A Polynomial Time Theory of Integer Programming

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Based on several papers joint with several co-authors including De Loera, Hemmecke, Lee, Romanchuk, Rothblum, Weismantel
(Non)-Linear Integer Programming

The problem is: \[ \min/\max \{ f(x) : Ax \leq b, \ l \leq x \leq u, \ x \in \mathbb{Z}^n \} \]

with data: \( A: \) integer \( m \times n \) matrix \quad \( b: \) right-hand side in \( \mathbb{Z}^m \)
\( l,u: \) lower/upper bounds in \( \mathbb{Z}^n \) \quad \( f: \) function from \( \mathbb{Z}^n \) to \( \mathbb{R} \)

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(Non)-Linear Integer Programming

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Has generic modeling power but NP-hard even for linear \( f(x)=wx \)
(Non)-Linear Integer Programming

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It is polynomial time solvable in \( \text{fixed dimension} \)
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Has generic modeling power but \textit{NP}-hard even for linear \( f(x) = wx \)

It is polynomial time solvable in \textit{fixed dimension}

Our theory enables \textit{polynomial time} solution of \textit{broad natural universal (non)-linear integer programs in variable dimension}

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Outline

Overview our theory of Graver bases for integer programming
Outline

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Drastically better complexity and fixed-parameter tractability
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Overview our theory of Graver bases for integer programming

Drastically better complexity and fixed-parameter tractability

Applications to multiway tables and huge multicommodity flows

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Overview: Graver Bases and Nonlinear Integer Programming
Background in my Book:

Theory of Graver bases for integer programming
(and more)

Available electronically from my homepage
(with kind permission of EMS)
Graver Bases

The Graver basis of an integer matrix $A$ is the finite set $G(A)$ of conformal-minimal nonzero integer vectors $x$ satisfying $Ax = 0$. 
Graver Bases

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The **Graver basis** of an integer matrix \( A \) is the finite set \( G(A) \) of conformal-minimal nonzero integer vectors \( x \) satisfying \( Ax = 0 \).

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**Example:** Consider \( A=(1\ 2\ 1) \). Then \( G(A) \) consists of
Graver Bases

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circuits: $\pm(2 \ -1 \ 0), \ \pm(1 \ 0 \ -1), \ \pm(0 \ 1 \ -2)$
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**Example:** Consider $A = (1 \ 2 \ 1)$. Then $G(A)$ consists of

- **circuits:** $\pm(2 \ -1 \ 0)$, $\pm(1 \ 0 \ -1)$, $\pm(0 \ 1 \ -2)$
- **non-circuits:** $\pm(1 \ -1 \ 1)$
Some Theorems on (Non)-Linear Integer Programming

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Some Theorems on 
(Non)-Linear Integer Programming

Theorem 1: Linear optimization in polytime with $G(A)$:

$$\max \{ wx : A x = b, \ l \leq x \leq u, \ x \in \mathbb{Z}^n \}$$
Some Theorems on
(Non)-Linear Integer Programming

Theorem 1: Linear optimization in polytime with $G(A)$:

$$\max \left\{ wx : Ax = b, \quad l \leq x \leq u, \quad x \in \mathbb{Z}^n \right\}$$

Reference: N-fold integer programming, (De Loera, Hemmecke, Onn, Weismantel)
Discrete Optimization (Volume in memory of George Dantzig)
Theorem 2: Multicriteria maximization in polytime with $G(A)$:

$$\max \{ f(Wx) : Ax = b, \ l \leq x \leq u, \ x \in \mathbb{Z}^n \}$$

where $W$ is $d \times n$ criteria matrix and $f$ convex function on $\mathbb{Z}^d$
which balances $d$ linear criteria or player utilities $W_i x$
Some Theorems on (Non)-Linear Integer Programming

Theorem 2: Multicriteria maximization in polytime with $G(A)$:

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Some Theorems on (Non)-Linear Integer Programming

**Theorem 3:** Separable convex minimization in polytime with $G(A)$:

$$\min \left\{ \sum f_i(x_i) : Ax = b, \ l \leq x \leq u, \ x \in \mathbb{Z}^n \right\}$$
Some Theorems on (Non)-Linear Integer Programming

Theorem 3: Separable convex minimization in polytime with $G(A)$:

$$\min \{ \sum f_i(x_i) : Ax = b, \ l \leq x \leq u, \ x \ in \ \mathbb{Z}^n \}$$

Reference: A polynomial oracle-time algorithm for convex integer minimization, (Hemmecke, Onn, Weismantel), Mathematical Programming
Some Theorems on (Non)-Linear Integer Programming

Theorem 4: Integer point closest to \( x \) in polytime with \( G(A) \):

\[
\min \{ |x - x|_p : A x = b, \quad 1 \leq x \leq u, \quad x \in \mathbb{Z}^n \}
\]

Reference: A polynomial oracle-time algorithm for convex integer minimization, (Hemmecke, Onn, Weismantel), Mathematical Programming
Some Theorems on (Non)-Linear Integer Programming

Theorem 5: Quadratic minimization in polytime with $G(A)$:

$$\min \{ x^T V x : A x = b, \ 1 \leq x \leq u, \ x \text{ in } \mathbb{Z}^n \}$$

where $V$ lies in cone $K_2(A)$ of possibly indefinite matrices, enabling minimization of some convex and some non-convex quadratics.
Some Theorems on
(Non)-Linear Integer Programming

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Reference: Quadratic Graver cones, quadratic integer minimization & extensions, (Lee, Onn, Romanchuk, Weismantel), Mathematical Programming

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Some Theorems on (Non)-Linear Integer Programming

Theorem 6: Polynomial minimization in polytime with $G(A)$:

$$\min \{ p(x) : Ax = b, \ 1 \leq x \leq u, \ x \in \mathbb{Z}^n \}$$

where $p$ is possibly indefinite polynomial of degree $d$ in cone $K_d(A)$, enabling minimization of some (non)-convex degree $d$ polynomials

Reference: Quadratic Graver cones, quadratic integer minimization & extensions, (Lee, Onn, Romanchuk, Weismantel), Mathematical Programming

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Some Theorems on (Non)-Linear Integer Programming

Theorem 7: Robust optimization in polytime with $G(A)$:

$$\min \{ \max \{ cx : d \leq c \leq e \} : Ax = b, \ l \leq x \leq u, \ x \in \mathbb{Z}^n \}$$

that is, minimum worst case cost where the cost of each variable can vary in an interval
Some Theorems on (Non)-Linear Integer Programming

Theorem 7: Robust optimization in polytime with $G(A)$:

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that is, minimum worst case cost where the cost of each variable can vary in an interval

Reference: Robust integer programming, (Onn), Operations Research Letters
Some Proofs
Proof of Theorem 3
(separable convex minimization)

To solve $\min \{ \sum f_i(x_i) : Ax = b, \ l \leq x \leq u, \ x \in \mathbb{Z}^n \}$

with the Graver basis $G(A)$

Do:
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Do:

1. Find initial point by auxiliary program
To solve \( \min \{ \sum f_i(x_i) : Ax = b, \ l \leq x \leq u, \ x \in \mathbb{Z}^n \} \) with the Graver basis \( G(A) \)

Do:

1. Find initial point by auxiliary program

2. Iteratively improve by Graver-best steps, that is, by best \( cz \) with \( c \in \mathbb{Z} \) and \( z \in G(A) \).
Proof of Theorem 3
(separable convex minimization)

To solve \( \min \{ \sum f_i(x_i) : Ax = b, \ l \leq x \leq u, \ x \in \mathbb{Z}^n \} \) with the Graver basis \( G(A) \)

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Using supermodality of \( f \) and integer Caratheodory theorem (Cook-Fonlupt-Schrijver, Sebo) we can show polytime convergence to some optimal solution.
Proof of Theorem 2
(multicriteria maximization)

Lemma: Can solve in polytime $\max \{ f(Wx) : x \in S \}$ with $S \in \mathbb{Z}^n$ if can do linear optimization over $S$ and have set $E$ of all edge-directions of $\text{conv}(S)$
Lemma: Can solve in polytime $\max \{ f(Wx) : x \in S \}$ with $S$ in $\mathbb{Z}^n$ if can do linear optimization over $S$ and have set $E$ of all edge-directions of $\text{conv}(S)$.

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Proof of Theorem 2
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Reference: Convex combinatorial optimization, (Onn, Rothblum), Journal of Discrete and Computational Geometry
Proof of Theorem 2
(multicriteria maximization)

To solve \( \max \{ f(Wx) : x \in S \} \)
with \( S := \{ x \in \mathbb{Z}^n : Ax = b, \ l \leq x \leq u \} \)
using the Graver basis \( G(A) \)

Do:
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Do:

1. Use \( G(A) \) to simulate linear-optimization oracle over \( S \) via Theorem 1
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2. Use the Graver basis as set \( E := G(A) \) of all edge-directions of \( \text{conv}(S) \)
Proof of Theorem 2
(multicriteria maximization)

To solve \( \max \{ f(Wx) : x \text{ in } S \} \)

with \( S := \{ x \text{ in } \mathbb{Z}^n : Ax = b, \, l \leq x \leq u \} \)

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Do:

1. Use \( G(A) \) to simulate linear-optimization oracle over \( S \) via Theorem 1

2. Use the Graver basis as set \( E := G(A) \) of all edge-directions of \( \text{conv}(S) \)

3. Apply the Lemma (use 1 for each vertex of zone("WG(A)) and pick best)
N-Fold Integer Programming
The $n$-fold product of an $(r,s) \times t$ bimatrix $A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$ is the $(r+ns) \times nt$ matrix $A^{(n)} = \begin{pmatrix} A_1 & A_1 & A_1 & \cdots & A_1 \\ A_2 & 0 & 0 & \cdots & 0 \\ 0 & A_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & A_2 \end{pmatrix}$. 

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The \( n \)-fold product of an \((r,s) \times t\) bimatrix \( A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \) is the \((r+ns) \times nt\) matrix

\[
A^{(n)} = \begin{pmatrix}
A_1 & A_1 & A_1 & \cdots & A_1 \\
A_2 & 0 & 0 & \cdots & 0 \\
0 & A_2 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & A_2 \\
\end{pmatrix}
\]

n

Lemma: For fixed \( A \) we can compute the Graver basis \( G(A^{(n)}) \) in polynomial time \( O(n^{g(A)}) \) with \( g(A) \) the Graver complexity of \( A \).
### N-Fold Products

The n-fold product of an \((r,s) \times t\) bimatrix \(A = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}\) is the \((r+ns) \times nt\) matrix

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\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & A_2 \\
\end{pmatrix}.
\]

**Proof very rough idea:** For any \(n\), any element \(x=(x^1, \ldots, x^n)\) in the Graver basis \(G(A^{(n)})\) has at most \(g(A)\) nonzero bricks \(x^k\) in \(\mathbb{Z}^t\).

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(Non)-Linear N-Fold Integer Programming

**Theorem:** for various $f$ can solve in polynomial time $O(n^{g(A)} L)$:

$$\min\{f(x) : A^{(n)} x = b, \ l \leq x \leq u, \ x \in \mathbb{Z}^{n_t}\}$$

$$A^{(n)} = \begin{pmatrix} A_1 & A_1 & A_1 & \cdots & A_1 \\ A_2 & 0 & 0 & \cdots & 0 \\ 0 & A_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & A_2 \end{pmatrix}$$

**References:** see *Nonlinear Discrete Optimization, (Onn), Zurich Lectures in Advanced Mathematics, European Mathematical Society*
Drastically Better Complexity
and
Fixed-Parameter Tractability
Cubic Running Time and Fixed-Parameter Tractability

Reference: N-fold integer programming in cubic time,
(Hemmecke, Onn, Romanchuk), Mathematical Programming
Cubic Running Time and Fixed-Parameter Tractability

**Theorem:** For any fixed bimatrix $A$, the following linear $n$-fold integer program is solvable in fixed-parameter time $O(n^3 L)$:

$$\max\{wx : A^{(n)}x = b, \ l \leq x \leq u, \ x \in \mathbb{Z}^{nt}\}$$

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Instead of $O(n^{g(A)} L)$

**Proof very rough idea:** in the iterative algorithm, at each iteration, can find a Graver-best step without computing the entire Graver basis.

**Reference:** N-fold integer programming in cubic time, (Hemmecke, Onn, Romanchuk), Mathematical Programming
Some Applications
Multiway Tables

*Complexity* of deciding the existence of \( l \times m \times n \) tables with given line sums:
Multiway Tables

**Complexity** of deciding the existence of \( l \times m \times n \) tables with given line sums:

- \( l, m, n \) variable:

- \( l \) fixed, \( m, n \) variable:

- \( l, m \) fixed, \( n \) variable:

- \( l, m, n \) fixed:
Multiway Tables

**Complexity** of deciding the existence of $l \times m \times n$ tables with given line sums:

- $l, m, n$ variable: NP-complete
  
  Three dimensional matching, Karp, 1972

- $l$ fixed, $m, n$ variable:

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- $l, m, n$ fixed: **Polytime**
  
  Integer programming in fixed dimension, Lenstra, 1982

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- $l, m, n$ variable: **NP-complete**
  
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- $l$ fixed, $m, n$ variable: **Universal for IP** (even with $l=3$)
  
  De Loera, Onn, 2006

- $l, m$ fixed, $n$ variable:

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**Multiway Tables**

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Much more generally, consider the multi-index transportation problem studied by Motzkin in 1952, of minimization over $m_1 \times \cdots \times m_k \times n$ tables with given margins:
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It is an $n$-fold program

$$\min \{ f(x) : A^{(n)} x = b, \ x \geq 0, \ x \text{ integer} \}$$

for suitable $A$ depending on $m_1, \ldots, m_k$ where:

- $A_1$ gives equations of margins summing over layers
- $A_2$ gives equations of margins summing within a single layer at a time
Much more generally, consider the multi-index transportation problem studied by Motzkin in 1952, of minimization over $m_1 \times \cdots \times m_k \times n$ tables with given margins:

**Corollary:** (Non)-linear optimization over $m_1 \times \cdots \times m_k \times n$ tables with given margins can be done in polynomial time $O(n^{g(m_1,\ldots,m_k)} L)$.
Multiway Tables

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**Corollary:** (Non)-linear optimization over $m_1 \times \cdots \times m_k \times n$ tables with given margins can be done in polynomial time $O(n^{g(m_1, \ldots, m_k)} L)$

In contrast: **Universality of three-way tables** (De Loera, Onn):
Every integer program is one over $3 \times m \times n$ tables with given line-sums

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**Better:** (Non)-linear optimization over $m_1 \times \cdots \times m_k \times n$ tables with given margins can be done in fixed-parameter cubic time $O(n^3 L)$
Multicommodity Flows

Find flow of $l$ commodities from $m$ servers to $n$ surfers satisfying given supplies $s_{i,k}$, demands $d_{j,k}$ and capacities $c_{i,j}$ of total bit size $L$.
Multicommodity Flows

Find flow of $l$ commodities from $m$ servers to $n$ surfers satisfying given supplies $s_{i,k}$, demands $d_{j,k}$ and capacities $c_{i,j}$ of total bit size $L$.

With $l=2$ or $m=3$ it is NP-complete so assume both $l,m$ are parameters.
Multicommodity Flows

Find flow of $l$ commodities from $m$ servers to $n$ surfers satisfying given supplies $s_{i,k}$, demands $d_{j,k}$ and capacities $c_{i,j}$ of total bit size $L$

2008: polynomial time $O(n^{g(l,m)} L)$ with Graver complexity $g(l,m)$ exponential in $l,m$ (De Loera, Hemmecke, Onn, Weismantel) (theory of $n$-fold IP)
**Multicommodity Flows**

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**2013**: fixed-parameter tractable $O(n^3 L)$ (Hemmecke, Onn, Lyubov Romanchuk)
Multicommodity Flows

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(Hemmecke, Onn, Lyubov Romanchuk)

2015: strongly polynomial $O(n^{g(l,m)})$
(De Loera, Hemmecke, Lee)
Multicommodity Flows

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usually the fastest

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Multicommodity Flows

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2015: strongly polynomial $O(n^{g(l,m)})$ (De Loera, Hemmecke, Lee)

Open: algorithm that is both fixed-parameter tractable and strongly polynomial?
Multicommodity Flows

Huge version: surfers come in huge clouds of \( t \) types

\[ n_1 + \ldots + n_t = n \]

binary encoded

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**Multicommodity Flows**

**Huge version:** surfers come in huge clouds of \( t \) types

\[ n_1 + \ldots + n_t = n \]

2016 (Onn): fixed-parameter tractable with parameters \( l, t \), variable \( m \), huge \( n \)
Multicommodity Flows

Huge version: surfers come in huge clouds of \( t \) types

2016 (Onn): fixed-parameter tractable with parameters \( l, t \), variable \( m \), huge \( n \)

Open: 4-dimensional huge tables are only known to be in \( \text{NP} \) intersect \( \text{coNP} \)
Some Further Developments in Theory and Applications

- Clustering and farmland consolidation, Borgwardt, Melamed, Onn
- Scheduling, Knop, Koutecky
- Stochastic integer programming, Hemmecke, Onn, Weismantel
- Portfolio optimization, Baumann, Trautmann
- Optimality certificates, Kobayashi, Murota, Saito, Weismantel
- Production scheduling, Andziulis, Dzemydien
- Block structured integer programs, Hemmecke, Köppe, Weismantel
- Matrix apportionment problems, Gaffke, Pukelsheim
- Strongly polynomial algorithms, De Loera, Hemmecke, Lee
- Rounding in integer nonlinear optimization, Hubner, Schobel
- Graver complexity, Berstein, Finhold, Hemmecke, Kudo, Nairn, Onn, Takemura
- Network design problems, Guyard, Laugier
- Games, Hemmecke, Nguyen, Onn, Ryan, Weismantel

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- The complexity of 3-way tables (SIAM J. Comp.)
- Convex combinatorial optimization (Disc. Comp. Geom.)
- Markov bases of 3-way tables (J. Symb. Comp.)
- All linear and integer programs are slim 3-way programs (SIAM J. Opt.)
- Graver complexity of integer programming (Annals Combin.)
- N-fold integer programming (Disc. Opt. in memory of Dantzig)
- Convex integer maximization via Graver bases (J. Pure App. Algebra)
- Polynomial oracle-time convex integer minimization (Math. Prog.)
- Theory and applications of n-fold integer programming (IMA Volume on MINLP)
- Quadratic Graver cones, quadratic integer minimization & extensions (Math. Prog.)
- Robust integer programming (Operations Research Letters)
- N-fold integer programming in cubic time (Math. Prog.)
- Huge tables and multicommodity flows are fixed-parameter tractable via unimodular integer Caratheodory (J. Computer and System Sciences)
Background in my Book:

Theory of Graver bases for integer programming

(and more)

Available electronically from my homepage

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