Multiway Tables and Integer Programming are Fixed-Parameter Tractable

Shmuel Onn

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Outline

1. **Overview** our theory of **Graver bases** for integer programming
   
   (with Berstein, De Loera, Hemmecke, Lee, Romanchuk, Rothblum, Weismantel)
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2. **Multiway tables** are polynomial time solvable
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3. Multiway tables and IP are fixed-parameter tractable
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4. Huge multiway tables - P versus NP and coNP
   (posted last week on the Arxiv)

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4. **Huge multiway tables** - **P versus NP** and **coNP**
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5. **Approximation hierarchy** for **integer programming**

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(Non)-Linear Integer Programming

The problem is: \[ \min \{ f(x) : Ax = b, \ l \leq x \leq u, \ x \in \mathbb{Z}^n \} \]

with data: \( A \): integer \( m \times n \) matrix \( b \): right-hand side in \( \mathbb{Z}^m \)
\( l, u \): lower/upper bounds in \( \mathbb{Z}^n \) \( f \): function from \( \mathbb{Z}^n \) to \( \mathbb{R} \)

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Our theory enables polynomial time solution of broad natural universal (non)-linear integer programs in variable dimension

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Graver Bases

and

Nonlinear Integer Programming
The **Graver basis** of an integer matrix $A$ is the finite set $G(A)$ of conformal-minimal nonzero integer vectors $x$ satisfying $Ax = 0$.

$x$ is conformal-minimal if no other $y$ in same orthant has all $|y_i| \leq |x_i|$.
**Graver Bases**

The **Graver basis** of an integer matrix $A$ is the finite set $G(A)$ of conformal-minimal nonzero integer vectors $x$ satisfying $Ax = 0$.

**Example:** Consider $A = (1 \ 2 \ 1)$. Then $G(A)$ consists of

- **circuits:** $\pm(2 \ -1 \ 0)$, $\pm(1 \ 0 \ -1)$, $\pm(0 \ 1 \ -2)$

- **non-circuits:** $\pm(1 \ -1 \ 1)$
Some Theorems on
(Non)-Linear Integer Programming

Theorem 1: linear optimization in polytime with $G(A)$:

$$\min \left\{ wx : Ax = b, \quad l \leq x \leq u, \quad x \in \mathbb{Z}^n \right\}$$

Reference: $N$-fold integer programming (De Loera, Hemmecke, Onn, Weismantel)
Discrete Optimization (Volume in memory of George Dantzig), 2008

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Some Theorems on 
(Non)-Linear Integer Programming

Theorem 2: separable convex minimization in polytime with $G(A)$:

$$\min \{ \sum f_i(x_i) : Ax = b, \ l \leq x \leq u, \ x \in \mathbb{Z}^n \}$$

Reference: A polynomial oracle-time algorithm for convex integer minimization 
(Hemmecke, Onn, Weismantel) Mathematical Programming, 2011
Some Theorems on
(Non)-Linear Integer Programming

Theorem 3: polynomial minimization in polytime with $G(A)$:

$$\min \{ p(x) : Ax = b, \ 1 \leq x \leq u, \ x \in \mathbb{Z}^n \}$$

for a certain class of possibly non-convex multivariate polynomials.

Reference: The quadratic Graver cone, quadratic integer minimization & extensions
(Lee, Onn, Romanchuk, Weismantel), Mathematical Programming, 2012

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The Iterative Algorithm
Proof of Theorem 2

To solve $\min \{ f(x) : Ax = b, \ l \leq x \leq u, \ x \in \mathbb{Z}^n \}$

with the Graver basis $G(A)$

Do:
Proof of Theorem 2

To solve \( \min \{ f(x) : A x = b, \ 1 \leq x \leq u, \ x \in \mathbb{Z}^n \} \) with the Graver basis \( G(A) \)

Do:

1. Find initial point by auxiliary program
Proof of Theorem 2

To solve \( \min \{ f(x) : Ax = b, \ l \leq x \leq u, \ x \in \mathbb{Z}^n \} \)

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Do:

1. Find initial point by auxiliary program

2. Iteratively improve by Graver-best steps, that is, by best \( cz \) with \( c \in \mathbb{Z} \) and \( z \in G(A) \).
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**Do:**

1. **Find initial point** by auxiliary program
2. **Iteratively improve** by **Graver-best steps**, that is, by best \( cz \) with \( c \in \mathbb{Z} \) and \( z \in G(A) \).

Using supermodality of \( f \) and integer Caratheodory theorem (Cook-Fonlupt-Schrijver, Sebo) we can show polytime convergence to some optimal solution.
Multiway Tables
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The multiway table problem of Motzkin (1952) concerns minimization over $m_1 \times \cdots \times m_k \times m_{k+1}$ tables with given margins.
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If all $m_i$ fixed then solvable by fixed dimension theory.
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If one side \( n \) is variable then we have the following:

**Lemma:** For fixed \( m_1, \ldots, m_k \) there is Graver complexity \( g := g(m_1, \ldots, m_k) \) such that the relevant Graver basis consists of all \( O(n^g) \) lifts of the relevant Graver basis of \( m_1 \times \cdots \times m_k \times g \) tables.
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**Lemma:** For fixed $m_1, \ldots, m_k$ there is Graver complexity $g := g(m_1, \ldots, m_k)$ such that the relevant Graver basis consists of all $O(n^g)$ lifts of the relevant Graver basis of $m_1 \times \cdots \times m_k \times g$ tables.

$g(3,3) = 9$, $g(3,4) = 27$, $g(3,5) = ?$  \quad $\Omega(2^m) = g(3,m) = O(6^m)$

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**Corollary:** (Non)-linear optimization over $m_1 \times \cdots \times m_k \times n$ tables with given margins is doable in polynomial time $O(n^{g(m_1, \ldots, m_k)} L)$.

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**Corollary:** (Non)-linear optimization over \( m_1 \times \cdots \times m_k \times n \) tables with given margins is doable in polynomial time \( O(n^{g(m_1,\ldots,m_k)} L) \).

However, if two sides are variable then have the **Universality Theorem:**
Every integer program is one over \( 3 \times m \times n \) tables with given line-sums.
Fixed-Parameter Tractability
Multiway Tables are Fixed-Parameter Tractable

Reference: N-fold integer programming in cubic time
(Hemmecke, Onn, Romanchuk) Mathematical Programming, 2013
Multiway Tables are Fixed-Parameter Tractable

**Theorem:** (Non)-linear optimization over $m_1 \times \ldots \times m_k \times n$ tables with given margins is doable in fixed-parameter cubic time $O(n^3 L)$. 
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Instead of $O(n^{g(m_1,\ldots,m_k)}L)$
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Now, have parameterization of all integer programming by universality: Every integer program is one over $3 \times m \times n$ tables with given line-sums.
**Multiway Tables are Fixed-Parameter Tractable**

**Theorem:** (Non)-linear optimization over $m_1 \times \cdots \times m_k \times n$ tables with given margins is doable in fixed-parameter cubic time $O(n^3 L)$.

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Now, have parameterization of all integer programming by *universality*: Every integer program is one over $3 \times m \times n$ tables with given line-sums.

**Corollary:** (nonlinear) integer programming is fixed-parameter tractable: for each fixed $m$ it is solvable in time $O(n^3 L)$ instead of $O(n^{g(3,m)} L)$. 
Proof: Better Way of Finding Graver-Best Steps

Let $x$ be a feasible $l \times m \times n$ table at some iteration of minimizing $w$ over such tables with given line-sums and let $G$ be the Graver basis.
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Let $x$ be a feasible $l \times m \times n$ table at some iteration of minimizing $w$ over such tables with given line-sums and let $G$ be the Graver basis.

A Graver-best step for $x$ is a table $h$ such that $x+h$ is feasible and at least as good as any feasible $x+cz$ with $c \in \mathbb{Z}$ and $z \in G$. 
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A *Graver-best step* for $x$ is a table $h$ such that $x+h$ is feasible and at least as good as any feasible $x+cz$ with $c \in \mathbb{Z}$ and $z \in G$.

Previous way: for each of $O(n^{g(l,m)})$ tables $z \in G$ find best $c \in \mathbb{Z}$
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Previous way: for each of $O(n^{g(l,m)})$ tables $z \in G$ find best $c \in \mathbb{Z}$

New way: for each $c$ find $ch$ at least as good as any $cz$ with $z \in G$
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Let $x$ be a feasible $l \times m \times n$ table at some iteration of minimizing $w$ over such tables with given line-sums and let $G$ be the Graver basis.

Let $V(l,m) := \{v : v$ is sum of at most $g(l,m)$ many $l \times m$ circuit matrices}$
Proof: Better Way of Finding Graver-Best Steps

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Example: for $3 \times 3 \times n$ tables, $V(3,3)$ consists of 42931 matrices such as

$$
\begin{bmatrix}
1 & -1 & 0 \\
0 & 1 & -1 \\
-1 & 0 & 1
\end{bmatrix}
$$

$$
\begin{bmatrix}
9 & -2 & -7 \\
-4 & 5 & -1 \\
-5 & -3 & 8
\end{bmatrix}
$$
**Proof: Better Way of Finding Graver-Best Steps**

Let $x$ be a feasible $l \times m \times n$ table at some iteration of minimizing $w$ over such tables with given line-sums and let $G$ be the Graver basis.

Let $V(l,m) := \{ v : v \text{ is sum of at most } g(l,m) \text{ many } l \times m \text{ circuit matrices} \}$

Note that $|V(l,m)| \leq c(l,m)^{g(l,m)}$ is (huge) constant.

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Let $V(l,m) := \{v : v$ is sum of at most $g(l,m)$ many $l \times m$ circuit matrices$\}$

Lemma: For every $n$ and every $l \times m \times n$ table $h$ in the Graver basis $G$, the sum of every subset of layers of $h$ is a matrix in $V(l,m)$. 

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Proof: Better Way of Finding Graver-Best Steps

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Let $V(l,m) := \{v: v \text{ is sum of at most } g(l,m) \text{ many } l \times m \text{ circuit matrices}\}$

For each $c \in \mathbb{Z}$ construct the following dynamic program:

- $S_0 = \{0\}$
- $S_{i-1} = V(l,m)$
- $S_i = V(l,m)$
- $S_n = \{0\}$

$v_{i-1}$ to $v_i$:
- $h_i := v_i - v_{i-1} \in V(l,m)$
- $x_i + ch_i \geq 0$
- Length $w_i h_i$
Lemma: Each feasible step $c_h = (c_{h1}, \ldots, c_{hn})$ with $h \in G$ gives dipath $(0, v^1, \ldots, v^n)$ in the dynamic program, with each $v^i = h^1 + \ldots + h^i \in V(l,m)$.

Proof: Better Way of Finding Graver-Best Steps
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Lemma: Each feasible step \( ch = (ch^1, \ldots, ch^n) \) with \( h \in G \) gives dipath \( (0, v^1, \ldots, v^n) \) in the dynamic program, with each \( v^i = h^1 + \ldots + h^i \in V(l,m) \).

Lemma: A Graver-best step for \( x \) can be computed in \( O(n^2) \) time, solving \( O(|V(l,m)| \cdot n) \) dynamic programs each in time \( O(|V(l,m)|^2 \cdot n) \).

\[
\begin{align*}
S_0 &= \{0\} & S_{i-1} &= V(l,m) & S_i &= V(l,m) & S_n &= \{0\} \\
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\end{align*}
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Lemma: Each feasible step \( ch = (ch^1, \ldots, ch^n) \) with \( h \in G \) gives dipath \( (0, v^1, \ldots, v^n) \) in the dynamic program, with each \( v^i = h^1 + \ldots + h^i \in V(l,m) \).
Proof: Better Way of Finding Graver-Best Steps

Example: for $3 \times 3 \times n$ tables, $V(3,3)$ consists of 42931 matrices such as

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\begin{pmatrix}
9 & -2 & -7 \\
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-5 & -3 & 8
\end{pmatrix}
\]

so finding a single Graver-best step in a single iteration involves some $10^{14}n^2$ arithmetic operations per iteration.
Huge Multiway Tables
Huge Multiway Tables

Consider the problem of deciding existence of an $l \times m \times n$ huge table with $t$ given types of row-sums and column-sums having $n_k$ layers of each type $k$ with the $n_k$ encoded in binary.
Huge Multiway Tables

Consider the problem of deciding existence of an \( l \times m \times n \) huge table with \( t \) given types of row-sums and column-sums having \( n_k \) layers of each type \( k \) with the \( n_k \) encoded in binary.

**Theorem:** For fixed \( t \) the problem is in \( P \).
For variable \( t \) it is in \( NP \) and \( coNP \).

**Reference:** Huge Multiway Table Problems (Onn), posted on the Arxiv last week
Graver Approximation Hierarchy
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Our algorithm naturally enables a hierarchy of approximations of the universal integer program over $3 \times m \times n$ tables with variable $m,n$. 
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At level $d$ of this hierarchy, we find approximated Graver-best steps in time $O(m^{9d} n^2)$, approximating $V(3,m)$ in the dynamic programs by

$$V_d(3,m) := \{ v : v \text{ is sum of at most } d \text{ many } 3 \times m \text{ circuit matrices} \}$$
**Graver Approximation Hierarchy**

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We apply Graver-best steps while possible. If the approximation is satisfactory then we stop, else we proceed to the next level.

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Some Bibliography
(available at http://ie.technion.ac.il/~onn)

- The complexity of 3-way tables (SIAM J. Comp.)
- Convex combinatorial optimization (Disc. Comp. Geom.)
- Markov bases of 3-way tables (J. Symb. Comp.)
- All linear and integer programs are slim 3-way programs (SIAM J. Opt.)
- Graver complexity of integer programming (Annals Combin.)
- N-fold integer programming (Disc. Opt. in memory of Dantzig)
- Convex integer maximization via Graver bases (J. Pure App. Algebra)
- Polynomial oracle-time convex integer minimization (Math. Prog.)
- The quadratic Graver cone, quadratic integer minimization & extensions (Math Prog.)
- N-fold integer programming in cubic time (Math. Prog.)
- Huge multiway table problems (Arxiv)
Comprehensive development is in my monograph available electronically from my homepage (with kind permission of EMS) (excluding fixed-parameter result and huge multiway table results)