A Polynomial Time Theory of Integer Programming

Shmuel Onn

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Based on several papers joint with several co-authors including
De Loera, Hemmecke, Lee, Romanchuk, Rothblum, Weismantel

(Non)-Linear Integer Programming

The problem is: $\min/\max \{ f(x) \ : \ Ax \leq b, \ \ l \leq x \leq u, \ \ x \text{ in } Z^n \}$

with data: A: integer $m \times n$ matrix

b: right-hand side in Z^m

l,u: lower/upper bounds in Z^n

f: function from Z^n to R

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Our theory enables polynomial time solution of broad natural universal (non)-linear integer programs in variable dimension

Outline

Overview our theory of Graver bases for integer programming

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Drastically better complexity and fixed-parameter tractability

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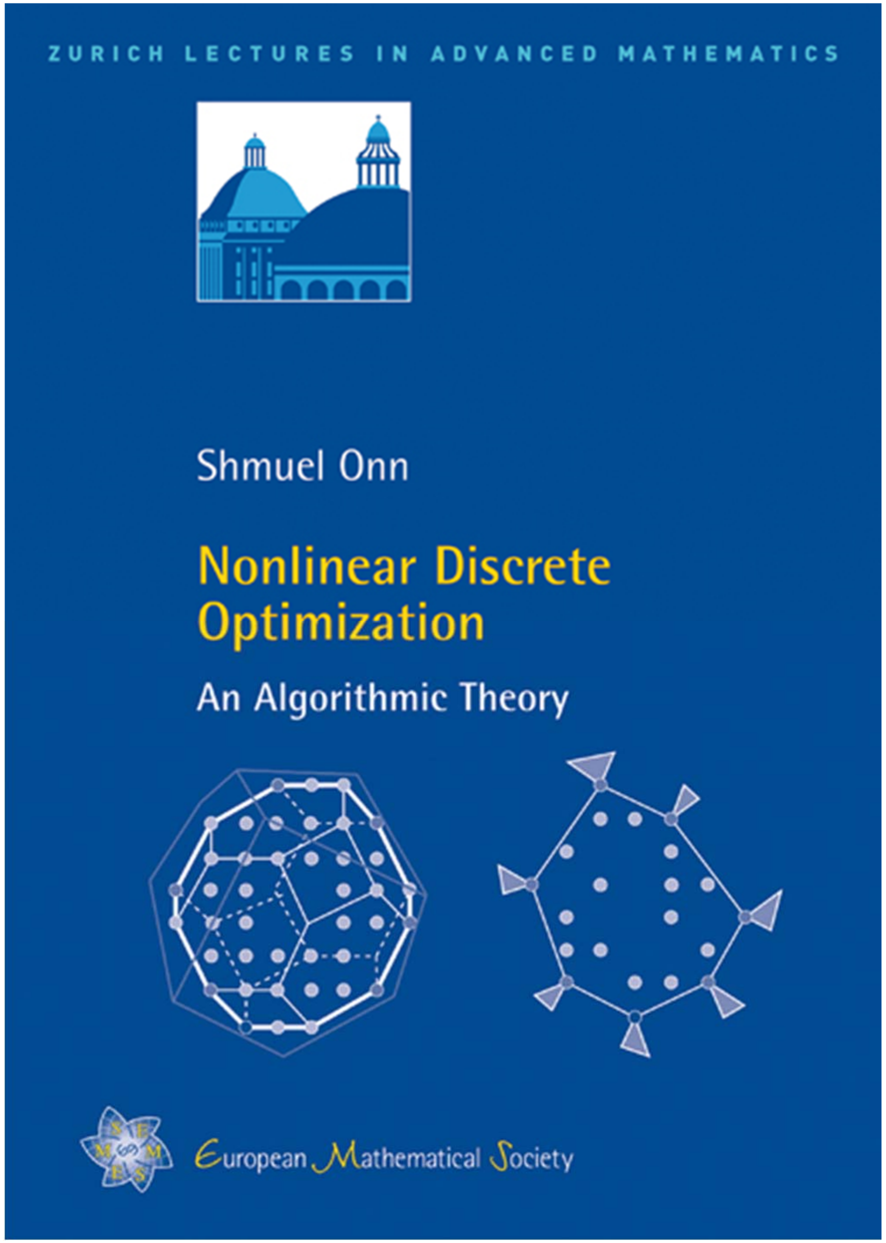
Drastically better complexity and fixed-parameter tractability

Applications to multiway tables and huge multicommodity flows

Overview: Graver Bases

and

Nonlinear Integer Programming



Background in my Book:

Theory of Graver bases for integer programming

(and more)

Available electronically from my homepage

(with kind permission of EMS)

Graver Bases

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non-circuits: $\pm(1\ {-1}\ 1)$

Some Theorems on
(Non)-Linear Integer Programming

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Some Theorems on (Non)-Linear Integer Programming

Theorem 1: Linear optimization in polytime with G(A):

$$\max \{ wx \ : \ Ax = b, \ \ l \le x \le u, \ \ x \text{ in } Z^n \}$$

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Reference: N-fold integer programming, (De Loera, Hemmecke, Onn, Weismantel)
Discrete Optimization (Volume in memory of George Dantzig)

Some Theorems on (Non)-Linear Integer Programming

Theorem 2: **Multicriteria maximization** in polytime with **G(A)**:

$$\max \{f(Wx) : Ax = b, \ l \le x \le u, \ x \text{ in } Z^n\}$$

where W is d x n criteria matrix and f convex function on Z^d which balances d linear criteria or player utilities $W_i x$

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Some Theorems on (Non)-Linear Integer Programming

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Reference: Convex integer maximization via Graver bases, (De Loera, Hemmecke, Onn, Rothblum, Weismantel), J. Pure & Applied Algebra

Some Theorems on
(Non)-Linear Integer Programming

Theorem 3: Separable convex minimization in polytime with $G(A)$:

$$\min\{\sum f_i(x_i) \ : \ Ax = b, \ \ l \le x \le u, \ \ x \text{ in } Z^n\}$$

Some Theorems on
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Reference: A polynomial oracle-time algorithm for convex integer minimization, (Hemmecke, Onn, Weismantel), Mathematical Programming

Some Theorems on (Non)-Linear Integer Programming

Theorem 4: **Integer point closest** to x in **polytime** with **G(A)**:

$$\min \{ |x - x|_p \; : \; Ax = b, \;\; l \le x \le u, \;\; x \in Z^n \}$$

Reference: *A polynomial oracle-time algorithm for convex integer minimization*, (Hemmecke, Onn, Weismantel), Mathematical Programming

Some Theorems on
(Non)-Linear Integer Programming

Theorem 5: Quadratic minimization in **polytime** with **$G(A)$**:

$$\min\{x^T V x \ :\ Ax = b,\ \ l \le x \le u,\ \ x \text{ in } Z^n\}$$

where V lies in cone $K_2(A)$ of possibly indefinite matrices, enabling minimization of some convex and some non-convex quadratics

Some Theorems on
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Reference: Quadratic Graver cones, quadratic integer minimization & extensions, (Lee, Onn, Romanchuk, Weismantel), Mathematical Programming

Some Theorems on
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Theorem 6: Polynomial minimization in polytime with **G(A)**:

$$\min \{p(x) \ : \ Ax = b, \ \ l \leq x \leq u, \ \ x \text{ in } Z^n\}$$

where p is possibly indefinite polynomial of degree d in cone $K_d(A)$, enabling minimization of some (non)-convex degree d polynomials

Reference: Quadratic Graver cones, quadratic integer minimization & extensions, (Lee, Onn, Romanchuk, Weismantel), Mathematical Programming

Some Theorems on
(Non)-Linear Integer Programming

Theorem 7: Robust optimization in polytime with $G(A)$:

$$\min\{\ \max\{\ cx : d \le c \le e\ \}\ :\ Ax = b,\ \ l \le x \le u,\ \ x \text{ in } Z^n\ \}$$

that is, minimum worst case cost where the cost of each variable can vary in an interval

Some Theorems on
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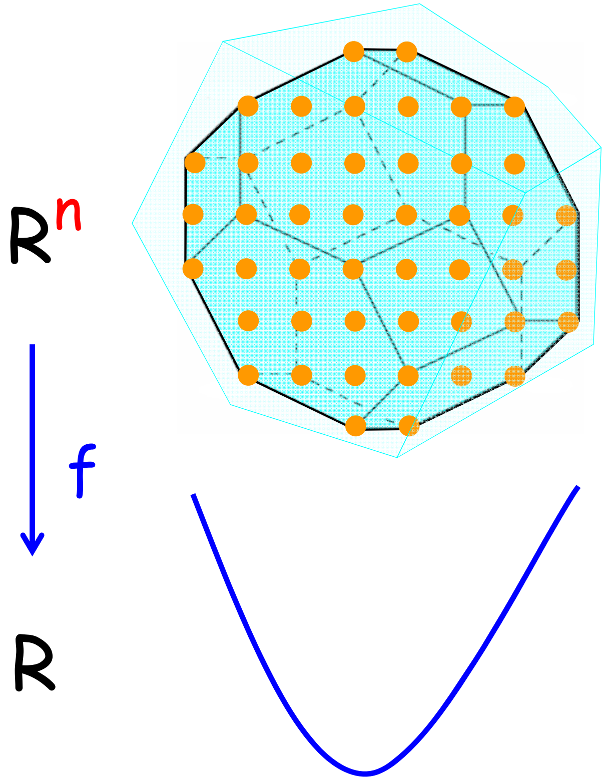
Reference: Robust integer programming, (Onn), Operations Research Letters

Some Proofs

Proof of Theorem 3
(separable convex minimization)



To solve $\min\{\sum f_i(x_i) : Ax = b,\ l \le x \le u,\ x \in Z^n\}$

with the Graver basis $G(A)$

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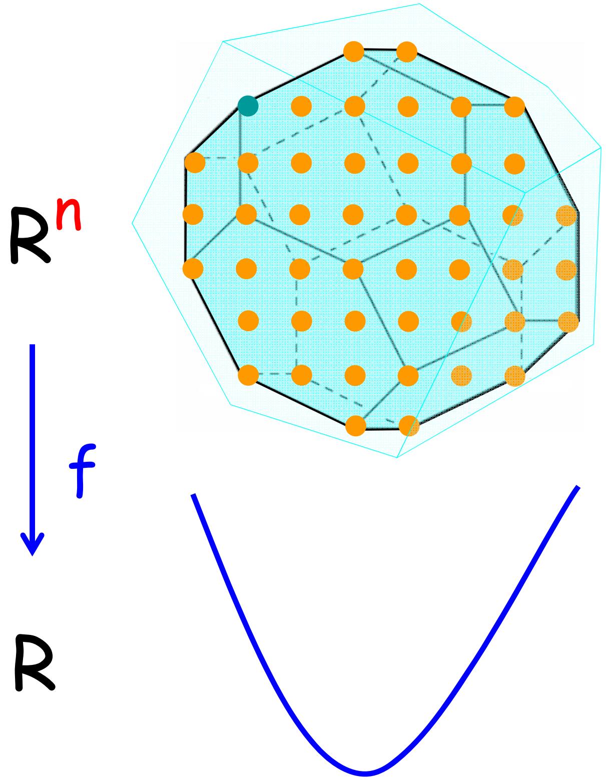

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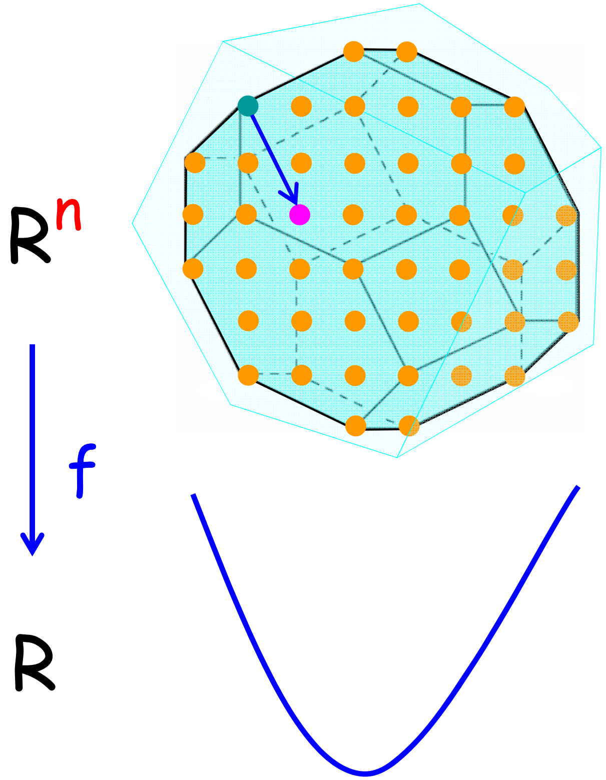

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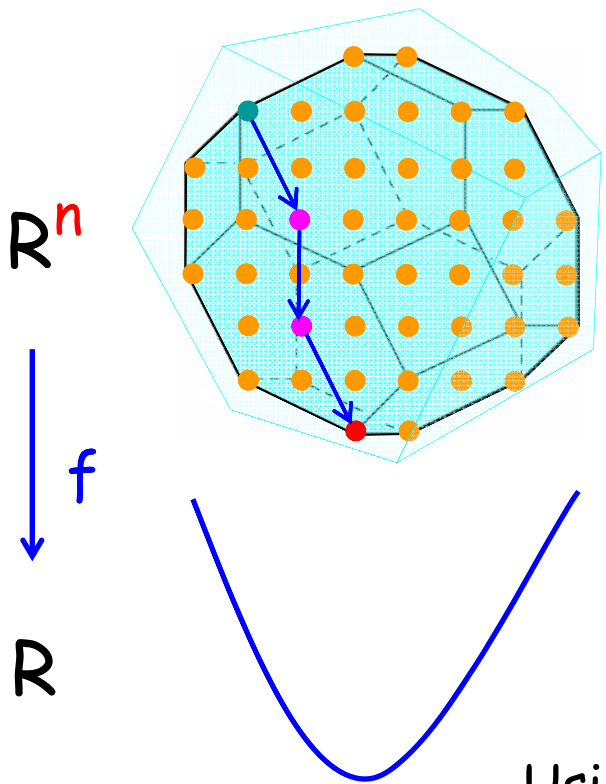

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Using supermodality of f and integer Caratheodory theorem (Cook-Fonlupt-Schrijver, Sebo) we can show polytime convergence to some optimal solution

Proof of Theorem 2
(multicriteria maximization)

Lemma: Can solve in polytime max { $f(Wx)$: x in S } with S in Z^n if can do linear optimization over S and have set E of all edge-directions of conv(S)

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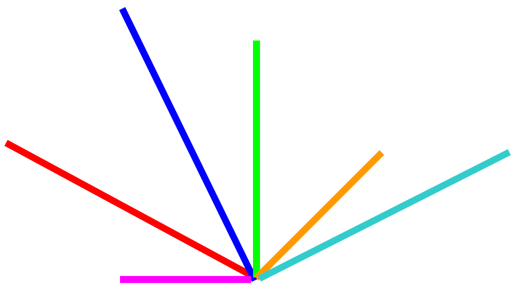

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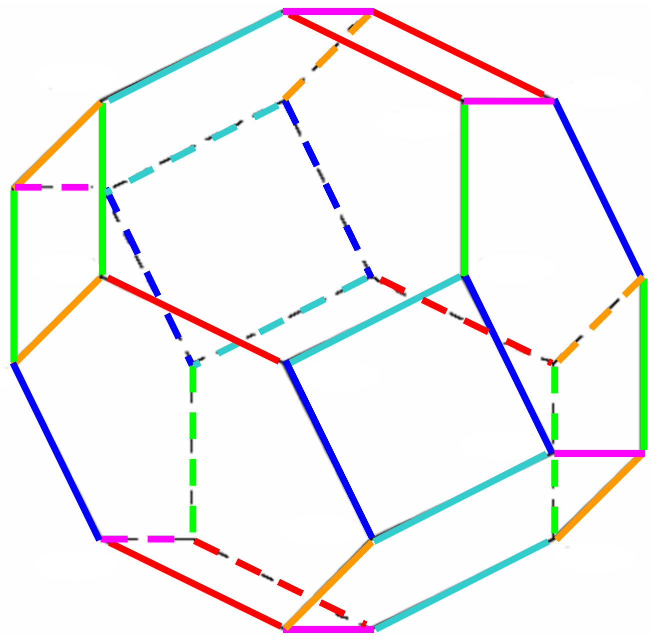

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Reference: Convex combinatorial optimization, (Onn, Rothblum), Journal of Discrete and Computational Geometry

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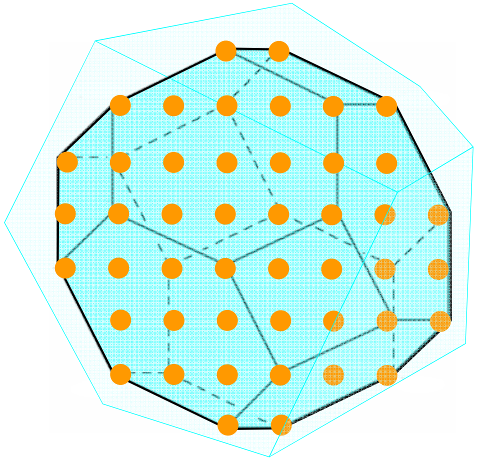

To solve $\max\{ f(Wx) \ : \ x \text{ in } S \}$

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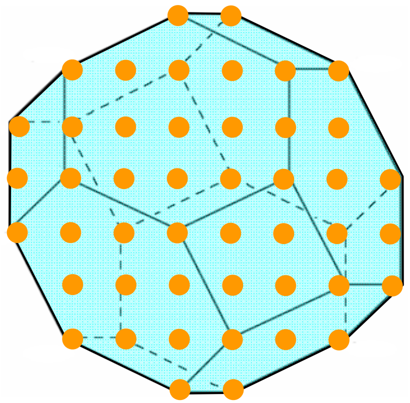

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1. Use $G(A)$ to simulate linear-optimization oracle over S via Theorem 1

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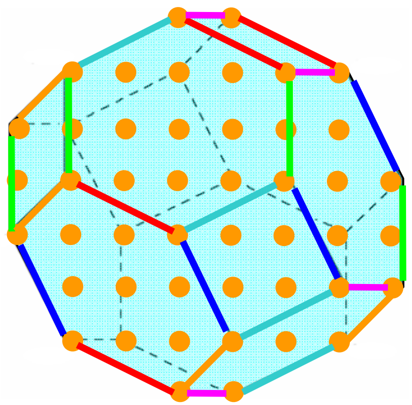

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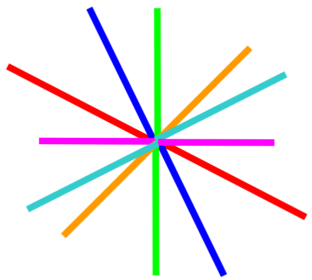

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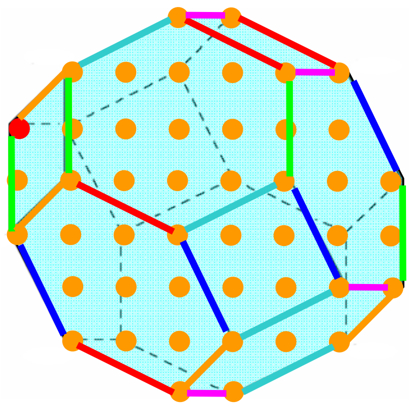

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2. Use the Graver basis as set $E := G(A)$ of all edge-directions of conv(S)

3. Apply the Lemma (use 1 for each vertex of zone($WG(A)$)) and pick best)

N-Fold Integer Programming

N-Fold Products

The n-fold product of an $(r,s) \times t$ bimatrix $A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$ is the $(r+ns) \times nt$ matrix

$$A^{(n)} = \underbrace{\begin{pmatrix} A_1 & A_1 & A_1 & \cdots & A_1 \\ A_2 & 0 & 0 & \cdots & 0 \\ 0 & A_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & A_2 \end{pmatrix}}_{n} .$$

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Lemma: For fixed A we can compute the Graver basis $G(A^{(n)})$ in polynomial time $O(n^{g(A)})$ with $g(A)$ the Graver complexity of A.

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Proof very rough idea: For any n, any element $x=(x^1,\ldots,x^n)$ in the Graver basis $G(A^{(n)})$ has at most $g(A)$ nonzero bricks x^k in Z^t.

(Non)-Linear N-Fold Integer Programming

Theorem: for various f can solve in polynomial time $O(n^{g(A)}L)$:

$$\min\{f(x) : A^{(n)}x = b,\ l \le x \le u,\ x \in Z^{nt}\}$$

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References: see Nonlinear Discrete Optimization, (Onn),
Zurich Lectures in Advanced Mathematics, European Mathematical Society

Drastically Better Complexity

and

Fixed-Parameter Tractability

Cubic Running Time and Fixed-Parameter Tractability

Reference: N-fold integer programming in cubic time,
(Hemmecke, Onn, Romanchuk), Mathematical Programming

Cubic Running Time and Fixed-Parameter Tractability

Theorem: For any fixed bimatrix A, the following linear n-fold integer program is solvable in fixed-parameter time $O(n^3 L)$:

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Proof very rough idea: in the iterative algorithm, at each iteration, can find a Graver-best step without computing the entire Graver basis.

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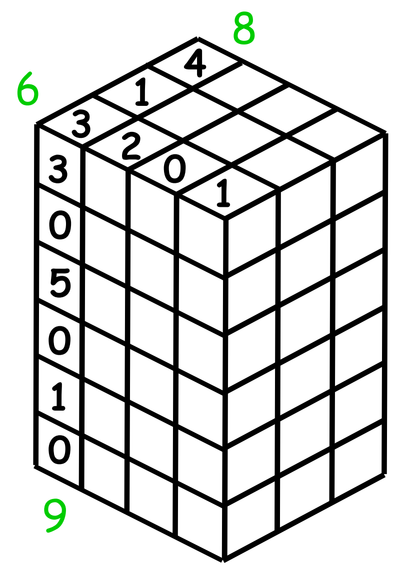
Some Applications

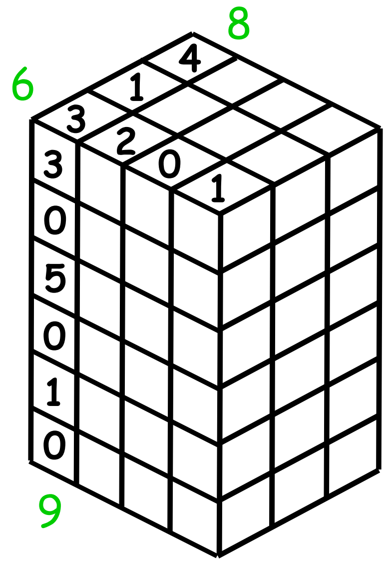
Multiway Tables

Complexity of deciding the existence of

$l \times m \times n$ tables with given line sums:



Multiway Tables



Complexity of deciding the existence of

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- l, m, n variable:

- l fixed, m, n variable:

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- l, m, n fixed:

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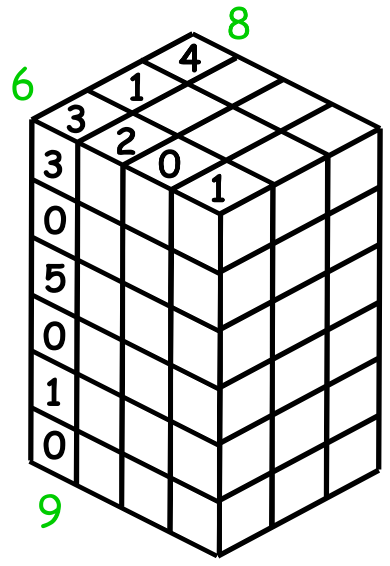

- l, m, n variable: NP-complete

  Three dimensional matching, Karp, 1972

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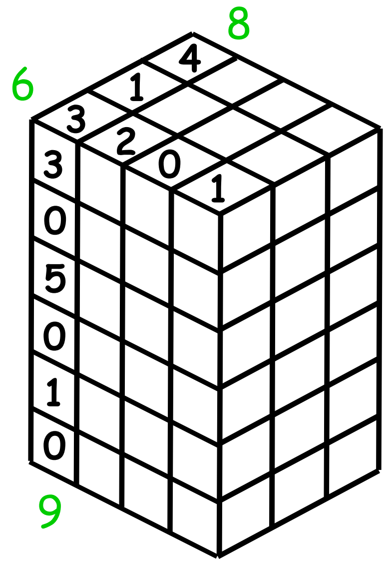

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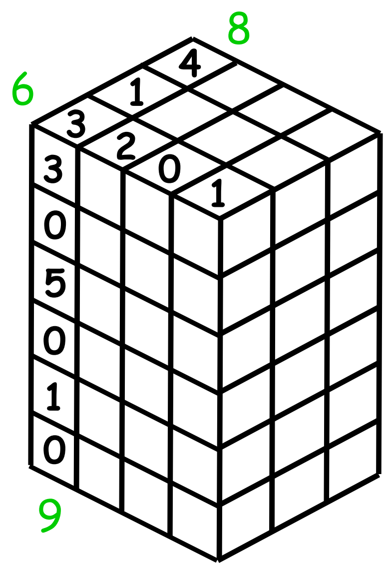
- l, m, n fixed: Polytime

  Integer programming in fixed dimension, Lenstra, 1982

Multiway Tables



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- l fixed, m, n variable: Universal for IP (even with $l=3$)

  De Loera, Onn, 2006

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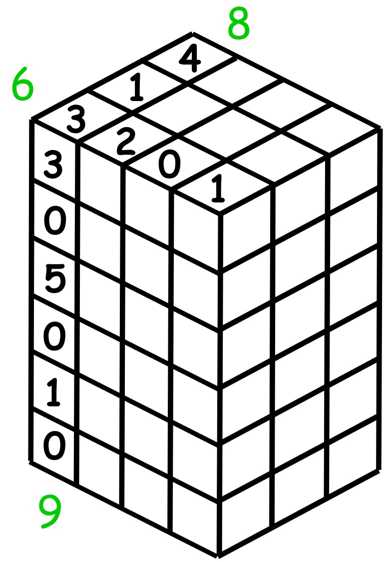

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- l, m fixed, n variable: Polytime

  Consequence of linear n-fold IP, De Loera, Hemmecke, Onn, Weismantel, 2008

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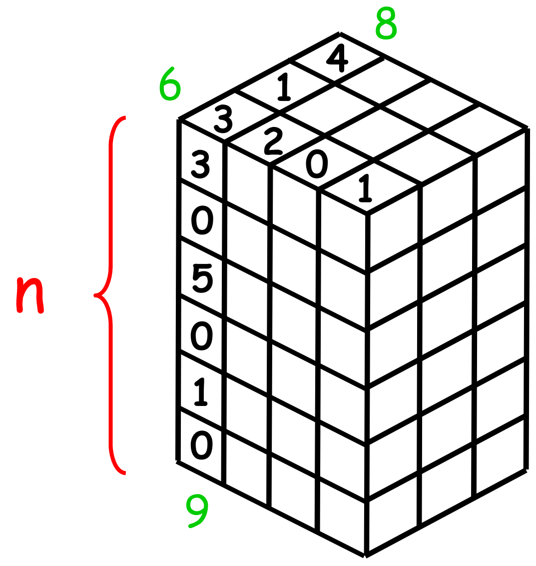
Multiway Tables

Much more generally, consider the multi-index transportation problem studied by Motzkin in 1952, of minimization over $m_1 \times \cdots \times m_k \times n$ tables with given margins:



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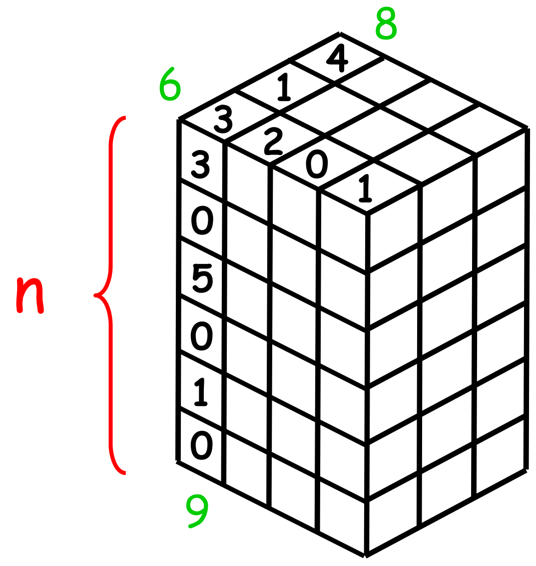

It is an n-fold program

$$\min\{f(x) \;:\; A^{(n)}x = b,\;\; x \geq 0,\;\; x \text{ integer}\}$$

$$A^{(n)} = \underbrace{\begin{pmatrix} A_1 & A_1 & A_1 & \cdots & A_1 \\ A_2 & 0 & 0 & \cdots & 0 \\ 0 & A_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & A_2 \end{pmatrix}}_{n}$$

for suitable A depending on $m_1, \ldots, m_k$ where:

- A_1 gives equations of margins summing over layers
- A_2 gives equations of margins summing within a single layer at a time

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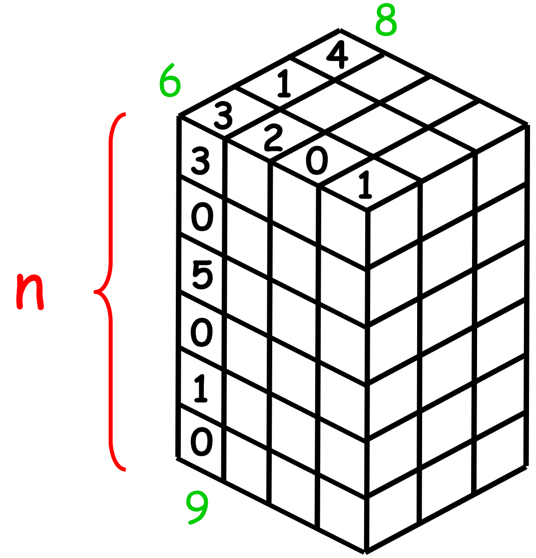

Corollary: (Non)-linear optimization over $m_1 \times \cdots \times m_k \times n$ tables with given margins can be done in polynomial time $O(n^{g(m_1,\ldots,m_k)} L)$

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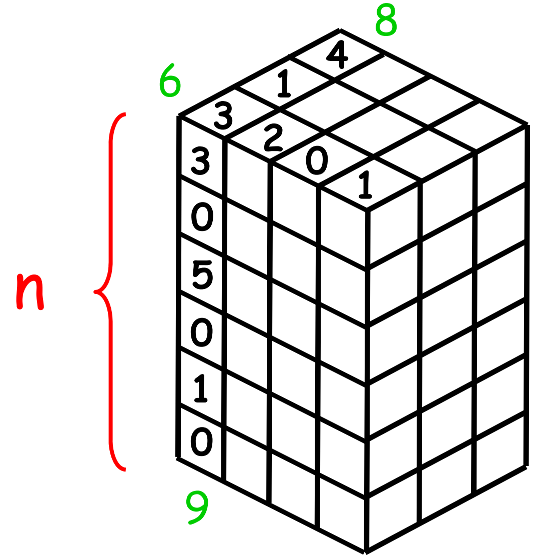
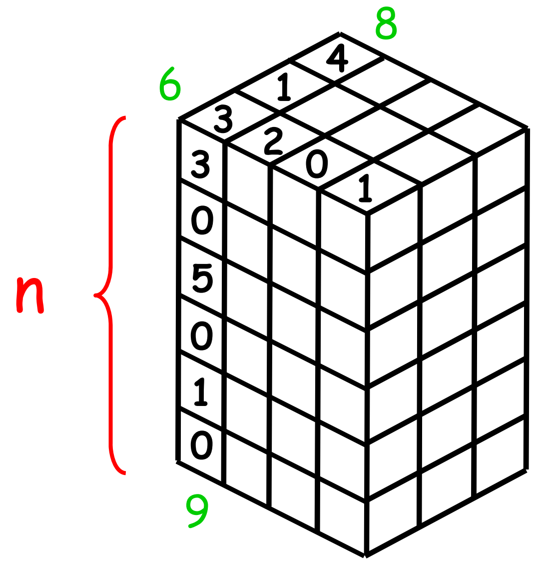


Corollary: (Non)-linear optimization over $m_1 \times \cdots \times m_k \times n$ tables with given margins can be done in polynomial time $O(n^{g(m_1,\ldots,m_k)} L)$

In contrast: **Universality of three-way tables** (De Loera, Onn):
Every integer program is one over $3 \times m \times n$ tables with given line-sums

Multiway Tables

Much more generally, consider the multi-index transportation problem studied by Motzkin in 1952, of minimization over $m_1 \times \cdots \times m_k \times n$ tables with given margins:

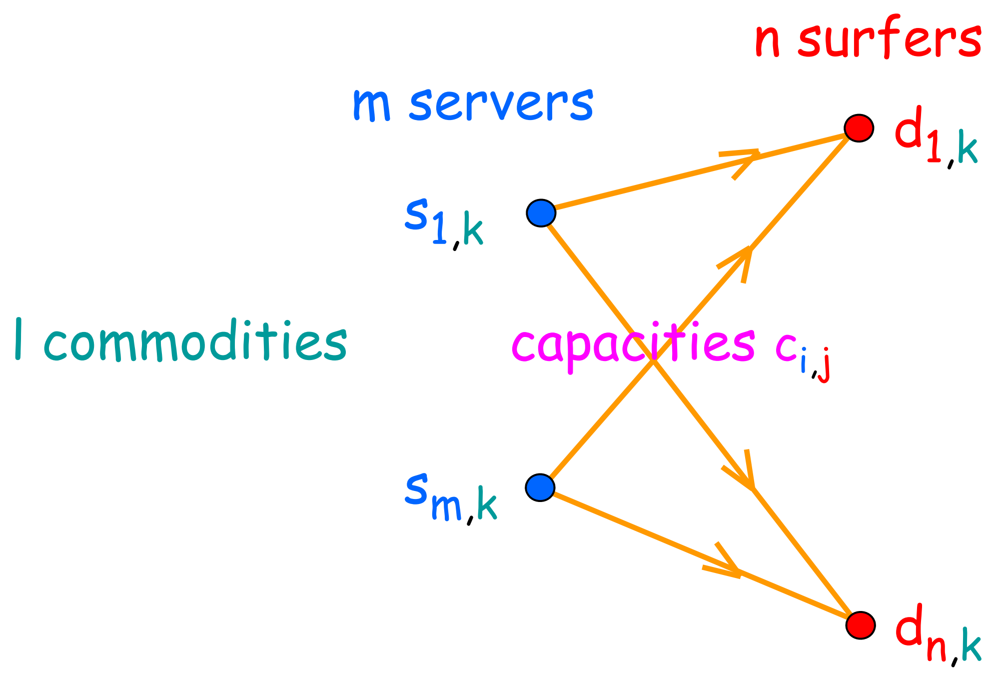


Corollary: (Non)-linear optimization over $m_1 \times \cdots \times m_k \times n$ tables with given margins can be done in polynomial time $O(n^{g(m_1,\ldots,m_k)} L)$

Better: (Non)-linear optimization over $m_1 \times \cdots \times m_k \times n$ tables with given margins can be done in fixed-parameter cubic time $O(n^3 L)$

Multicommodity Flows

Find flow of l commodities from m servers to n surfers satisfying given supplies $s_{i,k}$, demands $d_{j,k}$ and capacities $c_{i,j}$ of total bit size L



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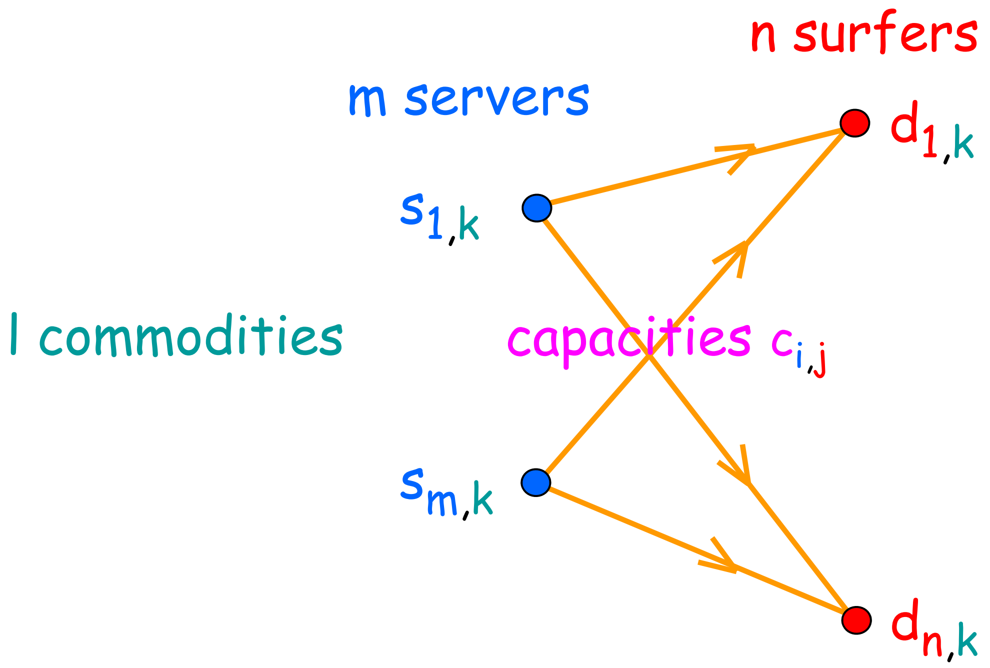

With l=2 or m=3 it is NP-complete so assume both l,m are parameters

Multicommodity Flows

Find flow of l commodities from m servers to n surfers satisfying given supplies $s_{i,k}$, demands $d_{j,k}$ and capacities $c_{i,j}$ of total bit size L

2008: polynomial time $O(n^{g(l,m)} L)$ with Graver complexity $g(l,m)$ exponential in l,m (De Loera, Hemmecke, Onn, Weismantel) (theory of n-fold IP)

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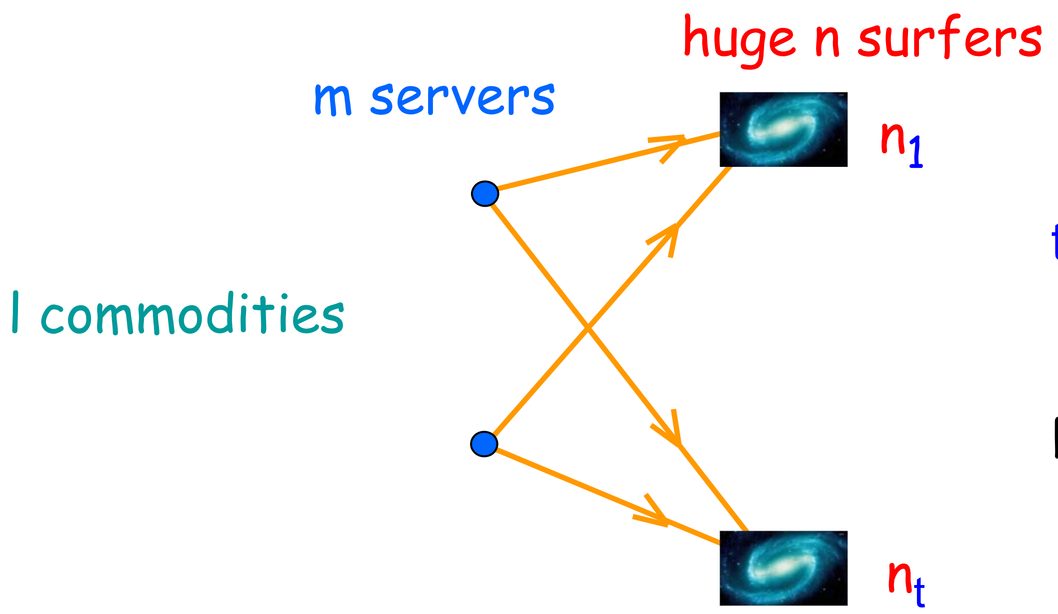
Open: algorithm that is both fixed-parameter tractable and strongly polynomial ?

Multicommodity Flows

Huge version: surfers come in huge clouds of t types



t surfer types

$n_1 + \ldots + n_t = n$

binary encoded

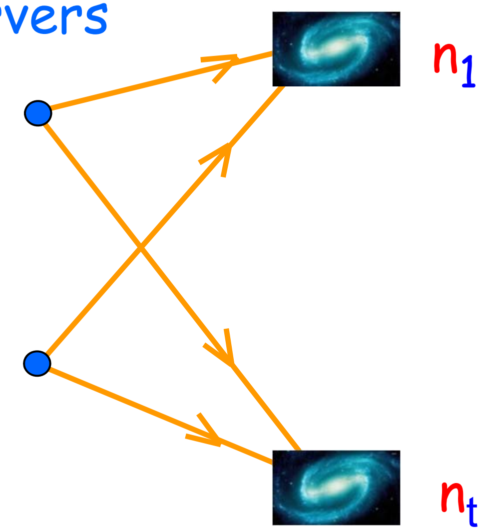
Multicommodity Flows

Huge version: surfers come in huge clouds of t types

huge n surfers

m servers

l commodities



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2016 (Onn): fixed-parameter tractable with parameters l,t, variable m, huge n

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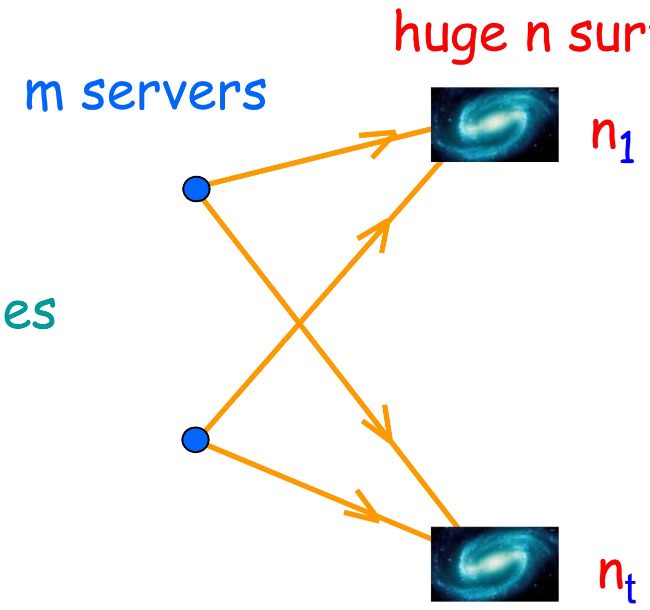

l commodities

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binary encoded

2016 (Onn): fixed-parameter tractable with parameters l,t, variable m, huge n

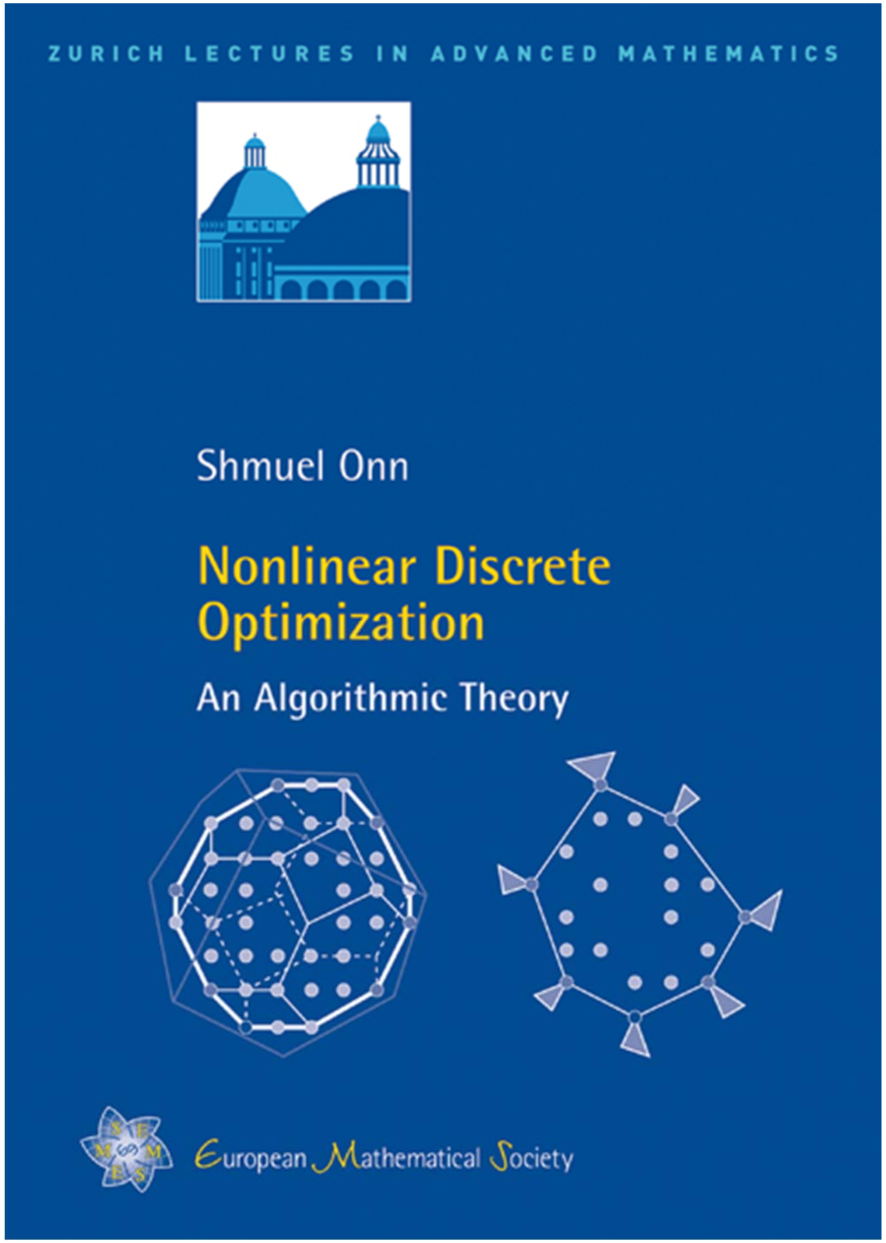
Open: 4-dimensional huge tables are only known to be in NP intersect coNP

Some Further Developments in Theory and Applications

- Clustering and farmland consolidation, Borgwardt, Melamed, Onn
- Scheduling, Knop, Koutecky
- Stochastic integer programming, Hemmecke, Onn, Weismantel
- Portfolio optimization, Baumann, Trautmann
- Optimality certificates, Kobayashi, Murota, Saito, Weismantel
- Production scheduling, Andziulis, Dzemydien
- Block structured integer programs, Hemmecke, Köppe, Weismantel
- Matrix apportionment problems, Gaffke, Pukelsheim
- Strongly polynomial algorithms, De Loera, Hemmecke, Lee
- Rounding in integer nonlinear optimization, Hubner, Schobel
- Graver complexity, Berstein, Finhold, Hemmecke, Kudo, Nairn, Onn, Takemura
- Network design problems, Guyard, Laugier
- Games, Hemmecke, Nguyen, Onn, Ryan, Weismantel

Bibliography (mostly available at http://ie.technion.ac.il/~onn)

- The complexity of 3-way tables (SIAM J. Comp.)
- Convex combinatorial optimization (Disc. Comp. Geom.)
- Markov bases of 3-way tables (J. Symb. Comp.)
- All linear and integer programs are slim 3-way programs (SIAM J. Opt.)
- Graver complexity of integer programming (Annals Combin.)
- N-fold integer programming (Disc. Opt. in memory of Dantzig)
- Convex integer maximization via Graver bases (J. Pure App. Algebra)
- Polynomial oracle-time convex integer minimization (Math. Prog.)
- Theory and applications of n-fold integer programming (IMA Volume on MINLP)
- Quadratic Graver cones, quadratic integer minimization & extensions (Math. Prog.)
- Robust integer programming (Operations Research Letters)
- N-fold integer programming in cubic time (Math. Prog.)
- Huge tables and multicommodity flows are fixed-parameter tractable via unimodular integer Caratheodory (J. Computer and System Sciences)



Background in my Book:

Theory of Graver bases
for integer programming

(and more)

Available electronically
from my homepage

(with kind permission of EMS)