Job Security, Stability and Production Efficiency

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Motivation

- Kelso and Crawford (1982) introduce the study of labor markets in a many-to-one matching model.
- Their solution concept: classic Gale-Shapley stability
  - Employees can choose to change jobs but will not.
  - Firms can choose to hire or fire employee but will not.
- Two problems:
  - Theoretical - existence of stable outcomes is limited
  - Realistic - firing employees is difficult in many job markets (e.g., job markets in Europe).
- IDEA - relax the notion of stability while capturing job security.

The Kelso and Crawford model (ECMA, 1982)

A job market is a tuple (N, M, v, b) (abbreviated (v, b)):

- N - finite set of firms, M - finite set of workers.
- $v = \{v^n\}_{n=1}^N$, $v^n : 2^M \rightarrow \Re_+$ firm n's production function, $v^n(\emptyset) = 0$ and $v(X) \leq v(Y)$ $\forall n$ and $\forall X \subset Y \subseteq M$.
- $b = \{b_m^n\}_{m \in M, n \in N}$ represent exogeneous a-priori workers' preferences over firms (in monetary values)

The Kelso and Crawford model - cont.

- An **assignment** $A = \{A^1, \dots, A^N\}$: firm n receives the set of workers A^n. Some workers may be left unassigned.
- An **allocation** is a pair (A, s), A is an assignment and $s \in \Re_+^M$.
- $u_m(A, s) = s_m - b_m^n$ for $m \in A^n$ is m's quasi-linear utility.
- $\Pi^n(A, s) = v^n(A^n) - \sum_{m \in A^n} s_m$ is the profit of firm n.

Assumption (adopted from Kelso an Crawford):
$\forall n,\ C \subset M,\ m \in M \setminus C,\quad v^n(m|C) \geq b_m^n$

> *"This is a natural restriction, since if a worker's marginal product, net of the salary required to compensate him or her for the disutility of work at a given firm, were negative, the firm could agree to let the worker do nothing for a salary of zero." (K &C)*

Definition

An allocation (A, s) is **individually rational (IR)** if
(1) $\Pi^n(A, s) = v^n(A^n) - \sum_{m \in A^n} s_m \geq 0 \ \forall n \in N$; and
(2) $u_m(A, s) = s_m - b^n_m \geq 0$ for all $n \in N$ and $m \in A^n$.

Definition

A coalition $\{n, C\}$ is **blocking** for an allocation (A, s) if exists $\hat{s} \in \Re^C_+$:

- $u_m(n, \hat{s}_m) \geq u_m(k, s_m) \ \ \forall k \in N, m \in A^k \cap C$
- $v^n(C) - \sum_{m \in C} \hat{s}_m \geq v^n(A^n) - \sum_{m \in A^n} s_m$

with at least one of the inequalities being strict.

Definition

An allocation (A, s) is **stable** if it is IR and there exist no blocking coalitions.

A new notion of stability

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A coalition $\{n, C\}$ is **JS-blocking** for an allocation (A, s) if exists $\hat{s} \in \Re^C_+$:

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- $v^n(C) - \sum_{m \in C} \hat{s}_m \geq v^n(A^n) - \sum_{m \in A^n} s_m$
- $A^n \subset C$

with at least one of the inequalities being strict.

Definition

An allocation (A, s) is **JS- stable** if it is IR and there exist no JS-blocking coalitions.

Results (1) - Efficiency

$P(A) = \sum_n (v^n(A^n) - \sum_{m \in A^n} b^n_m)$ $=$ *efficiency/welfare level* of A.
$\bar{A}$ is *efficient* if $P(\bar{A}) = \max_A P(A)$.

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Example

Two firms $\{1, 2\}$ and two workers $\{a, b\}$, $v_1(a) = 2, v_1(b) = 1$, $v_2(a) = 1, v_2(b) = 2, v_i(ab) = max(v_i(a), v_i(b))$. Workers are indifferent between the two firms.

Maximal welfare is 4 (by assigning a to 1 and b to 2).
(A, s) stable $\implies$ A is efficient (Kelso and Crawford, 1982).

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Theorem (A $\frac{1}{2}$-First Welfare Theorem)

(A, s) *JS-stable allocation* $\implies$ $P(A) \geq \frac{1}{2} \max_{\bar{A}} P(\bar{A})$.

Proof of first result

(for the case $b_m^n = 0$).

Let $\bar{A}$ be the optimal assignment and A a JS-stable assignment

Since firm n does not want to expand from A^n to $\bar{A}^n \cup A^n$:
$v^n(\bar{A}^n \cup A^n) - v^n(A^n) \leq \sum_{m \in \bar{A}^n \setminus A^n} s_m$.
Rerranging: $v^n(\bar{A}^n) \leq v^n(\bar{A}^n \cup A^n) \leq \sum_{m \in \bar{A}^n \setminus A^n} s_m + v^n(A^n)$

Summing over all firms:
$\sum_{i=1}^n v^n(\bar{A}^n) \leq \sum_{i=1}^n \sum_{m \in A^n} s_m + \sum_{i=1}^n v^n(A^n) \leq 2 \sum_{i=1}^n v^n(A^n)$
(last inequality follows from (IR) of A).

A class of production functions

Most of the current literature focuses on the set of production functions called **Gross Substitues (GS)**.

We consider the set of production functions called **Almost Fractionally Sub-additive (AFS)** - definition to follow.

- *GS* is a strict (tiny) subset of *AFS*.
- *AFS* allows for a significantly richer structure of substitutabilities, and even some complementarities.

Fractional Subadditivity (Feige 2009)

Definition

For any $C \subseteq M$, a vector of non-negative weights $\{\lambda_D\}_{D \subseteq C, D \neq \emptyset}$ is a **fractional cover** of C if for any $m \in C$, $\sum_{\{D \subseteq C : m \in D\}} \lambda_D = 1$.

Example

$C = \{a, b, c\}$. $\lambda_a = 1, \ \lambda_{bc} = 1$ is a fractional cover of C.
$\lambda_{ab} = \lambda_{ac} = \lambda_{bc} = \frac{1}{2}$ is also a fractional cover of C.

Fractional Subadditivity

Definition (Bondareva-Shapley, also in Feige 2009)

$v^n : 2^M \rightarrow \Re_+$ is **Fractionally Sub-additive on** $C \subseteq M$ if for any fractional cover $\{\lambda_D\}_{D \subseteq C, D \neq \emptyset}$ of C, $v(C) \leq \sum_{D \subseteq C, D \neq \emptyset} \lambda_D v(D)$.

$v^n : 2^M \rightarrow \Re_+$ is **Fractionally Sub-additive**, denoted $v \in FS$, if for any $C \subseteq M$, v is Fractionally Sub-additive on C.

A detour to cooperative GT

In cooperative game theory a Fractionally Sub-additive function is called a **(anti-) balanced cooperative game** (a-la Bondareva and Shapley).

Definition (Bondareva-Shapley, also in Dobzinski et al. 2010)

A non-negative vector of salaries s is called a **supporting salary vector** for a production function v and a set of workers $C \subseteq M$ if
(1) $\sum_{m \in C} s_m = v(C)$; and
(2) For any $T \subset C$, $\sum_{m \in T} s_m \leq v(T)$ (reversed CORE)

Theorem (**Bondareva-Shapley**)

A production function v is F.S. on $C \subseteq M$ if and only if there exists a supporting salary vector for (C, v).

Corollary: $v \in FS$ if and only if there exists a supporting salary vector for (C, v) for all $C \subseteq M$.

The connection of FS to JS-stability

Lemma

If $v^1, ..., v^N \in FS$ and then every efficient assignment $A = \{A^1, ..., A^N\}$ is JS-stable.

For example, (A, s) is a JS-stable allocation if for every firm n we set $\{s_m\}_{m \in A^n}$ to be a supporting salary vector for (v^n, A^n).

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Proof: IR is immediate. No firm k wants to add a single employee $m \in A^n$ since:

- $v^n(A^n) = \sum_{i \in A^n} s_i$
- $v^n(A^n \setminus \{m\}) \geq \sum_{i \in A^n \setminus \{m\}} s_i$
- Therefore $v^k(A^k \cup \{m\}) - v^k(A^k) \leq v^n(A^n) - v^n(A^n \setminus \{m\}) \leq s_m$

No firm wants to hire several employees using similar manipulations.

Defining AFS

Definition

$v^n : 2^M \to \Re_+$ is **Almost Fractionally Sub-additive**, denoted $v \in AFS$, if:

1. For any $C \subset M$ (excluding $C = M$) v is Fractionally Sub-additive on C; and
2. $v(M) \leq \frac{\sum_{m \in M} v(M \backslash m)}{|M|-1}$.

A production function is a vector of $2^M - 1$ real numbers and so can be viewed as an element of the $2^M - 1$-dimensional Euclidean space which induces a natural measure.

Theorem (Lehmann, Lehmann and Nisan, GEB 2006)

$GS \subset SUBMODULAR \subset FS$ ($\subset AFS$) and for some natural measure over the set of production functions GS has measure zero while $SUBMODULAR$ (and hence AFS) has positive measure

An example of a production function in AFS

Example

There are 3 workers, denoted a, b, c and let the production function v be defined by: $v(a) = v(b) = v(c) = 3$, $v(\{a, b\}) = v(\{a, c\}) = 6$, $v(\{b, c\}) = 4$, $v(\{a, b, c\}) = 8$.

Claim: $v \in AFS$ (but not in FS).

Observation: Worker a and the pair $\{b, c\}$ are complementarities.

Results (2)

Theorem (A Second Welfare Theorem)

If $v^n \in GS \quad \forall n \in N$ *and* A *is Pareto efficient*
then (A, s) *is stable for some* s *(Kelso and Crawford, 1982).*

Results (2)

Theorem (A Second Welfare Theorem)

If $v^n \in GS \;\; \forall n \in N$ and A is Pareto efficient then (A, s) is stable for some s (Kelso and Crawford, 1982).

If $v^n \in AFS \;\; \forall n \in N$ and A is efficient then (A, s) is JS-stable for some s.

Sketch of proof of second result

Case 1: In A, several firms receive a non-empty set of employees

Since $A^n \subset M$ (in particular, $A^n \neq M$) for all n, it follows that v^n is F.S. on A^n for all n.

Thus, there exists a supporting salary vector for (A^n, v^n) for all n and we already saw that setting salaries $\{s_m\}_{m \in A^n}$ to be supporting salaries for $v^n(A^n)$ yields a JS-stable outcome.

Sketch of proof of second result

Case 2: A assigns all workers to a single firm, firm n

Set $\forall m,\ s_m = v^n(M) - v^n(M \setminus \{m\})$.

- No firm k wants to hire a single employee m since:

  $$v^k(\{m\}) \le v^n(M) - v^n(M \setminus \{m\}) = s_m$$

  (inequality follows from efficient assignment).

  No firm wants to hire several employees using similar manipulations.

- IR holds since for $v \in AFS$,

  $$v(M) \ge \sum_{m \in M} (v(M) - v(M \setminus \{m\}))$$

  (rearranging the second condition of AFS).

Results (3) - Maximality of AFS

Theorem

If $\bar{v} \notin GS$ then there exists a job market $(v, 0)$, where $v^1 = \bar{v}$ and for all $n > 1$ $v^n \in GS$ such that no stable allocation exists (Gul & Stacchetti, JET 1999).

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If $\bar{v} \notin AFS$ then there exists a job market $(v, 0)$, where $v^1 = \bar{v}$ and for all $n > 1$ $v^n \in AFS$ such that no efficient JS-stable allocation of the market exists.

Structure of proof of third result

(proof somewhat more involved than previous proofs)

Definition

$v^n : 2^M \rightarrow \Re_+$ is **Symmetrically Fractionally Sub-additive (SFS)**, if $v(C) \leq \frac{\sum_{m \in C} v(C \setminus m)}{|C|-1}$ for all $C \subset M$.

Note that $AFS \subset SFS$.

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- If $\bar{v} \notin SFS$ then there exists a job market $(v, 0)$, where $v^1 = \bar{v}$ and for all $n > 1$ $v^n \in AFS$ such that no JS-stable allocation exists (in particular, any efficient allocation is not JS-stable).

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- If $\bar{v} \in SFS \setminus AFS$, no efficient JS-stable allocation exists (but there may exist inefficient JS-stable allocations).

Remark: We do not know if any tuple of production functions in SFS admit a JS-stable allocation; this is an (interesting?) open question.

Results (4) - JS-Stability and a natural auction game

The *SPIB (2nd-Price Item Bidding)* complete information game: Firms bid simultaneously for workers, each in a 2nd price auction.

Nash equilibrium with **weak no overbidding** satisfies: $v^n(A^n) \geq \sum_{m \in A^n} p^n_m$, where p^n_m is firm n's bid for worker m.

An assignment is called an **SPIB-assignment** if it is the outcome of a pure Nash equilibrium with no over-bidding of the SBIP-game.

Theorem

Any assignment is an SPIB-assignment if and only if it is a JS-stable assignment.

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- Ties existence of pure NE with weak no overbidding to existence of JS-stability
- Ties efficiency of JS-stability to efficiency of NE outcomes.
- Salaries in the two settings are not identical (typically, NE salaries are lower than JS-stable salaries).

Summary

- Study JS-stable allocations, a relaxation of the classic stable allocation concept.
- Show an approximate first welfare theorem.
- Show a second welfare theorem that holds for the class of AFS valuations. This class is significantly larger than GS.
- Show maximality of AFS: it is the largest class for which the second welfare theorem is guaranteed to hold.
- Show a connection to a second price auction game.
- Many extensions and open problems present themselves...