Shifted
Combinatorial Optimization

Shmuel Onn

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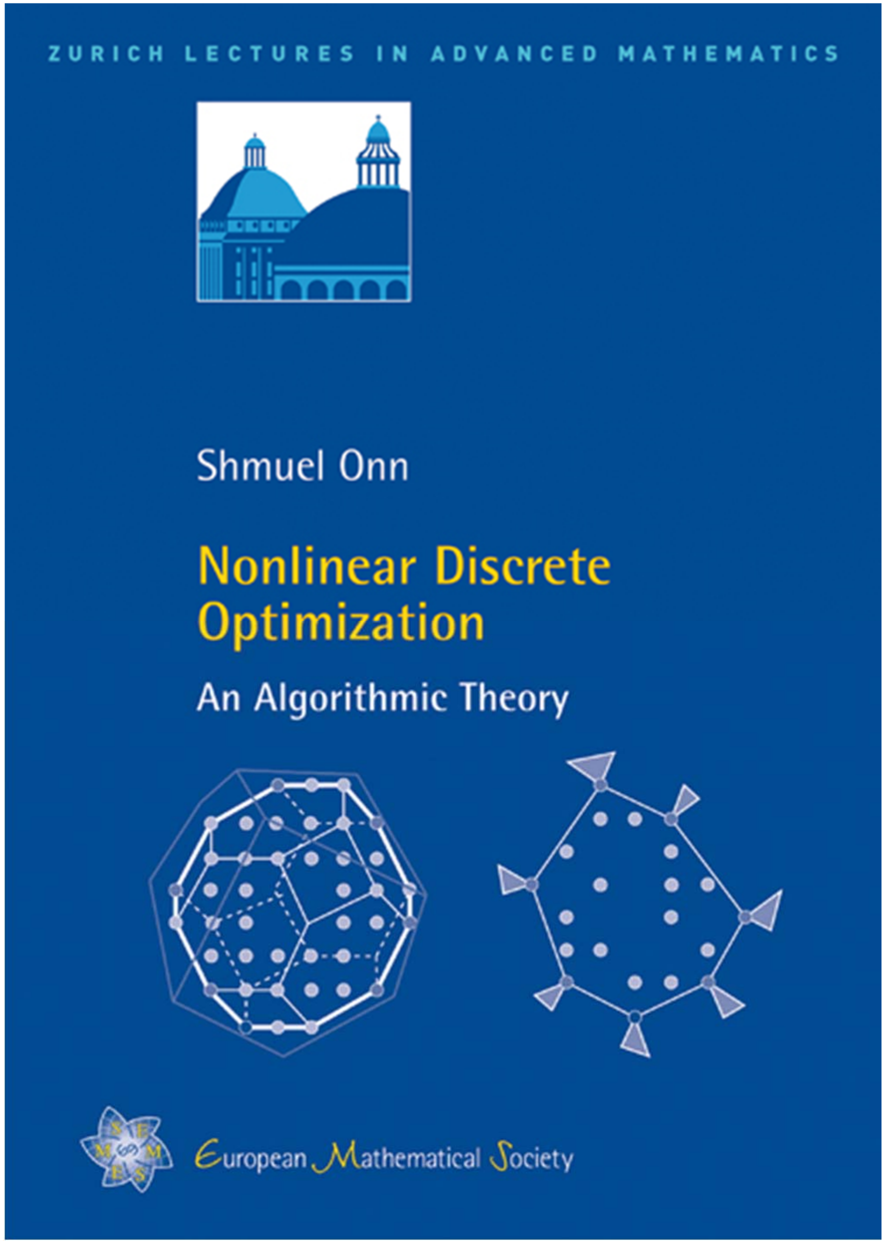

Acknowledgment and little advertisement

Monograph developed at ETH while delivering the Nachdiplom Lectures

Theory of Graver bases for integer programming

Available electronically from my homepage
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1. Shifted Combinatorial Optimization

2. Multicommodity Flows

Standard Combinatorial Optimization

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- Bases in a matroid over [d] given by independence oracle
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- Set $S = \{s^1, s^2, \ldots, s^m\} \subseteq \{0,1\}^d$ given explicitly

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For $S \subseteq \{0,1\}^d$ consider matrices consisting of n choices from S,

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Shifted Combinatorial Optimization: given $S \subseteq \{0,1\}^d$ and $c \in Z^{d \times n}$, solve

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  (set $c_{i,r} = -1$, $c_{i,j} = 0$, $i=1,\dots,d$, $j \neq r$)

- lexicographically minimize n-vulnerables, $(n-1)$-vulnerables, and so on

  (set $c_{i,j} = -(d+1)^{(j-1)}$, $i=1,\dots,d$, $j=1,\dots,n$)

Special Case: Minimum Shared Edges Problem

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- For n variable it is NP-hard (O-S-Z 2013)
- For n fixed it is polynomial time solvable (A-E-N-Y-Z 2014)
- For n parameter it is fixed-parameter tractable (F-K-N-S 2015)

Matchings, Dipaths, Inequalities

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matrix c is shifted if $c = \underline{c}$, that is, its rows are nonincreasing

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Proposition: for perfect matchings the shifting problem is NP-hard even for n=2 with shifted c and cubic graphs

Proof: Use shifting to find n=2 matchings with minimum shared edges.
If they are disjoint then the cubic graph is 3-edge-colorable.

(set $c_{i,1} = 0$, $c_{i,2} = -1$, $i=1, \ldots, d$)

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Matchings, Dipaths, Inequalities

Theorem 1: Shifting with shifted $c=\underline{c}$ is polynomial time solvable for any set $S=\{s\in\{0,1\}^d : As\leq b\}$ with A totally unimodular and b integer.

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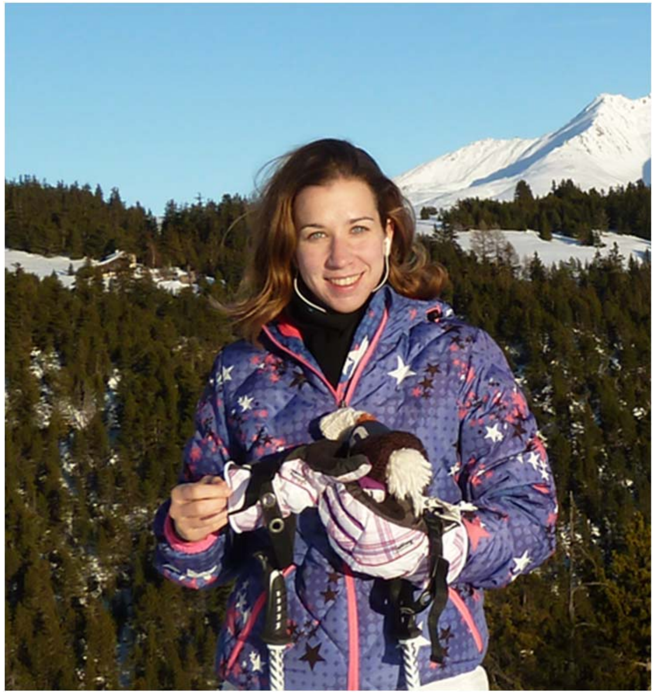

Reference: The unimodular intersection problem (Kaibel, Onn, Pauline Sarrabezolles), 2015

Outline of Proof of Theorem 1

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3. Any feasible x is no better than z in the original program, since

$$c\underline{z} = c\underline{y} \ge cy \ge c\underline{x}$$

since $\underline{z} = \underline{y}$, $c=\underline{c}$ is shifted, and $\underline{x}$ is feasible in the auxiliary program.

Matroids, Matroid Intersections

Theorem 2: Shifting with shifted $c=\underline{c}$ is polynomial time solvable for:

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- Intersection of strongly orderable matroids (transversals, gammoids)

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Proof: uses matroid union, matroid intersection, and other ingredients.

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Sets Given Explicitly

Reference: Shifted combinatorial optimization
(Gajarský, Hlinený, Martin Koutecký, Onn), in preparation

Sets Given Explicitly

Proposition: shifting is NP-hard already for explicitly given sets $S = \{s^1, s^2, ..., s^m\} \subseteq \{0,1\}^d$ and shifted $c = \underline{c} \in \{0,1\}^{d \times n}$

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Proof: Given graph with $d=3n$ vertices, let S be the set of indicators of triples of vertices inducing triangles.

Shift to find n triangles covering maximum vertices. If all are covered then they partition the vertex set.

(set $c_{i,1} = 1$, $c_{i,k} = 0$, $i=1, \ldots, d$, $k=2, \ldots, n$)

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Proof: uses convex integer minimization, Oertel, Wagner, Weismantel, 2014

Shifted Linear Programming

Corollary: to solve the shifting problem over any S with nondecreasing c, solve the linear problem with $w := \sum c^k$, obtain $s \in S$, return $x := [s, \ldots, s]$

Proposition: shifted linear programming over a nonempty polytope $S = \{s \in R^d : As \leq b\}$ is NP-hard already for $n=2$ and shifted $c=\underline{c}$

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Beyond

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Some work in progress: (Gajarský, Hlinený, Koutecký, Onn)

- Shifting logic defined properties in graphs of bounded treewidth

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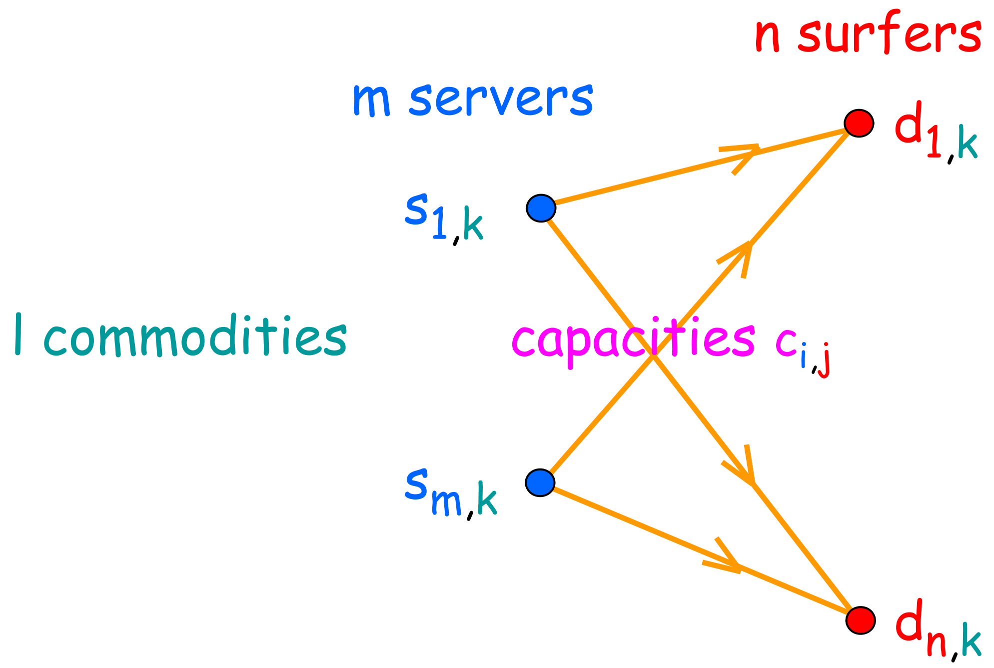
- shift your imagination !

Multicommodity Flows

Find flow of l commodities from m servers to n surfers satisfying given supplies $s_{i,k}$, demands $d_{j,k}$ and capacities $c_{i,j}$ of total bit size L



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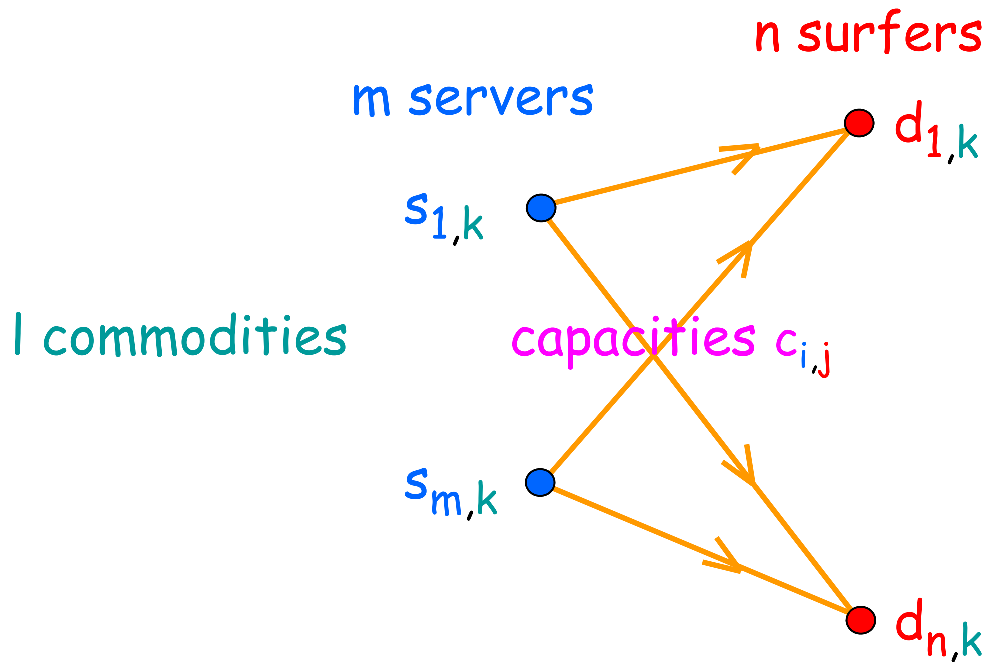

With $l=2$ or $m=3$ it is NP-complete so assume both l,m are parameters

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Open: algorithm that is both fixed-parameter tractable and strongly polynomial ?

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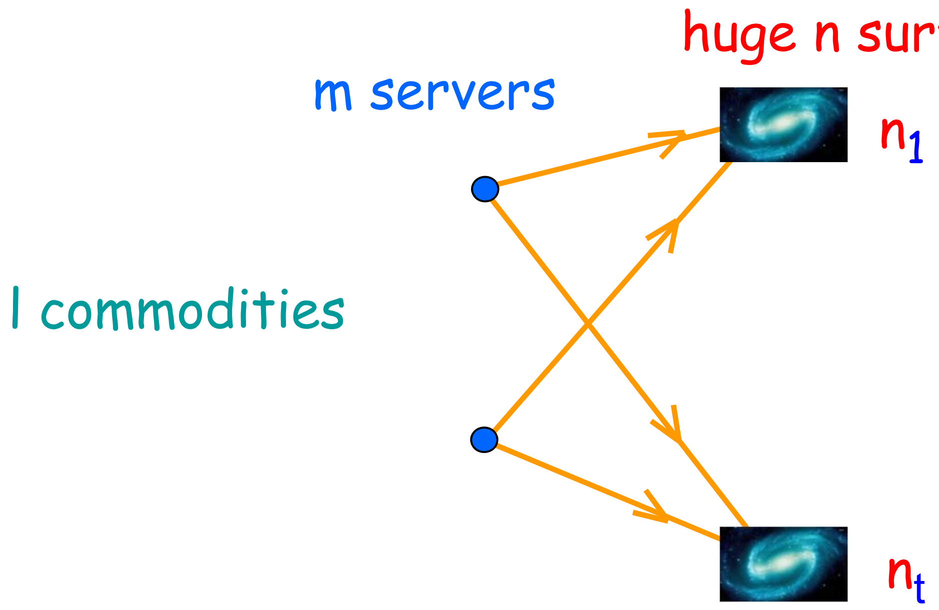
Huge version: surfers come in huge clouds of t types



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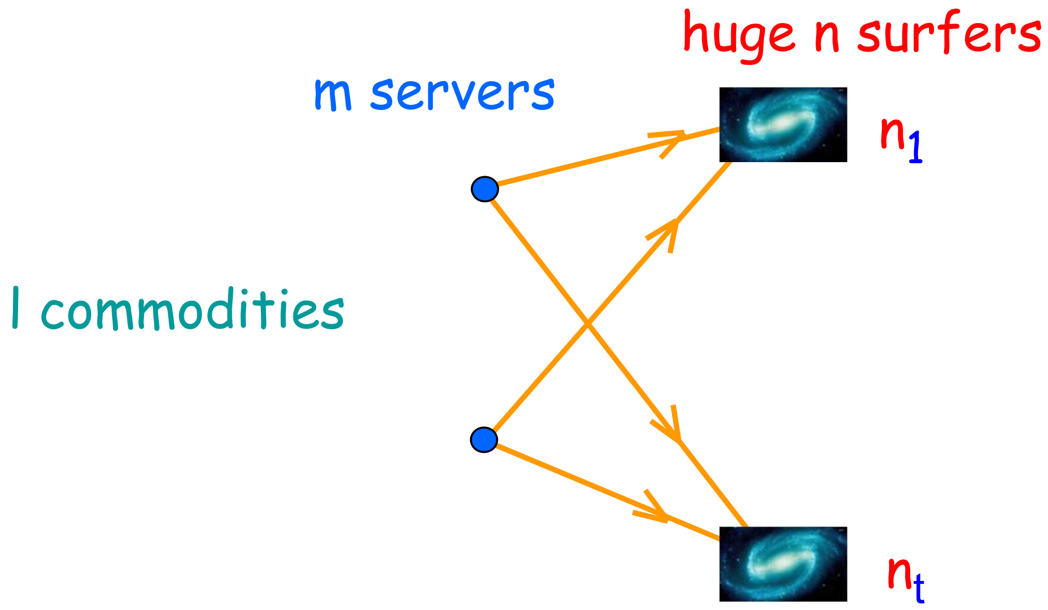

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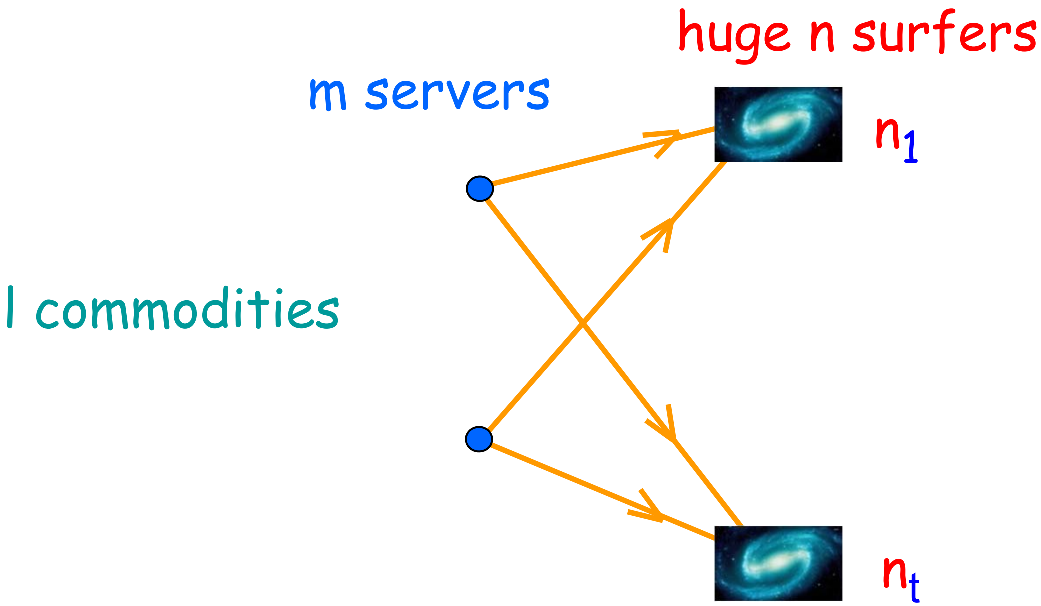

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Open: 4-dimensional huge tables are only known to be in NP intersect coNP