies our four properties. This impossibility naturaly extends for general valuations.

Definition 5.1: Define $difmax_i$ as the difference between the highest and the lowest value for alternatives of player i, and $difmin_i$ as the difference between the highest and the second highest value of player i.

Claim 10. *If $b_1 > difmax_1 > difmin_1 > difmax_2 > b_2 > difmin_2$ (assume also $difmax_2 - difmin_2 > b_2$), then the alternative chosen is player's 1 best alternative, $p_2 = 0$, $p_1 \leq difmax_2$.*

Proof : By PO, the best alternative of player 1 should be chosen. Suppose by contradiction that some alternative x (that is not player's 1 best alternative) is chosen. $U_1(x) = v_1(x) - p_1$, $U_2(x) = v_2(x) - p_2$. The following outcome is a Pareto improvement: $\tilde{x}$ =player's 1 best alternative, $\tilde{p_1} = p_1 + difmax_2$, $\tilde{p_2} = p_2 - difmax_2$. $U_1(\tilde{x}) = v_1(\tilde{x}) - p_1 - difmax_2$, and since $v_1(\tilde{x}) - difmax_2 > v_1(x)$, $U_1(\tilde{x}) > U_1(x)$. $U_2(\tilde{x}) = v_2(\tilde{x}) - p_2 + difmax_2$, and since $v_2(\tilde{x}) + difmax_2$ gives player 2 at least the highest value he can get, $U_2(\tilde{x}) > U_2(x)$. We improved for the players not damaging the auctioneer.

Now, we want to show that player 1 is able to pay $\tilde{p_1} = p_1 + difmax_2$. If x is "player's 1 worst alternative" then $p_1 \leq 0$, else player 1 can tell $v_1(x) = 0$ and so $b_1 > 0 + difmax_2$. If x is "player's 1 middle alternative" then $p_1 \leq v_1(\text{middle}) - v_1(\text{worst})$, else player 1 prefers his worst alternative and so $b_1 > v_1(\text{middle}) - v_1(\text{worst}) + difmax_2 = difmax_1 - difmin_1 + difmax_2$.

IR and SP imply $p_2 = 0$, $p_1 \leq difmax_2$ (else, player 1 could lower $difmax_1$ and $difmin_1$ to $difmax_2$ and tell his lowest and second highest values are equal to 0 - in that case, by IR, he wouldn't pay more than $difmax_2$) . □

Claim 11. *If $b_1 > \hat{difmax_2} > \hat{difmin_2} > \hat{difmax_1} > \hat{difmin_1} > b_2$, then the alternative chosen is player's 1 best alternative, $p_2 = 0$, $p_1 \leq b_2$.*

Proof : SP implies that the best alternative of player 1 should be chosen.

Else, if the true values of player 2 are according to the case of claim 10, then he can misreport to the case of claim 11. The increase in price is at most b_2 and the increase in value is at least $difmax_2 - difmin_2 > b_2$ (by definition of claim 10). Note that $difmax_2 - difmin_2$ is exactly the difference between the middle and the lowest value for alternatives of player 2, so by moving to claim 11 from claim 10 player 2 would obtain strictly positive utility.

By SP, $p_2 = 0$. IR and SP imply $p_1 \leq b_2$ (else, player 1 could lower $difmax_1$ and $difmin_1$ to b_2 and tell his lowest value and second highest values are equal to 0). □

Now, let us look at the Adaptive Clinching Auction of Dobzinski et al. (2008).

The general version of the auction is designed for m identical indivisible units for sale and n bidders that each one of them has a value v_i for each unit as well as a budget upper bound b_i on the total amount of money he may pay. The Adaptive Clinching Auction satisfies Individual Rationality (every truthful player obtains a non-negative utility), Incentive Compatibility (every truthful player cannot increase his utility by declaring anything but truth),optimality and always allocates all items.

The theorem in Dobzinski et al. imply:

Claim 12: For the case of two players, two identical items, and additive values, there is a unique mechanism that satisfies IR, SP, NPT and PO. In this mechanism, if $b_1 > 2v_2 > v_2 > 2v_1 > v_1 > b_2$, player 1 wins both items and pays $1.5b_2$.

A detailed explanation to Claim 12 is given in chapter 6.

Proof of theorem 2 (for $b_1 > 2b_2$) : Suppose by contradiction that when $b_1 > 2b_2$ there exists a general mechanism that satisfies IR, SP, PO and NPT. In particular, that general mechanism can be used to construct a mechanism with the same properties for the setting of 2 players and 2 units of an indivisible good. The input of such mechanism will be then as follows (when alternative a is "player 1 gets both units", b - "player 2 gets both units", c - "each player gets 1 unit"):

player	a	b	c	b_i
1	$2v_1$	0	v_1	b_1
2	0	$2v_2$	v_2	b_2

Table 5.2: values and budgets declared by the players

By Claim 11, if $b_1 > 2\hat{v}_2 > \hat{v}_2 > 2\hat{v}_1 > \hat{v}_1 > b_2$, then the constructed mechanism that satisfies all the desired properties would give the following outcome: player's 1 best alternative is chosen (player 1 gets both units), $p_2 = 0$, $p_1 \leq b_2$. But that outcome contradicts the outcome of the Adaptive Clinching Auction (player 1 gets both units, $p_2 = 0$, $p_1 = 1.5b_2$).

□

Chapter 6

The Adaptive Clinching Auction - The Mechanism

The mechanism keeps for every player i the current number of items q_i already allocated to him, his current total price for these items p_i and his remaining budget $B_i = b_i - p_i$; it also keeps the global unit price p and the global remaining number of items q. The price p gradually ascends as long as the total demand is strictly larger than the total supply, where the demand of the player is defined by:

$$D_i(p) = \begin{cases} \left\lfloor \frac{B_i}{p} \right\rfloor & , v_i > p \\ 0 & ,\text{otherwise} \end{cases}$$

The items are allocated to the player as soon as the total demand of the other players decreases strictly below the current total supply. The point when the price reaches the remaining budget of the player B_i is identified by using $D_i^+(p) = lim_{x \to p^+} D_i(x)$, as, for $p = B_i \prec v_i$, we have $D_i(p) \succ 0$ and $D_i^+(p) = 0$. The point when the price reaches the value v_i of the player is identified by using $D_i^-(p) = lim_{x \to p^-} D_i(x)$, as, for $p = v_i \leq B_i$, we have $D_i^-(p) \succ 0$ and $D_i(p) = 0$.

The Adaptive Clinching Auction (the mechanism):

1. While $\sum_i D_i^+(p) \succ q$,

   (a) If there exists a player i such that $D_{-i}^+(p) = \sum_{j \neq i} D_j^+(p) \prec q$ then allocate $q - D_{-i}^+(p)$ items to player i for a unit price p. Update all running variables (including the allocated and available quantities, the remaining budgets and the current demands) and repeat.

   (b) Otherwise increase the price p, recompute the demands and repeat.

2. Otherwise ($\sum_i D_i^-(p) \geq q \geq \sum_i D_i^+(p)$):

   (a) For every player i with $D_i^+(p) \succ 0$ allocate $D_i^+(p)$ units to player i for a unit-price p and update all running variables.

   (b) While $q \succ 0$ and there exists a player i with $D_i(p) \succ 0$, allocate $D_i(p)$ units to player i, for a unit-price p and update all running variables.

   (c) While $q \succ 0$ and there exists a player i with $D_i^-(p) \succ 0$, allocate $D_i^-(p)$ units to player i, for a unit-price p.

   (d) Terminate.

The case $b_1 > 2v_2 > v_2 > 2v_1 > v_1 > b_2$ (claim 12 explanation):

When the price is below $b_2/2$, each player demands 2 items, and so, for both players, the other player demands 2 items. Therefore no allocations will take place and the price will keep ascending. At $p = b_2/2$, $D_2^+(b_2/2) = 1$. Thus, player 1 "clinches" one item for a price $b_2/2$ (so the remaining budget of player 1 is updated to $b_1 - b_2/2 > b_2$, since by the definition of the case $b_1 > 2b_2$). Immediately after that, the demand of player 1 is updated to be 1. The available number of items is 1, and so no player can get any item. At a price b_2 the demand of player 2 reduces to 0, and player 1 "clinches" the second item for price b_2. To summarize, player 1 gets both items and pays $1.5b_2$, $p_2 = 0$.

Chapter 7

Conclusions

This research studies the problem of allocating multiple identical items that may be complements to budget-constrained bidders with private values. We show that there does not exist a deterministic mechanism that is individually rational, strategy-proof, Pareto-efficient, and that does not make positive transfers.

Our result implies that there really is *no* mechanism that satisfies the above desired properties, for almost all structures of valuations over some abstract set of outcomes. It gives an impossibility for any number of identical or non-identical items and any number of players with general valuations over the set of items. Similarly, it implies the same impossibility for the abstract social choice setting with at least three alternatives and an unrestricted domain of players' valuations.

Our result complements the previous studies. It actually extends the previously proved impossibilities to a more general case.

Future research should check more specific structures of valuations to understand if there are some more cases when building a mechanism with the desired properties is possible.

Another possible way to advance, given the above impossibilities, is to study the "constrained efficiency" problem of maximizing efficiency subject to Bayesian incentive-compatibility constraints. It is also important to understand the powers and limitations of randomized mechanisms for the case of a divisible good.

Chapter 8

Bibliography

1. Aggarwal G., Muthukrishman S., D.Pal and M.Pal (2009), *General auction mechanism for search advertising*, In Proc. of the 18th International Conference on World Wide Web (WWW'09).

2. Ashlagi I., Braverman M., Hassidim A., Lavi R. and Tennenholtz M. (2009), *Position Auctions with Budgets: Existence and Uniqueness*, Working paper.

3. Bhattacharya S., Conitzer V., Munagala K. and Xia L. (2010), *Incentive compatible budget elicitation in multi-unit auctions*, SODA '10 Proceedings of the 21st Annual ACM-SIAM Symposium on Discreet Algorithms.

4. Budish E., Cantillon E. (2010), *The multi-unit assignment problem: theory and evidence from course allocation at Harvard*, ECARES working paper 2010-002.

5. Dobzinski S., Lavi R. and Nisan N. (2008), *Multi-unit Auctions with Budget Limits*, In Proc. of the 49th Annual IEEE Symposium on Foundations of Computer Science (FOCS'08).

6. Fiat A., Leonardi S., Saia J. and Sankowski P. (2011), *Single valued combinatorial auctions with budgets*, In Proceedings of the 12th ACM Conference on Electronic Commerce (ACM-EC), pp. 223-232.

7. Gibbard A. (1973), *Manipulation of voting schemes: a general result*, Econometrica, 41, 587-601.

8. Groves T. (1973), *Incentives in teams*, Econometrica, pp. 617-631.

9. Hatfield J.W. (2009), *Strategy-proof, efficient, and nonbossy quota allocations*, Soc Choice Welf (2009) 33:505-515.

10. Maskin E.S. (2000), *Auctions, development and privatization: Efficient auctions with liquidity-constrained buyers*, European Economic Review, 44(4-6), 667-681.

11. Nisan N. (1994), *Introduction to Mechanism Desigh (for Computer Scientists)*, Algorithmic Game Theory, Chapter 9 (2007):209-241.

12. Satterthwaite M. (1975), *Strategy-proofness and arrow's condition: Existence and correspondence theorems for voting procedures and social welfare functions*, Journal of Economic Theory, pp.187-217.