Iterated Deletion of Dominated Strategies in Discrete Position Auctions

Research Thesis

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Abstract

In Game Theory, Iterated Deletion of Dominated Strategies is often used to converge a complex game. In some games, it provides a unique outcome for each of the participating players. In this study, we show its implementation in discrete position and VCG auctions, focusing on the linkage between Nash equilibrium strategies in position auction and the outcome of the deletion process. We setup an adjusted discrete position auction with n players and 2 slots and delete dominated strategies under players' beliefs. We then try to similarly apply it to genearl cases. We conclude that the deletion process leads to a unique outcome in the special case equal to the equilibrium strategies whereas in general cases there is no unique outcome.

Chapter 1

Introduction

Selling Internet ads positions via position auctions is widely used by search engines. Every year Google and Yahoo! generate a revenue of billions of dollars from conducting such auctions. These ads are based on keywords and are shown to Internet users whenever a specific keyword is typed. If a user then clicks on one of the ads, its advertiser then pays the search engine via a 'pay-per-click' model. Since the number of positions on the screen is limited and the allocation of the ads has an economic value to the advertisers, selling these positions through auctions is a natural choice.

Search engines use the Generalized Second Price auction (GSPa) in which each advertiser submits a bid for a specific keyword which she's willing to pay the search engine in case of a click on behalf of the Internet user. The player with the highest bid then wins the "best" position for her ad (i.e. her ad will be placed in the highest spot on the screen) and pays the search engine the second highest bid per click. The m^{th} highest bidder wins position m and pays the $(m+1)^{th}$ highest bid per click and so on.

The Generalized English auction (GEa) corresponds to GSP in an incomplete information environment. The price continuously increases over time and players decide whether to drop or stay. The last remaining player wins position *1* and pays the price at which the previous player dropped; The player who dropped 2^{nd} to last wins position *2* and pays the price at which the 3^{rd} to last player dropped and so on.

Edelman, Ostrowsky and Schwarz, (2005) analyze the ex post Nash equilibrium strategies of the GEa auction which are used widely in our study. We further investigate the GEa equlibrium strategies, by using Iterated Deletion of Dominated

Strategies Process (Deletion process). Osborne and Rubinstein (1994) describe the Deletion process as a "solution concept that looks at a game from the point of view of a single player". A player can calculate how should she *not* act, i.e. what are the actions that are *never* her best response to other players' actions. By only assuming that all other participants will follow the same rational process, a player can delete from further calculations all such actions (hers and others'). Then, all players face a smaller game - with less actions to choose from. The final outcome is a converged game with only those actions (for every player) that cannot be further deleted. The authors mention that this "solution concept is weaker than Nash equilibrium, since in many games it does not exclude any action from being deleted". However, there are games in which the Deletion process leads to significantly less complex span of possible actions to choose from or even to a unique outcome for each player. In such games, the expectation is that the result is more obvious both to the players and the auctioneer. Thus due to the ability of each player to reach the rational conclusion by herself without the need to understand complex situations and study the behavior of other players as often required in order to reach Nash equilibria.

There are very few works that study outcomes of the Deletion process in incomplete information games. An important study in this field is the one of Dekel and Wolinsky, (2001). The authors investigate the outcome of the Deletion process in Discrete First Price auctions in which the values and the valid bids are on a discrete grid [0-1]. They define a "Deletion of P-dominated strategies" (P-deletion process) in which strategies are deleted under specified players' beliefs (P) over others' values. The process stops when there are no longer P-dominated strategies. They conclude that when the number of players is large enough, there is a unique outcome in which each player bids slightly below her own value. We use the Deletion Process, similarly to the methodology described by the authors.

The main setups we conduct and our main results are as follows:

1. *Discrete Generalized Second Price auction* - we rely on the basic idea of GSPa presented by Edelman, Ostrowsky and Schwarz together with discrete grids presented by Dekel and Wolinsky. We then add some important adjustments to the bids' grids, players' beliefs and the payment rule. For a special case of n players and *2* slots we show that the GEa equilibrium strategy is a dominant strategy and prove in our *main result* that under the adjustments made it is also the only strategy that survives the P-deletion process. For a generalized

case of n players and m slots we show that the GEa equilibrium strategies are not dominant strategies and only one strategy is deleted.

2. *Discrete VCG mechanism with n players and m alternatives* - we use the Deletion process with no assumptions on players' beliefs. We show that although declaring true values is a dominant strategy, no strategy for any player is deleted.

Although an existence of a dominant strategy often provides the trigger for deleting strategies, we show in our results that it is not necessarily the unique outcome of the Deletion Process.

The structure of the thesis is:

In Chapter 2 we overview the general setting of GSPa, GEa and Discrete First Price auction. The chapter concludes with the methodology of the P-deletion process and previous results.

In Chapter 3 we formally define the model: adjusted Discrete GSPa with n players and *2* slots (GSPa(n,2) - corresponding to adjusted Discrete GEa(n,2)).

In Chapter 4 we show the equilibrium strategies of GEa being the unique outcome of the P-deletion process in adjusted Discrete GSPa(n,2). We then provide an assumptions' analysis and show their essentiality to the prove.

Chapter 5 deals with the implementation of the P-deletion process in adjusted Discrete GSPa(n,m) and Discrete Generalized VCG with n players and m alternatives mechanism (VCG(n,m)).

Chapter 6 details our final conclusions.

Chapter 2

Literature Survey

In this chapter we review the general setting of two auctions that build the adjusted Discrete GSPa(n,2) and describe the methodology of the P-deletion process. For each auction review previous results - the equilibrium strategies in GEa(n,m) and the outcomes of the P-deletion process in Discrete First Price auction.

2.1 Generalized English auction

Edelman, Ostrowsky and Schwarz, (2005) describe the following setting of Generalized Second Price auction. There are n players (advertisers) and m objects (slots - positions on the screen). Each player gets at most one slot. The expected number of clicks per period ("click through rate") on ad placed in slot $l \in (1..m)$ is α_l (regardless of one ad or another placed on this position or the location of other ads). $\alpha_1 > \alpha_2 > ... > \alpha_{m-1} > \alpha_m$. The private value per click to player i is v_i (regardless of the position in which her ad is displayed). Generally assume that $n \geq m$ (otherwise $m - n$ lowest slots can be ignored). Each player privately submits a bid for a slot. When the relevant keyword is typed in the search engine, the mechanism allocates the ads of the advertisers according to their last bids: the highest bidder wins the slot *1*, the second highest bidder wins slot *2* and so on until the m^{th} highest bidder gets the last slot. In a case of tie between bids, they are randomly ordered. Each player that won a slot, then pays the next highest bid. The utility of player i when winning slot l is $\alpha_l \cdot v_i - c^l$ where c_l, the payment, is the $(l+1)^{st}$ highest bid per click.

The correspondence of GSPa(n,m) auction to GEa(n,m) modeled similarly to the correspondence between Second Price and English auctions. This auction gradually increases a price parameter Q and players decide whether to drop or stay. Rename the players according to order at which they dropped so player *1* is the last remaining player. When player $l \leq k$ drops, she wins slot l and her payoff is the price at which player $l+1$ dropped. [1]

In chapter 3 we made some modifications in some of the original semantics for compatability in later definitions.

Result 1: *In the unique perfect Bayesian equilibrium of the Generalized English auction with strategies continuous in* v_i *a player* i *drops out at price* $c_i(l,h,v_i) = (\alpha_{l-1} - \alpha_l)v_i + b_{l+1}$

The price describes the equilibrium strategies with l players remaining. b_{l+1} is the price at which the previous player dropped and h is the history of all previous prices at which players dropped. $c_i(l,h,v_i)$ is player i's set of dropping prices as the auction proceeds and she didn't drop yet: i updates the price she will drop at each time another player drops. Important observations regarding the equilibrium:

1. Each player's resulting position and utility are the same as in the dominant-strategy equlibrium of the VCG mechanism.

2. This equilibrium is ex post: $c_i(l,h,v_i)$ is the best response to other bidders' strategies regardless of their realized values.

2.2 Discrete First Price auction and Iterated Deletion of P-dominated Strategies

Dekel and Wolinsky, (2001) define the following. Consider a private-value first price auction with n players and one item. Each player $i \in (1,2...n)$ knows her own value v_i of the item. Each player submits a bid. The highest bidder wins the item

[1] description made by Ashlagi, Braverman, Hassidim, Lavi and Tennenholtz, (2010) for analyzing Edelman Ostrowsky and Schwarz's GSPa results

and pays her own bid. In case of a tie between highest bids, the item is awarded with equal probability to one of the tied bidders. The values and bids are on a discrete grid: $V = (0, \frac{1}{m}, \frac{2}{m}, ..., 1 - \frac{1}{m}, 1)$. Let $d = \frac{1}{m}$. $s_i \in S_i$ is a strategy function from i's possible value, V, into the possible bids, V. Let b_i be player i's bid, b_{-i} be other players' bids and k be the number of bidders with the highest bid in a case of a tie. Player i's payoff with $v_i = v$ is then:

$$u_i[v, b_i, b_{-i}] = \begin{cases} v - b_i & , b_i > max(b_{-i}) \\ \frac{1}{k}(v - b_i) & , b_i = max(b_{-i}) \\ 0 & , b_i < max(b_{-i}) \end{cases}$$

Next, let $P_v \subset \Delta(V^{n-1})$ be the set of beliefs of type v over other types and P is the collection of the sets of P_v for each possible type. There are 2 restrictions on these beliefs:

1. Player i assigns a positive probability for her value to be the highest:

   $p_i(v_j < v \ \forall j \neq i | v_i = v) > 0 \ \forall v > 0$ and $p_i(v_j = 0 \ \forall j \neq i | v_i = 0) > 0$ for $v = 0$.

2. For n large enough, player i with $v_i = v$ assigns a small probability to the event that the number of other players with v is at most l given that their maximum value is at most v:

   $\forall n > N$: $p_i(\#\{j\colon\ v_j = v\} \leq l | v_j \leq v \ \forall j,\ v_i = v) < \frac{1}{n(l-1)+1}$.

Player i's expected payoff given $p_i \in P$, $v_i, b_i \in V$, $v_{-i} \in V^{n-1}$, $s_{-i} \in S_{-i}$ is: $\sum_{v_{-i} \in V^n} p_i(v_{-i} | v_i = v) u_i[v, b_i, s_{-i}(v_{-i})$

When dealing with Incomplete Information Games where players possess private values and beliefs regarding other players' values Dekel and Wolinsky (2001) define *P-Dominance* - dominance under players' beliefs.

Definition 1 (P-Dominated Strategies in an Incomplete Information Auction with Private Values): *Strategy $s'_i \in S_i$ is P-Dominated by strategy $s_i \in S_i$ for player i with $v_i = v$ and beliefs over other n-1 players' values $p_i \in P$ and $\forall s_{-i} \in S_{-i}$:*

$$\sum_{v_{-i} \in V^{n-1}} p_i(v_{-i} | v_i = v) u_i[v, s_i(v_i), s_{-i}(v_{-i})] > \sum_{v_{-i} \in V^{n-1}} p_i(v_{-i} | v_i = v) u'_i[v, s'_i(v_i), s_{-i}(v_{-i})]$$

The authors use the P-deletion process for the Discrete First Price auction. In each step, all players independtly delete P-dominated strategies for a specific value or a group of values and they are no longer included in further calculations. The game gradually converges and the set S_i gets smaller. The process stops when there are no longer P-dominated strategies for any of the values.

Although the process is described from the point of view of a single player i, the assumption is that all players go through the same rational process in which no one will bid a P-dominated strategy and so all players know the outcome for each value.

Every deletion for one value may make other strategies for this value or others become P-dominated, though they weren't P-dominated previously - that makes it different from an equilibrium in dominant strategies where the dominant strategy is the best response to others, regardless of their bids. Moreover, weak dominance demands weak inequality, whereas P-dominance demands srtict inequality under players' beliefs. Hence, the fact a player has a dominant strategy doesn't alone start the P-deletion process. That provides additional motivation for our study.

Result 2: *There exists N such that $\forall n > N$, the only strategy that survives the Iterated Deletion of P-Dominated Strategies for player i with $v_i = v > 0$, is bidding slightly below v: $s_i(v) = v - d$.*

At first, for any player i with $v_i = v$, $0 < v < 1$, bidding *1* is strictly dominated by bidding *0*. Note that in this case, the utility defined in this section is negative for all maximum bid of others when bidding *1* and non-negative when bidding *0* - regardless of players' beliefs. Thus, bidding *1* is strictly dominated rather than just P-dominated. By using the 1^{st} restriction on players' beliefs and by deleting P-Dominated strategies the authors show that any player i with $v_i = v$, $0 < v < 1$ will bid strictly below her value. $v_i = 0$ will bid *0*.

Next, authors assume to the contrary that for $v_i = 1$ the highest bid that survives the P-Deletion process is lower than $1 - d$. They then conradict the assumption by showing that for n large enough and when using the 2^{nd} restriction, the minimum payoff from bidding $1 - d$ is strictly higher than the maximum payoff from bidding less, thus proving the result for $v_i = 1$. The next step is showing that $v_i = 1 - d$ will bid $1 - 2d$ by using the converged auction and the positive probability that there are no types *1* among other players (the 1^{st} restriction). Iterating similarly leads to the desired result $\forall v > 0$, $v \in V$.

Chapter 3

The Discrete GSP(n,2)

In this chapter we study the Discrete Generalized Second Price auction with n players and *2* slots. It is based on the general setting of both continuous GSPa and the Discrete First Price auctions described in previous chapter with several adjustments. We show that the auction has a dominant strategy which is the equilibrium strategy of the GEa auction. As required for the analysis of the P-deletion process, the following is written from a perspective of a single player i.

3.1 Setting

3.1.1 General Description

Consider Discrete Generalized Second Price auction with $n>2$ players and *2* items (slots) whis is **equivalent to Generalized English auction** described in Chapter 2 with $m = 2$ slots. The expected number of clicks per period on slot *1* or *2* is α_1 or α_2, respectively. $\alpha_2 > \alpha_1$ (slot *2* placed 'better' than slot *1*). The private value per click of player i is v_i, $i \in (1, 2..n)$. The values are on a discrete grid: $V = (0, d, 2d, ..., 1 - d, 1)$, where d is some fracture unit $\frac{1}{q}$. Each player has a set of beliefs $p_i(\cdot|v_i = v) \in P$ over other players' values. The auction is built from *2* phases - one phase for each slot.

3.1.2 Phase 1

In phase *1*, each player privately submits a bid for slot *1*, $b_{i,1}$, $i \in (1, 2..n)$. The bids for slot *1* are on a discrete grid: $B_1 = (0, \alpha_1 d, \alpha_1(2d), ..., \alpha_1 - \alpha_1 d, \alpha_1)$. B_1 reflects players' possible utility from winning slot *1*: $\alpha_1 v$, $v \in V$. Only the 3^{rd} highest bid is revealed and determined as the price of slot *1* - $c^1 \in B_1$. Two highest bidders proceed to phase *2*, while the others remain with no payoff. In a case of tie between bids, each of the tied bidders wins promotion to phase *2* with equal probability *s*. For every player *i*, let b^1_{-i} be the 2^{nd} highest bid for slot *1* when excluding *i*'s bid. We use b^1_{-i} later.

3.1.3 Phase 2

In phase *2*, two remaining players privately submit their bids for slot *2* - $b_{i,2}$. The bids for slot *2* are on a discrete grid, depended on c^1: $B_2(c^1) \subseteq (\frac{\alpha_2}{\alpha_1} \cdot c^1, (\alpha_2 - \alpha_1) \cdot 0 + c^1, (\alpha_2 - \alpha_1)d + c^1, ..., (\alpha_2 - \alpha_1) \cdot 1 + c^1)$. $\frac{\alpha_2}{\alpha_1} \cdot c^1$ is a leap from B_1 and it is the **minimum valid bid** for slot *2*, i.e. c^1 **determines what bids in** B_2 **are valid**. B_2 reflects the additional utility from winning slot *2* insted of slot *1* - $(\alpha_2 - \alpha_1)v$. Both bids are revealed. The 2^{nd} highest bid is determined as the price of slot *2* - $c^2 \in B_2$, and its bidder wins slot *1* and pays c^1. The highest bidder wins the slot *2* (the 'best' slot) and pays c^2. In a case of tie, both bidders have a probability of $\frac{1}{2}$ to win each slot. For every player *i*, let b^2_{-i} be the highest bid for slot *2* when excluding *i*'s bid. A Bidding Strategy function from *i*'s possible values, *V*, into possible bids, B_1 and B_2 is $s_{i,(1,2)}(v_i) \in S_i$.

3.1.4 Outcome for player i

For every player *i*, her possible payments pay_i depend on her bidding and the previously defined b^1_{-i}, b^2_{-i}. In case of being one of the tied biddders, we add two possible reductions of the payments:

1. ϵ^1=reduction of slot *1*'s payment for player *i* in case of being one of the tied bidders both in phase *1* and *2*: $[b_{i,1} = b^1_{-i}$ and $b_{i,2} = b^2_{-i}]$.

2. ϵ^2=reduction of slot *2*'s payment for player *i* in case of being one of the tied bidders in phase *2*, given promotion to phase *2*: $b_{i,2} = b^2_{-i}$.

We assume that: $0 < (n-1)\cdot\epsilon^2 < \epsilon^1 < \frac{d}{n}$. To summarize, i's payment function:

$$pay_i(b_{i,1}, b_{i,2}, b^1_{-i}, b^2_{-i}) = \begin{cases} (1) b^2_{-i} & b_{1,i} > b^1_{-i}, b_{2,i} > b^2_{-i} \\ (2) \frac{1}{2}(b^2_{-i} - \epsilon^2 + b^1_{-i}) & b_{1,i} > b^1_{-i}, b_{2,i} = b^2_{-i} \\ (3) b^1_{-i} & b_{1,i} > b^1_{-i}, b_{2,i} < b^2_{-i} \\ (4) s\cdot b^2_{-i} & b_{1,i} = b^1_{-i}, b_{2,i} > b^2_{-i} \\ (5) \frac{s}{2}(b^2_{-i} - \epsilon^2 + b^1_{-i} - \epsilon^1) & b_{1,i} = b^1_{-i}, b_{2,i} = b^2_{-i} \\ (6) s\cdot b^1_{-i} & b_{1,i} = b^1_{-i}, b_{2,i} < b^2_{-i} \\ (7) 0 & b_{1,i} < b^1_{-i} \end{cases}$$

Cases (1),(4) - winning slot *2*; Cases (2),(5) - winning one of the slots; cases (3),(6) - winning slot *1*; case (7) - i doesn't win any slot.

Next, i's Utility function:

$$u_i[v_i, s_i(v_i), s_{-i}(v_{-i})] = \begin{cases} \alpha_2 v_i - b^2_{-i} & b_{1,i} > b^1_{-i}, b_{2,i} > b^2_{-i} \\ \frac{1}{2}(\alpha_2 v_i - b^2_{-i} + \epsilon^2 + \alpha_1 v_i - b^1_{-i}) & b_{1,i} > b^1_{-i}, b_{2,i} = b^2_{-i} \\ \alpha_1 v_i - b^1_{-i} & b_{1,i} > b^1_{-i}, b_{2,i} < b^2_{-i} \\ s(\alpha_2 v_i - b^2_{-i}) & b_{1,i} = b^1_{-i}, b_{2,i} > b^2_{-i} \\ \frac{s}{2}(\alpha_2 v_i - b^2_{-i} + \epsilon^2 + \alpha_1 v_i - b^1_{-i} + \epsilon^1) & b_{1,i} = b^1_{-i}, b_{2,i} = b^2_{-i} \\ s(\alpha_1 v_i - b^1_{-i}) & b_{1,i} = b^1_{-i}, b_{2,i} < b^2_{-i} \\ 0 & b_{1,i} < b^1_{-i} \end{cases}$$

Note that in case of a tie, u_i is the expected utility for player i.

Player i's expected payoff, given $v_i = v$, $p_i \in P$, $s_i(v_i)$,$s_{-i}(v_{-i}) \in S$:

$$\sum_{v_{-i}\in V^{n-1}} p_i(v_{-i}|v_i = v) u_i[v, s_i(v), s_{-i}(v_{-i})]$$

For further analysis for a single player i, let $v_i = v$, $p_i = p$, $b_{i,1} = b_1$, $b_{i,2} = b_2$, $b^1_{-i} = b^1$, $b^2_{-i} = b^2$ $s_{i,1}(v) = s_1(v)$, $s_{i,2}(v) = s_2(v)$ and $u_i = u$.

Claim 1: *The strategy* $[s_1(v) = b_1 = \alpha_1 v,\ s_2(v) = b_2 = (\alpha_2 - \alpha_1)v + b^1]$ *yields a* *non-negative payoff:* $\forall v_i = v \in V$, $\forall s_{-i} \in S_{-i}$ *and regardless of* P:

$$u[v, (\alpha_1 v, (\alpha_2 - \alpha_1)v + b^1), s_{-i}(v_{-i})] \geq 0$$

Proof:

$$u_i[v, \alpha_1 v, (\alpha_2 - \alpha_1)v + b^1, s_{-i}(v_{-i})] = \begin{cases} (1)\alpha_2 v - b^2 > 0 & \alpha_1 v > b^1, (\alpha_2 - \alpha_1)v + b^1 > b^2 \\ (2)\alpha_1 v - b^1 + \frac{\epsilon^2}{2} > 0 & \alpha_1 v > b^1, (\alpha_2 - \alpha_1)v + b^1 = b^2 \\ (3)\alpha_1 v - b^1 > 0 & \alpha_1 v > b^1, (\alpha_2 - \alpha_1)v + b^1 < b^2 \\ (4)s(\alpha_2 v - b^2) > 0 & \alpha_1 v = b^1, (\alpha_2 - \alpha_1)v + b^1 > b^2 \\ (5)\frac{s}{2}(\epsilon^2 + \epsilon^1) > 0 & \alpha_1 v = b^1, (\alpha_2 - \alpha_1)v + b^1 = b^2 \\ (6)0 & \alpha_1 v = b^1, (\alpha_2 - \alpha_1)v + b^1 < b^2 \\ (7)0 & \alpha_1 v < b^1 \end{cases}$$

3.2 Dominant Strategy

Claim 2: *The strategy* $[s_1(v) = b_1, s_2(v) = b'_2 \neq (\alpha_2 - \alpha_1)v + b^1]$ *is dominated by* $[s_1(v) = b_1, s_2(v) = b_2 = (\alpha_2 - \alpha_1)v + b^1]$: $\forall v_i = v \in V$, $\forall b_1 \in B_1$, $\forall s_{-i} \in S_{-i}$ *and regardless of* P:

$$u[v, b_1, b_2, s_{-i}(v_{-i})] \geq u'[v, b_1, b'_2, s_{-i}(v_{-i})]$$

Proof: We compare u and u' for every outcome of the auction, given that player i won promotion ($b_1 \geq b^1$). For simplicity, we assume that $b_1 > b^1$ (case of $b_1 = b^1$ doesn't change the result). For $b^1 > b_1$, $u = u' = 0$.

$\underline{b'_2 < b_2}$: for $b'_2 > b^2$, $u = u' = \alpha_2 v - b^2$; for $b'_2 = b^2$, $u = \alpha_2 v - b^2 > u' = \frac{1}{2}(\alpha_2 v - b^2 + \epsilon^2 + \alpha_1 v - b^1)$ [2]; for $b_2 > b^2 > b'_2$, $u = \alpha_2 v - b^2 > \alpha_2 v - (\alpha_2 v -$

[2] $(\alpha_2 - \alpha_1)v + b^1 > b^2 \Rightarrow \alpha_2 v - b^2 > \alpha_1 v - b^1$ and due to restrictions on $\epsilon^2 \Rightarrow \alpha_2 v - b^2 > \frac{1}{2}(\alpha_2 v - b^2 + \epsilon^2 + \alpha_1 v - b^1)$

$\alpha_1 v + b^1) = \alpha_1 v - b^1 = u'$; for $b^2 = b_2$, $u = \frac{1}{2}(\alpha_2 v - b^2 + \epsilon^2 + \alpha_1 v - b^1) = \alpha_1 v - b^1 + \frac{\epsilon^2}{2} > u' = \alpha_1 v - b^1$; for $b^2 > b_2$, $u = u' = \alpha_1 v - b^1$.

$\underline{b'_2 > b_2}$: for $b_2 > b^2$, $u = u' = \alpha_2 v - b^2$; for $b_2 = b^2$, $u = \alpha_1 v - b^1 + \frac{\epsilon^2}{2} > u' = \alpha_2 v - (\alpha_2 v - \alpha_1 v + b^1) = \alpha_1 v - b^1$; for $b'_2 > b^2 > b_2$, $u = \alpha_1 v - b^1 > u' = \alpha_2 v - b^2$ [3]; for $b^2 = b'_2$, $u = \alpha_1 v - b^1 > u' = \frac{1}{2}(\alpha_2 v - b^2 + \epsilon^2 + \alpha_1 v - b^1)$ [4]; for $b^2 > b'_2$, $u = u' = \alpha_1 v - b^1$. $\square$.

Claim 3: *The strategy* $[s_1(v) = b'_1 \neq \alpha_1 v, s_2(v) = b_2 = (\alpha_2 - \alpha_1)v + b^1]$ *is dominated by* $[s_1(v) = b_1 = \alpha_1 v, s_2(v) = b_2 = (\alpha_2 - \alpha_1)v + b^1]$*:* $\forall v_i = v \in V$, $\forall s_{-i} \in S_{-i}$ *and regardless of* P:

$$u[v, b_1, b_2, s_{-i}(v_{-i})] \geq u'[v, b'_1, b_2, s_{-i}(v_{-i})]$$

Proof: We compare promotion probabilities to phase 2 from bidding $[b_1, b_2]$ and $[b'_1, b_2]$. Note that $u \geq u'$ by Claim 2 given that promotion probability from bidding $[b_1, b_2]$ is not lower than from bidding $[b'_1, b_2]$.

$\underline{b'_1 < b_1}$: for $b^1 < b'_1$, both probabilities equal 1; for $b_1 \geq b^1 \geq b'_1$, $[b_1, b_2]$ provides a higher promotion probability; for $b^1 > \alpha_1 v$, both probabilities equal 0.

$\underline{b'_1 > b_1}$: for $b^1 < b_1$, both probabilities equal 1; for $b^1 = b_1$, the lowest valid bid for slot 2 is $\alpha_2 v$ which equals to b_2. In this case, for $b^2 = \alpha_2 v$, $u = \frac{s}{2}(\epsilon^2 + \epsilon^1)$ while $u' = \alpha_2 v - b^2 = \alpha_2 v - \alpha_2 v = 0$ for $b'_2 > \alpha_2 v$ and $\frac{\epsilon^2}{2}$ for $b'_2 = \alpha_2 v$; [5] for $b^1 > b_1$, $u = 0$ and then for $b^1 > b'_1$, $u' = 0$; for $b'_1 > b^1$, the initial bid for slot 2 is higher than $\alpha_2 v$ and the utility becomes negative. $\square$.

Conclusion 1 (from Claims 2-3) : *The strategy* $[s_1(v) = \alpha_1 v, s_2(v) = (\alpha_2 - \alpha_1)v + b^1]$ *is a dominant strategy:* $\forall v_i = v \in V$, $\forall s_{-i} \in S_{-i}$ *and regardless of* P:

$$u[v, s_1(v), s_2(v), s_{-i}(v_{-i})] \geq u'[v, s'_1(v), s'_2(v), s_{-i}(v_{-i})]$$

$\forall s'_1(v) \in S_i$, $\forall s'_2(v) \in S_i$ *where* $s_1(v) \neq s'_1(v)$ *or* $s_2(v) \neq s'_2(v)$.

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[3] $b^2 > b_2 \Rightarrow u' = \alpha_2 v - b^2 < \alpha_2 v - [(\alpha_2 - \alpha_1)v + b^1] = \alpha_1 v - b^1$

[4] $b^2 > b_2 \Rightarrow \alpha_1 v - b^1 > \alpha_2 v - b^2 \Rightarrow u' < \alpha_1 v - b^1 + \frac{\epsilon^2}{2}$ and due to restrictions on $\epsilon^2 \Rightarrow u' < \alpha_1 v - b^1$

[5] since $(n-1) \cdot \epsilon^2 < \epsilon^1$ and s is at least $\frac{1}{n}$ then $\frac{s}{2}(\epsilon^2 + \epsilon^1) > \frac{\epsilon^2}{2}$

Chapter 4

Main Result

We begin the chapter with restrictions we made on players' beliefs - P in adjusted Discrete GSPa(n,2) introduced in Chapter 3. We proceed with stating and proving our main result regarding being GEa equilibrium strategy the unique outcome of the P-deletion process in this auction. The chapter ends with assumptions' analysis made for the model.

4.1 Restrictions on Players' Beliefs

1. Player i assigns a positive probability for every maximum value over other players' values: $p_i(max_{-i}v_j = v|v_i) > 0$, $\forall v \in V$.

2. Player i assigns a positive probability for each value to be shared by at least *2* other players: $p_i[\#(j \neq i$ s.t.: $v_j = v|v_i) \geq 2] > 0$, $\forall v \in V$.

4.2 Stating Main Result

Theorem 1 : *In the adjusted Discrete Generalized Second Price auction with* n *players and 2 slots, the only strategy that survives Iterated Deletion of P-dominated strategies for player* i *with* $v_i = v$, $\forall p \in P$ *and* $\forall s_{-i} \in S_{-i}$ *is* *[*$s_1(v) = \alpha_1 v,\ s_2(v) = (\alpha_2 - \alpha_1)v + b^1$*]*.

The proof is built from 2 stages. In the first stage, we show that $\forall v \in V$ all strategies s.t. $s_1(v) \neq \alpha_1 v$ are P-dominated at some stage of the P-deletion process i.e. $s_1(v) = \alpha_1 v \; \forall v \in V$. The result is shown first for $v_i = 1$. We then prove two general Claims regarding the converged game and proceed iterating for $v_i = 1 - d$ and so on until $v_i = 0$.

In the second stage, we rely on the deletion so far and go on with the process to show that $\forall v \in V$ all $[s_1(v) = \alpha_1 v,\; s_2 \neq (\alpha_2 - \alpha_1)v + b^1]$ are P-dominated i.e. $s_2(v) = (\alpha_2 - \alpha_1)v + b^1$. The result is shown first for $v_i = 0$. We then prove two general Claims regarding the converged game and proceed iterating for $v_i = d$ and so on until $v_i = 1$.

Let S be the set of all strategies for all values before starting the process and S^a, is the set of the remaining strategies that were not yet deleted in the coverged game after ending step a.

4.3 Proof of Theorem 1

4.3.1 $s_1(v) = \alpha_1 v$, $\forall v \in V$

Let $v_i = 1$ and $[s_1(1) = b_1 = \alpha_1, s_2(1) = b_2 = \alpha_2 - \alpha_1 + b^1]$. Since $[b_1, b_2]$ are hte highest valid bids, only cases (1),(2),(4),(5) are possible in the Utility function derived in Claim 1 for which $u[1, b_1, b_2, s_{-i}(v_{-i})] > 0 \; \forall s_{-i} \in S_{-i}$.

Claim 4: *For $v_i = 1$, $[0, b_2']$, $b_2' \in B_2(c^1)$ is strictly dominated by $[b_1, b_2]$: $\forall s_{-i} \in S_{-i}$, and regardless of P:*

$$u[1, \alpha_1, \alpha_2 - \alpha_1 + b^1, s_{-i}(v_{-i})] > u'[1, 0, b_2', s_{-i}(v_{-i})]$$

Proof: We analyze possible outcomes of phase 1: $b^1 = 0$ and $b^1 > 0$. For $b^1 > 0$, $u > 0$, whereas $u' = 0$ since player i doesn't win promotion to phase *2*; for $b^1 = 0$, the promotion probabilities from bidding $[b_1, b_2]$ and $[0, b_2']$ are *1* and s, respectively, $\forall b^1 \in B_1$. Since $s < 1$ and $[b_1, b_2]$ is a dominant strategy by Conclusion 1, $u > u'$. Therefore, $s_1(1) = 0$ is deleted from further calculations:

$[s_1(1) = 0] \notin S^1_{-i}$ and $s_1(1) \geq \alpha_1 d$. $\square$.

Claim 5: *In the converged game, $[\alpha_1 d, b_2']$, $b_2' \in B_2(c^1)$ is P-dominated by $[b_1,b_2]$ for $v_i = 1$: $\forall p(\cdot|v_i = 1) \in \mathrm{P}$ and $\forall\, s_{-i} \in S^1_{-i}$:*

$$\sum_{v_{-i}\in V^{n-1}} p(v_{-i}|v_i = 1)u[1, b_1, b_2, s_{-i}(v_{-i})] > \sum_{v_{-i}\in V^{n-1}} p(v_{-i}|v_i = 1)u'[1, \alpha_1 d, b_2', s_{-i}(v_{-i})]$$

Proof: Both sides are rearranged as follows:

$$\underbrace{\sum_{v_{-i} s.t. \#(j\neq i:v_j=1)<2} p(v_{-i}|v_i = 1)u[1, b_1, b_2, s_{-i}(v_{-i})]}_{A} + \underbrace{\sum_{v_{-i} s.t. \#(j\neq i:v_j=1)\geq 2} p(v_{-i}|v_i = 1)u[1, b_1, b_2, s_{-i}(v_{-i})]}_{Z} >$$

$$\underbrace{\sum_{v_{-i} s.t. \#(j\neq i:v_j=1)<2} p(v_{-i}|v_i = 1)u'[1, \alpha_1 d, b_2', s_{-i}(v_{-i})]}_{A'} + \underbrace{\sum_{v_{-i} s.t. \#(j\neq i:v_j=1)\geq 2} p(v_{-i}|v_i = 1)u'[1, \alpha_1 d, b_2', s_{-i}(v_{-i})]}_{Z'}$$

$A \geq A'$ since $u \geq u'$ by Conclusion 1 as $[s_1(1) = b_1, s_2(1) = b_2]$ is a dominant strategy. For v_{-i} s.t. $\#(j \neq i : v_j = 1) \geq 2$, $b^1 \geq \alpha_1 d$ by Claim 4. For $b^1 = \alpha_1 d$, the promotion probabilities from bidding $[b_1, b_2]$ and $[\alpha_1 d, b_2']$ are *1* and s, respectively $\forall b^1 \in B_1$ and similarly to Claim 4 - $u > u'$; for $b^1 > \alpha_1 d$, the promotion probabilities from bidding $[b_1, b_2]$ are positive $\forall b^1 \in B_1$ (since b_1 is the maximum valid bid) with $u > 0$ and equal *0* $\forall b^1 \in B_1$ for $[0, b_2']$ with $u' = 0$. Hence, $A + A' > Z + Z'$, $[s_1(1) = \alpha_1 d] \notin S^2_{-i}$ and $s_1(1) \geq \alpha_1(2d)$. $\square$.

Iterating for $v_i = 1$: we use the same rearrangemnt of the inequality, the outcome of previous steps and the dominance feature together with higher promotion probabilities from bidding $[b_1, b_2]$ to delete other bids, increasing in b_1' till $b_1' = \alpha_1$ and we reach that $\boxed{s_1(1) = \alpha_1}$.

Next, let $c \cdot d$ be the minimum value for which at some stage x the deletion process leads to $s_1(c \cdot d) = \alpha_1(c \cdot d)$. $c \cdot d \in V$ and $x>2$.

Claim 6: *$[\alpha_1(cd), b_2']$, $b_2' \in B_2(c^1)$ is P-dominated by $[b_1 = \alpha_1 v, b_2 = (\alpha_2 - \alpha_1)v + b^1]$ $\forall v < cd$, $v \in V$: $\forall p(\cdot|v_i = v) \in P$ and $\forall s_{-i} \in S^x_{-i}$:*

$$\sum_{v_{-i}\in V^{n-1}} p(v_{-i}|v_i = v)u[v, b_1, b_2, s_{-i}(v_{-i})] > \sum_{v_{-i}\in V^{n-1}} p(v_{-i}|v_i = v)u'[v, \alpha_1(cd), b_2', s_{-i}(v_{-i})]$$

Proof: Both sides are rearranged as follows: A and A' for $max_{-i}v_j \neq cd$ - $A \geq A'$; Z and Z' for $max_{-i}v_j = cd$. For $max_{-i}v_j = cd$ and $\#(j \neq i : v_j = cd) \geq 2$, $Z \geq 0$ by Claim 1 as $[b_1,b_2]$ yields a non-negative utility whereas $[\alpha_1(cd), b'_2]$ leads to $b^1 = \alpha_1(cd)$ for which $b^2 \geq \alpha_2(cd)$ (the leap $\frac{\alpha_2}{\alpha_1}\alpha_1(cd) = \alpha_2(cd)$).

$$u' = \begin{cases} (1)\frac{s}{2}[(\alpha_2+\alpha_1)(v-cd)+\epsilon^2+\epsilon^1] < 0 & b^2 = b'_2 = \alpha_2(cd) \\ s\alpha_1(v-cd) < 0 & b^2 > b'_2 \\ (2)\alpha_2(v-cd) < 0 & b'_2 > b^2 = \alpha_2(cd) \end{cases}$$

(1),(2): Note that for $b^2 > \alpha_2(cd)$, u' decreases. Hence, $Z > Z'$, $[s_1(v) = \alpha_1(cd)] \notin S^{x+1}_{-i}$ and $s_1(v) \leq \alpha_1[(c-1)d]$ $\forall v_i = v < cd$, $v \in V$. □.

Next, let $c \cdot d$ be the minimum value for which at some stage y the deletion process leads to $s_1(c \cdot d) = \alpha_1(c \cdot d)$ and $\alpha_1(z \cdot d) \leq s_1[(c-1) \cdot d]$. $z \cdot d < (c-1) \cdot d < 1$, $z \cdot d, (c-1) \cdot d, c \cdot d, \in V$ and y>2.

Claim 7: *$[z \cdot d, b'_2]$, $b'_2 \in B_2(c^1)$ is P-dominated by $[b_1 = \alpha_1 v, b_2 = (\alpha_2 - \alpha_1)v + b^1]$ for $v_i = (c-1) \cdot d$: $\forall p(\cdot|v_i = (c-1) \cdot d) \in$ P and $\forall\, s_{-i} \in S^y_{-i}$:*

$$\sum_{v_{-i} \in V^{n-1}} p[v_{-i}|(c-1)d]u[(c-1)d, b_1, b_2, s_{-i}(v_{-i})] > \sum_{v_{-i} \in V^{n-1}} p[v_{-i}|(c-1)d]u'[(c-1)d, \alpha_1(zd), b'_2, s_{-i}(v_{-i})]$$

Proof: Rearrangement made: A and A' for $max_{-i}v_j \neq (c-1)d$ - $A \geq A'$; Z and Z' for $max_{-i}v_j = (c-1)d$. For $max_{-i}v_j = (c-1)d$ and v_{-i} s.t. $\#[j \neq i : v_j = (c-1)d] \geq 2$, $\alpha_1(zd) \leq b^1 \leq \alpha_1[(c-1)d]$ by Claim 6 and the assumption. Then: for $\alpha_1(zd) \leq b^1 < \alpha_1[(c-1)d]$, $[b_1,b_2]$ yields a positive utility with higher promotion probability $\forall b^1 \in B_1$; for $b^1 = \alpha_1[(c-1)d]$, $b^2 = b_2 = \alpha_2[(c-1)d]$ [6] for which $u = \frac{s}{2}(\epsilon^2 + \epsilon^1) > u' = 0$, $Z > Z'$, $[s_1[(c-1)d] = zd] \notin S^{y+1}_{-i}$ and $s_1[(c-1)d] \geq \alpha_1[(z+1)d]$. □.

Now, let $v_i = 1 - d$ with $[b_1 = \alpha_1(1-d), b_2 = (\alpha_2 - \alpha_1)(1-d) + b^1]$. First, $[\alpha_1, b'_2]$, $b'_2 \in B_2(c^1)$ is P-dominated by $[b_1, b_2]$ by Claim 6 where $cd = 1$ and so $0 \leq s_1(1-d) \leq \alpha_1(1-d)$. Next, $[0, b'_2]$, $b'_2 \in B_2(c^1)$ is P-dominated by $[b_1, b_2]$ by Claim 7 where $cd = 1$, $(c-1)d = 1-d$ and $zd = 0$. Iterating when using Claim 7, we

[6] the other promoting player bids either $b^* = \alpha_2[(c-1)d]$ or $b^{*'} > \alpha_2[(c-1)d]$. b^* yields a strictly higher utility for any opponent's bid

conclude that $s_1(1-d) = \alpha_1(1-d)$. Similarly, we show that $s_1(1-2d) = \alpha_1(1-2d)$ for which $cd = 1-d$. Iterating while decreasing player's value till $v_i = 0$, we reach that $\boxed{s_1(v) = \alpha_1 v}$, $\forall v \in V$, $\forall p \in P$.

4.3.2 $s_2(v) = (\alpha_2 - \alpha_1)v + b^1$, $\forall v \in V$

For simplicity, assume that the P-deletion process is now at step $a = 3$. Let $v_i = 0$ and $[s_1(0) = b_1 = 0, s_2(0) = b_2 = (\alpha_2 - \alpha_1) \cdot 0 + b^1 = b^1]$. Note that $v_i = 0$ may win promotion only in case of $b^1 = 0$.

Claim 8: *For* $v_i = 0$, $[0, b'_2 = \alpha_2 - \alpha_1]$, *is strictly dominated by* $[b_1, b_2]$*:* $\forall s_{-i} \in S^3_{-i}$ *and regardless of* P:

$$u[0, 0, b^1, s_{-i}(v_{-i})] > u'[0, 0, \alpha_2 - \alpha_1, s_{-i}(v_{-i})]$$

Proof: We analyze possible outcomes of phase 2, given that $b^1 = 0$: $b^2 = 0$ and $b^2 > 0$. For $b^2 = 0$, $u = \frac{s}{2}(\epsilon^2 + \epsilon^1)$ (since $b^1 = b_1$ and $b^2 = b_2$) whereas $u' = s(\alpha_2 \cdot 0 - 0) = 0$; for $0 < b^2 \leq b'_2$, $u = 0 > u' = s(\alpha_2 \cdot 0 - b^2)$; for $b^2 = b'_2$, $u = 0 > u' = \frac{s}{2}(\alpha_2 \cdot 0 - \alpha_2 + \alpha_1 + \epsilon^2 + 0 + \epsilon^1) < 0$. b^2 cannot be higher than b'_2 since b'_2 is the maximum valid bid, given $b^1 = 0$ and so Z>Z', $[s_2(0) = \alpha_2 - \alpha_1] \notin S^3_{-i}$ and $s_2(0) \leq (\alpha_2 - \alpha_1)(1-d)$. $\square$.

Claim 9: *In the converged game,* $[0, b'_2 = (\alpha_2 - \alpha_1)(1-d)]$, *is P-dominated by* $[b_1, b_2]$ *for* $v_i = 0$*:* $\forall p(\cdot|v_i = 0) \in \mathrm{P}$ *and* $\forall\, s_{-i} \in S^3_{-i}$*:*

$$\sum_{v_{-i} \in V^{n-1}} p(v_{-i}|v_i = 0) u[0, b_1, b_2, s_{-i}(v_{-i})] > \sum_{v_{-i} \in V^{n-1}} p(v_{-i}|v_i = 0) u'[0, 0, b'_2, s_{-i}(v_{-i})]$$

Proof: Rearrangement made: A and A' for $max_{-i} v_j > 0$ - $A \geq A'$; Z and Z' for $max_{-i} v_j = 0$ for which $b^2 \leq (\alpha_2 - \alpha_1)(1-d)$; for $0 < b^2 \leq b'_2$, $u = 0 > u' = s(\alpha_2 \cdot 0 - b^2)$; for $b^2 = b'_2$: $u = 0 > u' = \frac{s}{2}[\alpha 2 \cdot 0 - (\alpha_2 - \alpha_1)(1-d) + \epsilon^2 + 0 + \epsilon^1]$. $Z > Z'$, $s_2(0) = (\alpha_2 - \alpha_1)(1-d) \notin S^4_{-i}$ and $s_2(0) \leq (\alpha_2 - \alpha_1)(1-2d)$. $\square$.

Iterating for $v_i = 0$ while decreasing in b'_2 till $b'_2 = b^1 = 0$, we reach that $\boxed{s_2(0) = b^1 = 0}$.

Next, let $c \cdot d$ be the minimum value for which at some stage x the deletion process leads to $s_2(c \cdot d) = (\alpha_2 - \alpha_1)(c \cdot d) + b^1$. $c \cdot d \in V$ and $x>4$.

Claim 10: *[$b'_1 = \alpha_1 v, b'_2 = (\alpha_2 - \alpha_1)cd + b^1$] is P-dominated by [$b_1 = \alpha_1 v, b_2 = (\alpha_2 - \alpha_1)v + b^1$] for $v_i = v$, $v > c \cdot d$, $v \in V$: $\forall p(\cdot|v_i = v) \in$ P and $\forall\, s_{-i} \in S^x_{-i}$:*

$$\sum_{v_{-i} \in V^{n-1}} p(v_{-i}|v_i = v) u[v, b_1, b_2, s_{-i}(v_{-i})] > \sum_{v_{-i} \in V^{n-1}} p(v_{-i}|v_i = v) u'[v, \alpha_1 v, (\alpha_2 - \alpha_1)cd + b^1, s_{-i}(v_{-i})]$$

Proof: Rearrangement made: A and A' for $max_{-i} v_j \neq cd$ - $A \geq A'$; Z and Z' for $max_{-i} v_j = cd$. For $max_{-i} v_j = cd$ and v_{-i} s.t. $\#(j \neq i : v_j = cd) \geq 2$, $b^1 = \alpha_1(cd)$, $b^2 = (\alpha_2 - \alpha_1)cd + b^1 = \alpha_2(cd)$, $u = \alpha_2 v - \alpha_2(cd) > u' = \frac{1}{2}[\alpha_2 v - \alpha_2(cd) + \epsilon^2 + \alpha_1 v - \alpha_1(cd)]$. Then, $Z > Z'$, $[s_2(v) = (\alpha_2 - \alpha_1)cd + b^1] \notin S^{x+1}_{-i}$ and $s_2(v) \geq (\alpha_2 - \alpha_1)[(c+1)d]$ $\forall v_i = v > cd$, $v \in V$. □.

Now, let $c \cdot d$ be the minimum value for which at some stage y the deletion process leads to $s_2(c \cdot d) = (\alpha_2 - \alpha_1)(c \cdot d) + b^1$ and $s_2[(c+1)d] \leq (\alpha_2 - \alpha_1) z \cdot d + b^1$. $z \cdot d > (c+1) \cdot d > 0$, $z \cdot d, c \cdot d, (c+1) \cdot d, \in V$ and $y>4$.

Claim 11: *[$\alpha_1(c+1)d, b'_2 = (\alpha_2 - \alpha_1)zd + b^1$] is P-dominated by [$b_1 = \alpha_1(c+1)d, b_2 = (\alpha_2 - \alpha_1)(c+1)d + b^1$] for $v = (c+1)d$: $\forall p[\cdot|v_i = (c+1)d] \in$ P and $\forall\, s_{-i} \in S^y_{-i}$:*

$$\sum_{v_{-i} \in V^{n-1}} p[v_{-i}|(c+1)d] u[(c+1)d, b_1, b_2, s_{-i}(v_{-i})] > \sum_{v_{-i} \in V^{n-1}} p[v_{-i}|(c+1)d] u'[(c+1)d, b'_1, b'_2, s_{-i}(v_{-i})]$$

Proof: Rearrangement made: A and A' for $max_{-i} v_j \neq (c+1)d$ - $A \geq A'$; Z and Z' for $max_{-i} v_j = (c+1)d$. For $max_{-i} v_j = (c+1)d$ and v_{-i} s.t. $\#[j \neq i : v_j = (c+1)d] \geq 2$, $(\alpha_2 - \alpha_1)(c+1)d + b^1 \leq b^2 \leq (\alpha_2 - \alpha_1)zd + b^1$ by Claim 10 and the assumption. Then: for $b^2 = \alpha_2(c+1)d$, $u = \frac{s}{2}(\epsilon^2 + \epsilon^1) > u' = s[\alpha_2(c+1)d - \alpha_2(c+1)d] = 0$; for $\alpha_2(c+1)d < b^2 < b'_2$, $u = 0 > u' = s[\alpha_2(c+1)d - b^2]$; for $b^2 = b'_2$: $u = 0 > u' = \frac{s}{2}[\alpha_2(c+1)d - \alpha_2(zd) + \alpha_1(zd) - \alpha_1(c+1) \cdot d + \epsilon^2 + 0 + \epsilon^1]$, $Z > Z'$, $s_2(c+1)d = (\alpha_2 - \alpha_1)zd + b^1 \notin S^{y+1}_{-i}$ and $s_2(c+1)d \leq (\alpha_2 - \alpha_1)(z-1)d + b^1$. □.

Now, let $v_i = d$ with $[b_1 = \alpha_1 d, b_2 = (\alpha_2 - \alpha_1)d + b^1]$. First, $[\alpha_1 d, b^1]$ is P-dominated by $[b_1, b_2]$ by Claim 10 where $cd = 0$ and so $(\alpha_2 - \alpha_1)d + b^1 \leq s_2(d) \leq$

$\alpha_2 - \alpha_1 + b^1$. Next, $[\alpha_1 d, \alpha_2 - \alpha_1 + b^1]$ is P-dominated by $[b_1, b_2]$ by Claim 11 where $cd = 0$, $(c+1)d = d$ and $zd = 1$. Iterating when using Claim 11, we conclude that $s_2(d) = (\alpha_2 - \alpha_1)d + b^1$. Similarly, we show that $s_2(2d) = (\alpha_2 - \alpha_1)(2d) + b^1$ for which $cd = d$. Iterating while increasing player's value till $v_i = 1$, we reach that $\boxed{s_1(v) = \alpha_1 v, s_2(v) = (\alpha_2 - \alpha_1)v + b^1}$, $\forall v \in V$, $\forall p \in P$. The process stops since all possible strategies were analyzed and we reach the unique outcome as stated in Theorem 1. $\square$.

4.4 Assumptions' Analysis

In this Section we analyze the assumptions made in Chapter 3 and their essentiality to the course of proving Theorem 1. We analyze the borders of B, restrictions on players' beliefs P and the reduction of slots' payments - ϵ^1, ϵ^2.

4.4.1 Borders of B_1

After showing that $s_1(1) = \alpha_1$, we proceed to $v = 1 - d$. We use the assumption in Claim 6 that $\alpha_1(1-d)$ is the second highest valid bid for slot *1*. Thus, the difference between two valid following bids for slot *1* is required to be $\alpha_1 d$ and we can't follow the Discrete First Price auction's setting, where the bids are on the same discrete grid as players' values per click: $V = (0, d, 2d, ..., 1-d, 1)$. In this case, the difference between two following bids is d.

4.4.2 Borders of B_2

The $\frac{\alpha_2}{\alpha_1} \cdot c^1$ *leap* - the restriction on the lowest valid bid for slot *2* to $\frac{\alpha_2}{\alpha_1} \cdot c^1$ is used in Claim 2 to prove the dominance of $[s_1(v) = \alpha_1 v, s_2(v) = (\alpha_2 - \alpha_1)v + c^1]$ in case of bidding more than $\alpha_1 v$ for the 1^{st} slot. The dominance stands at the basis of proving Theorem 1 and denies players bidding $b_1 > \alpha_1 v$ since they must bid mpore than $\alpha_2 v$ in phase *2* and their utility becomes negative.

Claim 12: *Let* $B_2 = (c^1, c^1 + \alpha_1 d, c^1 + 2\alpha_1 d, ..., \alpha_2 - \alpha_1 d, \alpha_2)$ *then strategy* $[s_1(v) = \alpha_1 v, s_2(v) = (\alpha_2 - \alpha_1)v + c^1]$ *is not a dominant strategy.*

Proof: Let $v < 1$, $b_1 = \alpha_1 v$, $b'_1 = b^1 = b^2 = \alpha_1$ (possible since there is no leap) and $b'_2 = \alpha_2 v$. Then, $u = 0$ whereas $u'_1 \geq \alpha_2 v - \alpha_1$. Without further assumptions on α_1, α_2, we can't exclude that $u'_1 > u_1 = 0$. $\square$.

Note that the reductions of payments or players' beliefs are irelevant in this case, so their existence or abscence doesn't change the result.

Valid Bids in B_2 - Allowing the minimal difference in B_2 to be $\alpha_1 d$ (as in B_1) denies us from assuring that the strategy $s_2(v) = (\alpha_2 - \alpha_1)v + b^1$ is a valid bid. Let $v > v'$ and $b^1 = \alpha_1 v'$. Then the next valid bid for slot *2* is $\alpha_2 v' + \alpha_1 d$. Without further restrictions on α_1 and α_2, we can't exclude that $s_2(v) = (\alpha_2 - \alpha_1)v + \alpha_1 v' < \alpha_2 v' + \alpha_1 d$. This disqualifies changing the difference to $\alpha_2 d$ too, which could be a logical sequence as well.

Note that the analysis of B's borders emphasizes the need of its finiteness, given that V is finite.

4.4.3 Restrictions on Players' Beliefs

Recall that Dekel and Wolinsky formulate two restrictions on players' beliefs:

1. Each player assigns a positive probability that her value is the highest among all players.

2. For sufficiently large n, each player assigns a small probability that no more than l ($l \leq n$) of the players share the same value with her, conditional on all players having values not larger than v.

We don't use the second restriction. The first one, however is expanded to a more general view: player i assigns a positive probability for every maximum value over other players; Additional restriction, regarding at least *2* players sharing the same value was added. These restrictions are used widely in our prove both in deleting previous outcomes for other values and proceeding the P-deletion process for the next value.

4.4.4 Adjusting the payment rule - ϵ^1, ϵ^2

In GSPa, any player that wins a slot pays the next highest bid. Changing this payment rule by reducing the slots' prices in case of a tie is somewhat the most crucial adjustment to be made, with regard to the GSPa's setup. However, the ϵ^1 and ϵ^2 are restricted to small numbers and influence on the payoff in tie situations only.

Whereas ϵ^2 is crucial for proving the inequalities throughout the proof, ϵ^1 maintains the dominance feature once ϵ^2 is added to the auction's calculations. Again, recall Claim 2: for $b'_1 > b^1 = b_1 = \alpha_1 v$ and $b'_2 > b^2 = \alpha_2 v = b_2$, $u = \frac{s}{2}(\epsilon^2 + \epsilon^1)$ whereas $u' = \frac{\epsilon^2}{2}$. When excluding ϵ^1, $u' > u$, and the dominance does not hold. Since in this case u is at most $\frac{\epsilon^2+\epsilon^1}{2n}$, adding ϵ_1 with the proper restriction of $(n-1)\epsilon^2 < \epsilon^1$ leads to $u > u'$.

Reduction of payments for tie on 0 - this makes our mechanism to actually pay the players when $v = 0$ wins a slot. We show that $s_2(0) = 0$ by proving first that $[0, \alpha_2 - \alpha_1]$ is strictly dominated by $[0, 0]$: for $b^2 = 0$, $[0, 0]$ yields $\frac{s}{2}(\epsilon^2 + \epsilon^1)$ whereas $[0, \alpha_2 - \alpha_1]$ yields *0*. Without the payment to the players, both bids yield *0* and we can't delete $[0, \alpha_2 - \alpha_1]$. Moreover, it is easy to see that changing the ϵ in case of a tie on *0* to $\delta << \epsilon^2$ will not change the result.

Chapter 5

Deletion Process in Discrete General Auctions

In this chapter we run the Deletion process in Discrete General auctions: generalized version of our main aution - GSPa with n players and m slots and a Discrete VCG mechanism.

5.1 Discrete GSPa(n,m)

5.1.1 General Description

Consider Discrete GSPa with $n>2$ players and $m \geq 3$ items (slots) whis is **equivalent to GEa**. The expected number of clicks per period on slot $l \in (1..m)$ is α_l. $\alpha_m > \alpha_{m-1} > ... > \alpha_1$. The values' grid remains V. Each player has a set of beliefs $p_i(\cdot|v_i = v) \in P$ over other players' values. The auction is built from m phases.

5.1.2 Phases 1-m

In phase *1*, each player privately submits a bid for slot *1*, $b_{i,1}$, $i \in (1, 2..n)$. The bids for slot *1* are on a discrete grid: $B_1 = (0, \alpha_1 d, \alpha_1(2d), ..., \alpha_1 - \alpha_1 d, \alpha_1)$. Only the $(m+1)^{th}$ highest bid is revealed and determined as the price slot *1* - $c^1 \in B_1$. m bidders proceed to phase *2*, while the others remain with no payoff. In each of the following phases $l > 1$, there are $m-l+2$ remaining players that bid for slot l. The lowest bid is revealed and determined as the price of the slot l - $c^l \in B_l(c^{l_1})$ where $B_l(c^{l-1}) \subseteq [\frac{\alpha_l}{\alpha_{l-1}} \cdot c^{l-1}, (\alpha_l - \alpha_{l-1}) \cdot 0 + c^{l-1}, (\alpha_l - \alpha_{l-1})d + c^{l-1} ... (\alpha_l - \alpha_{l-1}) \cdot 1 + c^{l-1}]$. Again, $\frac{\alpha_l}{\alpha_{l-1}} \cdot c^{l-1}$ determines the valid bids and the grid reflects players' potential additional utility from winning slot l over slot *l-1*. In case of a tie between bids in phase l, $l \in (1..m-1)$, each of the tied bidders wins promotion to phase *l+1* with equal probability s_l. For every player i, let b^1_{-i} be the m^{th} highest bid for slot *1* when excluding i's bid and b^l_{-i} be the l^{m+l-1} highest bid for slot l when excluding i's bid, $l > 1$. A Bidding Strategy function from i's possible values, V, into possible bids, $B_1, B_2, .., B_m$ is $s_{i,(1,2,..,m)}(v_i) \in S_i$.

5.1.3 Outcome for player i

Similarly to adjusted Discrete GSPa(n,2), we add two possible reductions of the payments:

1. ϵ^{m-1}=reduction of payment for slot m-1 when $[b_{i,m-1} = b^{m-1}_{-i}$ and $b_{i,m} = b^m_{-i}]$, given promotion to phase m.

2. ϵ^2=reduction of payment for slot m when $b_{i,m} = b^m_{-i}$, given promotion to phase m.

We assume that: $0 < (n-1)\epsilon^m < \epsilon^{m-1} < \frac{d}{n}$.

Let l be the last phase i participates in, then her Utility function $u_i[v_i, l, s_i(v_i), s_{-i}(v_{-i})]$:

$$u_i = \begin{cases} \prod_1^{m-1} s_l(\alpha_m v_i - b^m_{-i}) & l = m, b_{i,m} > b^m_{-i} \\ \frac{1}{2}\prod_1^{m-1} s_l(\alpha_m v_i - b^m_{-i} + \epsilon^m + \alpha_{m-1} v_i - b^{m-1}_{-i} + \epsilon^{m-1}) & l = m, b_{i,m} = b^m_{-i}, b_{i,m-1} = b^{m-1}_{-i} \\ \frac{1}{2}\prod_1^{m-1} s_l(\alpha_m v_i - b^m_{-i} + \epsilon^m + \alpha_{m-1} v_i - b^{m-1}_{-i}) & l = m, b_{i,m} = b^m_{-i}, b_{i,m-1} > b^{m-1}_{-i} \\ \prod_1^{m-1} s_l(\alpha_{m-1} v_i - b^{m-1}_{-i}) & l = m, b_{i,m} < b^m_{-i} \\ (\prod_1^{l-1} s_l)(\alpha_{l-1} v_i - b^{l-1}_{-i}) & 1 < l < m \\ 0 & l = 1 \end{cases}$$

For further analysis for a single player i, let $v_i = v$, $p_i = p$, $b_{i,l} = b_l$, $b^l_{-i} = b^l$ and $s_{i,l}(v) = s_l(v)\ \forall l \in (1..m)$, and $u_i = u$.

5.1.4 Dominance

While in the adjusted GSPa(n,2), $[s_1(v) = \alpha_1 v, s_2(v) = (\alpha_2 - \alpha_1)v + b^1, .., s_m(v) = (\alpha_m - \alpha_{m-1})v + b^{m-1}]$ is a dominant strategy, it is not the case in GSPa(n,m). The intuition for this difference is the aggregate probability $\prod_1^m s_l$ which reduces the expected utility of promoting players. While in GSPa(n,2) this term equals $\frac{1}{2n}$ in the worse case (all players bid the same in the phase *1* and both of the players bid the same in phase *2*), in GSPa(n,m) the worse case brings it down to $\frac{1}{n} \cdot \frac{1}{m!}$ (a player wins promotion to the phase m while in every phase all the remaining players bid the same as she does).

Claim 13: *The strategy* $[s_1(v) = \alpha_1 v, s_2(v) = (\alpha_2 - \alpha_1)v + b^1, .., s_m(v) = (\alpha_m - \alpha_{m-1})v + b^{m-1}]$ *is not a dominant strategy* $\forall v_i = v \in V$.

Proof: Consider the following strategies of other players: in phase *1*, *m-2* players bid above $\alpha_1(v-d)$, one player bids $\alpha_1(v-d)$ and *n-m+1* players bid below $\alpha_1(v-d)$. Hence, $b^1 = \alpha_1(v-d)$; in phase *2*, all promoting players bid $\alpha_2 v - \alpha_1 d$. Hence, $b^2 = \alpha_2 v - \alpha_1 d$; In phase *3* all remaining players bid $\alpha_3(v+d) - \alpha_2 d - \alpha_1 d$. Hence, $b^3 = \alpha_3(v+d) - \alpha_2 d - \alpha_1 d$. In phases *4-m* all remaining players bid according to the strategy. We separate to 3 cases:

1. i *bids according to the strategy*: $b_1 = \alpha_1 v$, $b_2 = (\alpha_2 - \alpha_1)v + b^1 = \alpha_2 v - \alpha_1 d$ and $b_3 = (\alpha_3 - \alpha_2)v + b^2 = \alpha_3 v - \alpha_1 d$. i reaches phase *3* with probability of $1 \cdot \frac{1}{m}$ and wins slot *2*. $u = \frac{1}{m}(\alpha_2 v - \alpha_2 v + \alpha_1 d) = \frac{1}{m}\alpha_1 d$.
2. *At some phase* i *bids higher than the strategy*: $b_1 = \alpha_1 v$, $b_2 = \alpha_2(v+d) - 2\alpha_1 d$ and $b_3 = \alpha_3 v - \alpha_1 d$ i reaches phase *3* with probability of *1* and wins slot 2. $u = \alpha_2 v - \alpha_2 v + \alpha_1 d = \alpha_1 d$ - higher than in the first case.
3. *At some phase* i *bids lower than the strategy*: $b_1 = \alpha_1 v$, $b_2 = \alpha_2(v-d)$. i reaches the phase *2* with probability of *1* and wins slot *1*. $u = \alpha_1 v - \alpha_1 v + \alpha_1 d = \alpha_1 d$ - higher than in the first case. □

Claim 14: *In the adjusted Discrete GSPa(n,m), all strategies survive Iterated Deletion of Dominated Strategies, except* $[s_1(0) = \alpha_1 v, s_2(0) = (\alpha_2 - \alpha_1)v + b^1, .., s_m(0) = (\alpha_m - \alpha_{m-1})v + b^{m-1}]$.

Proof: First, bidding $[b'_1 = \alpha_1, b'_2 = (\alpha_2 - \alpha_1) + b^1, .., b'_m = (\alpha_m - \alpha_{m-1}) + b^{m-1}]$ is strictly dominated by $[b_1 = b_2 = ... = b_m = 0]$ for $v = 0$. For $b^l = 0$, $\forall l \in (1..m)$, $u = \frac{1}{2}\prod_1^{m-1} \frac{1}{k_l+1}(\alpha_m v - b^m + \epsilon^m + \alpha_{m-1} v - b^{m-1} + \epsilon^{m-1}) > u' = 0$; for $b^l > 0$, $l \in (1..m)$, $u = 0 > u'$.

Next, all other bids $(b'_l < maxB_l,\ l \in (1..m))$ *are not* dominated by $[b_1 = b_2 = ... = b_m = 0]$: If $b^l > 0$, both bids yield *0*. Hence, $[b_1 = b_2 = ... = b_m = 0]$ *is not* dominated too.

Now, none of the remaining possible bids is dominated for $v = 0$. For $b_l > 0$, $b'_l > 0$ and $b^l = 0\ \forall l \in (1..m)$, both bids yield *0*. For $b_l > 0$, $b'_l = 0$, $l \in (1..m)$ and $b^l = 0\ \forall l \in (1..m)$, u might be higher that u', but for $b^l > 0$, $u' > u$.

Finally, $\forall v > 0$, no bids are dominated: due to the $\prod_1^m \frac{1}{k_l+1}$ term, for any *2* bids there might be *2* types of outcome for which one of them $u > u'$ and for other $u' > u$. If both bids promote win the m^{th} slot, the term defines the expected payoff; If a player drops with bidding b_l, her utility might be higher than bidding b'_l if from the droping phase $b'_l = b^l$; And it mighr be lower, otherwise. □.

Note that the proof doesn't use P which would be defined similarly to GSPa(n,m).

5.2 Discrete VCG Mechanism

General Description

Consider a VCG mechanism with n players and m alternatives. Each player $i \in (1, 2...n)$ has her own set of private values $\{v_{i,a}\}_{a=1}^m$ where $a \in (1, 2...m)$ is the index of an alternative. Each player submits her bids for each of the alternatives - $b_{i,(1..m)}$. The values and bids are on a discrete grid: $V = B = (0, d, 2d...1 - d, 1)$. A function from i's possible values, V, into the possible bids, B is $s_{i,(1..m)} \in S_{i,(1..m)}$. For each alternative a *social welfare* (sw) is calculated: $sw_a = \sum_{i=1}^n b_{i,a}$. The alternative with the $max_a(sw)$ is chosen. In case of a tie between alternatives with highest sw, each is chosen with equal probability s. The payment of each player for the chosen alternative(s) are according to the VCS prices: for every alternative player i pays the *social harm* she caused others with her bidding:

$$c_{i,(1..m)}[v_{i,a}, s_{i,(1..m)}, s_{-i,(1..m)}] = max_{a,-i}(sw) - s[\sum_{a:max(sw)=sw_a} sw_{a,-i}]$$

where $max(sw)$ is the maximum social welfare over all alterntives, $sw_{a,-i} = sw_a - b_{a,j}$ is the social welfare of alternative a when excluding player i's bid and $max_{a,-i}(sw)$ is the maximum social welfare when excluding i's bids. Player i's Utility function is then:

$$u_i[v_{i,(1..m)}, s_{i,(1..m)}, s_{-i,(1..m)}] = s \sum_{a:max(sw)=sw_a} [v_{i,a} + sw_{a,-i}] - max_{a,-i}(sw)$$

Claim 15: *The VCG mechanism described is truthful* $\forall v_i = v \in V$ *and* $\forall\, s_{-i} \in S_{-i}$.

Proof: We need to show the dominance of truthful bidding for player i, i.e.:

$$u_i[v_{i,(1..m)}, b_{i,1} = v_{i,1}, b_{i,2} = v_{i,2}, ..., b_{i,m} = v_{i,m}, s_{-i,(1..m)}] \geq \\ u'_i[v_{i,(1..m)}, b'_{i,1}, b'_{i,2}, ..., b'_{i,m}, s_{-i,(1..m)}]$$

$\forall v_i = v \in V$ and $\forall\, s_{-i} \in S_{-i}$ where at least one of the b''s is not truthful.

The prove is similar to this of regular VCG mechanism. [7]

$u_i = s \sum_{a:max(sw)=sw_a} [v_{i,a} + sw_{a,-i}] - max_{a,-i}(sw)$

$u'_i = s' \sum_{a':max'(sw)=sw'_a} [v'_{i,a'} + sw'_{a',-i}] - max_{a',-i}(sw)$.

Since other players' bids are fixed, $max_{a,-i}(sw)$ is the same in both cases. Therefore, it is sufficient to show that:

$$\frac{1}{k} \sum_{a:max(sw)=sw_a} [v_{i,a} + sw_{a,-i}] \geq \frac{1}{k'} \sum_{a':max'(sw)=sw'_{a'}} [v'_{i,a'} + sw'_{a',-i}]$$

The left side equals *max(sw)* when bidding true values. The right side is at most *max(sw)* since it's a linear combination of *social welfare* of alternatives for which $s' \cdot sw'_{a'} \leq s' \cdot max(sw)$. □.

[7] N.Nissan, Introduction to Mechanism Design for Computer Scientists, p.16

5.2.1 Iterated Deletion of Dominated Strategies

Claim 6: *In VCG mechanism described all strategies survive the Iterated Deletion of Dominated Strategies.*

Proof: Since the VCG mechanism is truthful it is sufficient to show that there is no strategy for player i that is *strictly dominated* by truthful bidding and so the P-deletion process with technique described, cannot be strated. [5] This is indeed the case since for every outcome of the auction in which the difference between two highest sw, when excluding player i's bidding is higher then $max\{v_{i,a}\}_{a=1}^{m}$, i's payoff will remain the same, simply because her bidding has no direct influence, but rather a marginal one on the chosen alternatives by the VCG mechanism. □.

Note that in VCG(n,m) bidding according to the strategy with a maximum value possible does not assure an influence on the auction's outcome. This is why applying similar beliefs P as in GSPa(n,2) will not resolve the issue. Moreover, the ability of player i to cause a tie between alternatives with her bidding is limited as well - due to the assignment methodology and the large number of players. Thus, reducing the payments in the form of ϵ doesn't start the Deletion process.

[5] if a dominant strategy doesn't delete any other strategy, no strategy is deleted

Chapter 6

Conclusions

In our study we implement the Iterated Deletion of P-dominated Strategies in Discrete Position auctions. We setup a Genearlized Second Price auction with n players and *2* slots with private values that corresponds to Generalized English auction. We make several important adjustments in the borders bids' grids, players' beliefs and the payment rule. We then run the P-deletion process on players' strategies and show the existence of a unique outcome. We end with investigating the outcomes of the process in Discrete Generalized Second Price auction with m slots and Discrete VCG mechanism. The summary of our main results is as follows:

- Adjusting bids' grids in Discrete GSPa(n,2) provides the necessary conditions for the equilibrium strategy $s_1(v) = \alpha_1 v$, $s_2(v) = (\alpha_2 - \alpha_1)v + c^1$ to be a dominant strategy.
- In the adjusted Discrete GSPa(n,2) the equilibrium strategy $s_1(v) = \alpha_1 v$, $s_2(v) = (\alpha_2 - \alpha_1)v + c^1$ is a unique outcome of the P-deletion process $\forall v$,v$\in V$. The dominance feature leads to this outcome.
- In the adjusted Discrete GSPa(n,m), the equilibrium strategy is not dominant and only one strategy for $v=0$ is deleted: $[s_1(0) = \alpha_1 v, s_2(0) = (\alpha_2 - \alpha_1)v + b^1, .., s_m(0) = (\alpha_m - \alpha_{m-1})v + b^{m-1}]$.
- In Discrete Generalized VCG mechanism, all strategies survive the deletion of dominated strategies, despite the existence of a dominant strategy (declaring true values). Similar adjustments to those made for *2* slots do not change the result for the P-deletion process.

Our main result contributes to the practice of investigating auctions through outcomes of the Deletion process. Reaching a unique outcome strengthtens the Nash equilibrium concept. However, if a unique outcome exists, it is not necessarily simple for the participating players to realize it. This observation is strengthened by Dekel and Wolinsky result: bidding slightly below one's value in Discrete First Price auction is a logical result; proving it, however is not simple when expecting a rational player to reach this conclusion by herself. In our case, significant adjustments are made, including adjusting the payment rule which deviates from the original GSPa's payment rule in case of a tie between bids. And yet, despite the existence of a dominant strategy, the proof is not trivial. Moreover, dominant strategies do not assure that any strategy can be deleted. Researching for less complex setups will result in a lighter demand for analytical skills from the players and lead to the desired outcome during the auction more easily. One of the future adjustments that might reduce the need in other modifications - α's - the expected number of clicks per period. The auctioneer can influence it by for example ephasizing the text of the ads placed in a specific location.

We use the dominant strategy as a leverage for running the Deletion process and show that this strategy is its unique outcome. The dominance feature, together with the adjustments made provide the conditions for strict dominance in some cases allowing the start of the whole process. Further research might investigate general discrete and continuous auctions with dominant strategies and the minimal adjustments required for deleting strategies, even when the outcome is not unique. The path to generalization might begin from a special case, such as VCG mechanism with *2* players and m alternatives where every bid of each of the players has a high probability to influence on the auction's outcome.

Another possible use of the Deletion process is in auctions with no dominant strategy, such as Dekel and Wolinsky's Discrete First Price auction. Although the authors use strict dominance feature to start the process, there is no equilibrium in dominant strategies. Defining the conditions required to delete strategies in such auctions without first leading to a dominant strategy is interesting, for GEa in particular.

Chapter 7

Bibliography

1. Ashlagi I., Braverman M., Hassidim A., Lavi R., Tennenholtz M. (2010), *Position Auctions with Budgets: Existence and Uniqueness*, The B.E. Journal of Theoretical Economics (2010): Volume 10, Issue 1, Article 20.

2. Dekel E., Wolinsky A. (2001), *Rationalizable Outcomes of Large Private-Value First-Price Discrete Auctions*, Games and Economic Behavior 43 (2003): 175-188.

3. Edelman B., Ostrowsky M., Schwartz M. (2005), *Internet Advertising and the Generalized Second Price Auction: Selling Billions of Dollars Worth of Keywords*, National Bureau of Economic Research, Working Paper 11765.

4. Nisan N., *Introduction to Mechanism Design (for Computer Scientists)*:15-16.

5. Osborn M.J., Rubinstein A. (1994), *Rationalizability and Iterated Elimination of Dominated Actions*, A Course in Game Theory (1994):53-66.