A Note on Lavi, Mu'alem, and Nisan (2009, SCF)

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In this short note we correct a minor mistake in the first proof of Lavi, Mu'alem and Nisan (2009, SCF). To be specific, the authors mistakenly take $C = C(x, y)$ instead of $C = \mathring{C}(x, y)$ in the proof of Claim 6.[1] In particular, they reason that $\alpha/2 \notin C$ implies $\alpha/2 + \gamma(x, y) \cdot \vec{1} \notin P(x, y)$ in the second line on p. 418. However, this is incorrect because one can only conclude that $\alpha/2 + \gamma(x, y) \cdot \vec{1} \notin \mathring{P}(x, y)$; it is still possible that $\alpha/2 + \gamma(x, y) \cdot \vec{1}$ belongs to the boundary of $P(x, y)$, and hence also belongs to $P(x, y)$. In the following, we will stick to the notations of the paper and provide a correct proof of this claim.

Proof (of Claim 6): The first part of the proof is correct; thus we only need to show that $\forall \alpha \in C$, we have $\alpha/2 \in C$ as well. Assume $\alpha \in C$, but $\alpha/2 \notin C$. Then, by the definition of interior[2] and Claim 5, we have

$$\begin{aligned} &\exists\, \varepsilon > 0, \text{ s.t. } \alpha - \varepsilon \in C(x, y), \\ \iff &\exists\, \varepsilon > 0, \text{ s.t. } \alpha - \varepsilon + \gamma(x, y) \cdot \vec{1} \in P(x, y) \end{aligned} \tag{1}$$

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[1] Claim 6 establishes that set C is convex, see p.417-18.

[2] In the paper, the interior of set C is defined as $\mathring{C} = \{\alpha \in C \,|\, \alpha - \varepsilon \in C \text{ for some } \varepsilon > \vec{0}\}$, see line 12 on p. 417. This definition is not standard, but, with the other properties of set C, is equivalent to the standard definition.

Moreover,

$$\begin{aligned}
&\forall \tilde{\varepsilon} > 0,\ \frac{\alpha}{2} - \tilde{\varepsilon} \notin C(z, y), \\
\iff &\forall \tilde{\varepsilon} > 0,\ \frac{\alpha}{2} - \tilde{\varepsilon} + \gamma(z, y) \cdot \vec{1} \notin P(z, y), \\
\implies &\forall \tilde{\varepsilon} > 0,\ -\frac{\alpha}{2} + \tilde{\varepsilon} + \gamma(y, z) \cdot \vec{1} \in P(y, z),
\end{aligned} \tag{2}$$

where the second implication follows Claim 2 and 4. In particular, we can pick $\tilde{\varepsilon} = \varepsilon/4$. Now, (2) becomes

$$\left(-\frac{\alpha}{2} + \frac{\varepsilon}{2}\right) - \frac{\varepsilon}{4} + \gamma(y, z) \cdot \vec{1} \in P(y, z). \tag{3}$$

Summing up (1) and (3) and applying Claim 3 and Claim 4, one obtains

$$\begin{aligned}
&(\alpha - \frac{\alpha}{2} + \frac{\varepsilon}{2}) - (\varepsilon + \frac{\varepsilon}{4})/2 + \gamma(x, z) \cdot \vec{1} \in P(x, z) \\
\iff &\frac{\alpha}{2} - \frac{\varepsilon}{8} + \gamma(x, z) \cdot \vec{1} \in P(x, z) \\
\iff &\frac{\alpha}{2} - \frac{\varepsilon}{8} \in C(x, z)
\end{aligned}$$

In other words, there exists $\varepsilon' = \varepsilon/8 > 0$, s.t. $\alpha/2 - \varepsilon' \in C(x, z)$. Hence, by the definition of interior and Claim 5, we have

$$\frac{\alpha}{2} \in \mathring{C}(x, z) = C.$$

We reach a contradiction. Hence, set C is indeed convex. □

References

[1] Lavi, Ron, Ahuva Mu'alem and Noam Nisan "Two Simplified Proofs for Roberts' Theorem," *Social Choice Welfare* 32 (2009), 407-423.