Integer Programming in Parameter Tractable Strongly Polynomial Time

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Based on ICALP 2018 paper with Martin Koutecký and Asaf Levin

and on broad extended version in preparation with Eisenbrand, Hunkenschröder, Klein, Koutecký, Levin
Integer Programming

\[ \text{IP: } \max \{ w^T x : A x = b, \ l \leq x \leq u, \ x \in \mathbb{Z}^n \} \]

- \( A \): integer \( m \times n \) matrix
- \( b \): right-hand side in \( \mathbb{Z}^m \)
- \( l, u \): lower/upper bounds in \( \mathbb{Z}^n \)
- \( w \): profit vector in \( \mathbb{Z}^n \)
Integer Programming

IP: \[
\max \{ \, wx \mid Ax = b, \ l \leq x \leq u, \ x \in \mathbb{Z}^n \, \}
\]

Arithmetic input: \(n\) unary
**Integer Programming**

**IP:** \[ \max \{ wx : Ax = b, \ l \leq x \leq u, \ x \in \mathbb{Z}^n \} \]

*Arithmetic input: n unary*

*Numeric input: w, b, l, u binary*
Integer Programming

IP: \[
\max \{ \mathbf{w} \cdot \mathbf{x} : \mathbf{A} \cdot \mathbf{x} = \mathbf{b}, \ l \leq \mathbf{x} \leq \mathbf{u}, \ \mathbf{x} \in \mathbb{Z}^n \}
\]

Arithmetic input: \( n \) unary

Numeric input: \( \mathbf{w}, \mathbf{b}, \mathbf{l}, \mathbf{u} \) binary

Parametric input: \( a,d \) where \( a=|A|_{\infty} \) and \( d=\min\{\text{td}(\mathbf{A}), \text{td}(\mathbf{A}^\top)\} \)
**Integer Programming**

**IP:** \[ \max \{ wx : Ax = b, \ l \leq x \leq u, \ x \in \mathbb{Z}^n \} \]

*Arithmetic input: n unary*

*Numeric input: w,b,l,u binary*

*Parametric input: a,d where a=|A|_\infty and d=\min\{td(A), td(A^T)\}*

**Theorem (Koutecký-Levin-Onn):** Integer Programming is solvable in parameter tractable and strongly polynomial time \( f(a,d) \text{ poly}(n) \)
Integer Programming

IP: \[ \max \{ wx : Ax = b, \ l \leq x \leq u, \ x \in \mathbb{Z}^n \} \]

Arithmetic input: \( n \) unary

Numeric input: \( w, b, l, u \) binary

Parametric input: \( a, d \) where \( a = |A|_\infty \) and \( d = \min\{\text{td}(A), \text{td}(A^T)\} \)

**Theorem** (Koutecký-Levin-Onn): Integer Programming is solvable in parameter tractable and strongly polynomial time \( f(a,d) \text{poly}(n) \)

The tree-depth \( \text{td}(A) \) is the smallest height of a rooted tree \( T \) on \([n]\) such that if \( A_{i,j}, A_{i,k} \) are nonzero then \( j, k \) are comparable in \( T \)

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\textbf{Integer Programming}

\[ \text{IP: } \max \{ wx : Ax = b, \ l \leq x \leq u, \ x \in \mathbb{Z}^n \} \]

\textbf{Arithmetic input: } n \text{ unary}

\textbf{Numeric input: } w, b, l, u \text{ binary}

\textbf{Parametric input: } a, d \text{ where } a = |A|_{\infty} \text{ and } d = \min\{ \text{td}(A), \text{td}(A^T) \}

\textbf{Theorem (Koutecký-Levin-Onn): } Integer Programming is solvable in parameter tractable and strongly polynomial time \( f(a,d) \text{ poly}(n) \)

We hope this will provide a new tool that may allow to establish new FPT results for various combinatorial optimization problems and will be happy to learn of any such progress that may occur.

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Some Consequences
The n-fold product of $s_1 \times t$ block $A_1$ and $s_2 \times t$ block $A_2$ is

\[
A = \begin{pmatrix}
A_1 & A_1 & A_1 & \cdots & A_1 \\
A_2 & 0 & 0 & \cdots & 0 \\
0 & A_2 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & A_2
\end{pmatrix}^n.
\]
Consider $n$-fold integer programming over $s_i \times t$ blocks $A_i$ for $i=1,2$:

$$\max \{wx : Ax = b, \quad l \leq x \leq u, \quad x \in \mathbb{Z}^{nt}\}$$

$$A = \begin{pmatrix}
A_1 & A_1 & A_1 & \cdots & A_1 \\
A_2 & 0 & 0 & \cdots & 0 \\
0 & A_2 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & A_2 \\
\end{pmatrix}$$
N-Fold Integer Programming

Consider \( n \)-fold integer programming over \( s_i \times t \) blocks \( A_i \) for \( i=1,2 \):

\[
\max \{ wx : Ax = b, \ l \leq x \leq u, \ x \in Z^{nt} \}
\]

\[
A = \begin{pmatrix}
A_1 & A_1 & A_1 & \cdots & A_1 \\
A_2 & 0 & 0 & \cdots & 0 \\
0 & A_2 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & A_2
\end{pmatrix}
\]

**Arithmetic input:** \( n,t \) unary

**Numeric input:** \( w,b,l,u \) binary

**Parametric input:** \( s_i,a = |A|_\infty \)
N-Fold Integer Programming

Corollary:
Parameter tractable and strongly polynomial $f(s_i,a) \text{poly}(n,t)$

$$\max \{wx : Ax = b, \quad l \leq x \leq u, \quad x \in \mathbb{Z}^{nt}\}$$

$$A = \begin{pmatrix} A_1 & A_1 & A_1 & \cdots & A_1 \\ A_2 & 0 & 0 & \cdots & 0 \\ 0 & A_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & A_2 \end{pmatrix}$$

Arithmetic input: $n,t$ unary

Numeric input: $w,b,l,u$ binary

Parametric input: $s_i,a = |A|_\infty$
N-Fold Integer Programming

Corollary:
Parameter tractable and strongly polynomial $f(s_i,a) \text{poly}(n,t)$

Proof: Follows from Theorem since $d \leq td(A^T) \leq s_1 + s_2$

\[ A = \begin{pmatrix} A_1 & A_1 & A_1 & \cdots & A_1 \\ A_2 & 0 & 0 & \cdots & 0 \\ 0 & A_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & A_2 \end{pmatrix} \]

Arithmetic input: $n,t$ unary

Numeric input: $w,b,l,u$ binary

Parametric input: $s_i,a = |A|_\infty$
For optimization over $m_1 \times \cdots \times m_k \times n$ tables with given margins, studied by Motzkin in 1952, have:
Multiway Tables

For optimization over $m_1 \times \cdots \times m_k \times n$ tables with given margins, studied by Motzkin in 1952, have:

**Corollary:**

Parameter tractable and strongly polynomial $f(m_1, \ldots, m_k) poly(n)$
Multiway Tables

For optimization over $m_1 \times \cdots \times m_k \times n$ tables with given margins, studied by Motzkin in 1952, have:

**Corollary:**

Parameter tractable and strongly polynomial $f(m_1, \ldots, m_k) \text{poly}(n)$

**Proof:** It is an $n$-fold IP with $a=1$ and $s_1, s_2$ depending only on $m_1, \ldots, m_k.$
Multiway Tables

For optimization over \( m_1 \times \cdots \times m_k \times n \) tables with given margins, studied by Motzkin in 1952, have:

Corollary:

Parameter tractable and strongly polynomial \( f(m_1, \ldots, m_k)\text{poly}(n) \)

Proof: It is an \( n \)-fold IP with \( a=1 \) and \( s_1, s_2 \) depending only on \( m_1, \ldots, m_k \)

In Contrast - Universality of Three-Way Tables (De Loera-Onn):

Every integer program is one over \( 3 \times m \times n \) tables with line-sums
Tree-Fold Integer Programming

(Chen-Marx, SODA 2018): Tree-fold integer programs have a matrix with several blocks in tree structure, parameterized by $s_i, a = |A|_\infty$

$$A = \begin{bmatrix}
A_1 & A_1 & A_1 & A_1 & A_1 & A_1 & A_1 & A_1 & A_1 & A_1 \\
A_2 & A_2 & A_2 & A_2 & A_2 & A_2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_2 & A_2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_2 \\
A_3 & A_3 & A_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & A_3 & A_3 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & A_3 & A_3 & A_3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & A_3 & A_3 & A_3 & A_3 & 0 \\
A_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_4 \\
0 & A_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & A_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & A_4 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & A_4 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & A_4 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & A_4 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & A_4 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_4 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_4
\end{bmatrix}$$

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Tree-Fold Integer Programming

**Corollary:**
Parameter tractable and strongly polynomial $f(s_i,a)\ poly(n,t)$

*Arithmetic input: $n,t$ unary*

*Numeric input: $w,b,l,u$ binary*

*Parametric input: $s_i,a=|A|_\infty$*
Tree-Fold Integer Programming

Corollary:
Parameter tractable and strongly polynomial \( f(s_i,a) \text{ poly}(n,t) \)

Proof: Follows from Theorem since \( d \leq td(A^T) \leq \Sigma s_i \)

Arithmetic input: \( n,t \) unary

Numeric input: \( w,b,l,u \) binary

Parametric input: \( s_i,a=|A|_\infty \)
Multistage Stochastic Integer Programming

These programs have a matrix the transpose of a tree-fold matrix with several blocks in tree structure, parameterized by $s_i,a=|A|_\infty$.

$\begin{bmatrix}
A_1 & A_1 & A_1 & A_1 & A_1 & A_1 & A_1 & A_1 & A_1 & A_1 \\
A_2 & A_2 & A_2 & A_2 & A_2 & A_2 & A_2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & A_2 & A_2 & A_2 \\
A_3 & A_3 & A_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & A_3 & A_3 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & A_3 & A_3 & A_3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & A_3 & A_3 & A_3 & A_3 & 0 \\
A_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & A_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & A_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & A_4 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & A_4 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & A_4 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & A_4 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & A_4 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_4 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_4
\end{bmatrix}$

$A^T =$

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Multistage Stochastic Integer Programming

**Corollary:**
Parameter tractable and strongly polynomial $f(s_i,a) \text{poly}(n,t)$

*Arithmetic input:* $n,t$ unary

*Numeric input:* $w,b,l,u$ binary

*Parametric input:* $s_i,a = |A|_{\infty}$
Corollary:
Parameter tractable and strongly polynomial $f(s_i,a) \text{ poly}(n,t)$

Proof: Follows from Theorem since $d \leq td(A) \leq \sum s_i$

Arithmetic input: $n,t$ unary

Numeric input: $w,b,l,u$ binary

Parametric input: $s_i,a=|A|_\infty$
Proof Sketch
The Theorem

**Theorem** (Koutecký-Levin-Onn): Integer Programming is solvable in Parameter tractable and strongly polynomial time $f(a,d)\ poly(n)$

$$\text{IP: } \max \{ wx : Ax = b, \ l \leq x \leq u, \ x \in \mathbb{Z}^n \}$$

**Arithmetic input:** $n$ unary

**Numeric input:** $w,b,l,u$ binary

**Parametric input:** $a,d$ where $a=|A|_{\infty}$ and $d=\min\{td(A), \ td(A^T)\}$
Few Graver-Best Steps Suffice

IP: \( \max \{ wx : Ax = b, \ l \leq x \leq u, \ x \in \mathbb{Z}^n \} \)
Few Graver-Best Steps Suffice

IP: \[ \max \{ wx : Ax = b, \ l \leq x \leq u, \ x \in \mathbb{Z}^n \} \]

The Graver basis of an integer matrix \( A \) is the finite set \( G(A) \) of conformal-minimal nonzero integer vectors \( x \) satisfying \( Ax = 0 \)
Few Graver-Best Steps Suffice

\[ \text{IP: } \max \{ wx : Ax = b, \ l \leq x \leq u, \ x \in \mathbb{Z}^n \} \]

The \textit{Graver basis} of an integer matrix \( A \) is the finite set \( G(A) \) of \textit{conformal-minimal} nonzero integer vectors \( x \) satisfying \( Ax = 0 \)

\( x \) is conformal to \( y \) if \( x_iy_i \geq 0 \) (same orthant) and \( |x_i| \leq |y_i| \) for all \( i \)

Example: For \( A=(1 \ 2 \ 1) \) have \( G(A) = \pm \{(2 \ -1 \ 0), (1 \ 0 \ -1), (0 \ 1 \ -2), (1 \ -1 \ 1)\} \)
Few Graver-Best Steps Suffice

IP: \[ \max \{ wx : Ax = b, \ l \leq x \leq u, \ x \in \mathbb{Z}^n \} \]

The Graver basis of an integer matrix A is the finite set \( G(A) \) of conformal-minimal nonzero integer vectors \( x \) satisfying \( Ax = 0 \).

A Graver-best step for feasible \( x \) is \( h \) such that \( x+h \) is feasible and at least as good as any feasible \( x+cg \) with \( c \in \mathbb{Z} \) and \( g \in G(A) \).
Few Graver-Best Steps Suffice

**IP:** \[ \max \{ wx : Ax = b, \ l \leq x \leq u, \ x \in \mathbb{Z}^n \} \]

The *Graver basis* of an integer matrix \( A \) is the finite set \( G(A) \) of conformal-minimal nonzero integer vectors \( x \) satisfying \( Ax = 0 \).

A *Graver-best step* for feasible \( x \) is \( h \) such that \( x+h \) is feasible and at least as good as any feasible \( x+cg \) with \( c \in \mathbb{Z} \) and \( g \in G(A) \).

**Lemma:** IP is solvable using poly\((n)\) many Graver-best steps.
Few Graver-Best Steps Suffice

\[ \text{IP: } \max \{ wx : Ax = b, \ l \leq x \leq u, \ x \in \mathbb{Z}^n \} \]

The **Graver basis** of an integer matrix \( A \) is the finite set \( G(A) \) of conformal-minimal nonzero integer vectors \( x \) satisfying \( Ax = 0 \)

A **Graver-best step** for feasible \( x \) is \( h \) such that \( x + h \) is feasible and at least as good as any feasible \( x + cg \) with \( c \in \mathbb{Z} \) and \( g \in G(A) \)

**Lemma**: IP is solvable using poly\((n)\) many Graver-best steps

**Proof**: Using the integer Caratheodory theorem can show polynomial convergence. Using a theorem on proximity of the integer and real optimal solutions and reducing the numerical data \( w, b, l, u \) by applying a result of Frank-Tardos we can obtain the strongly polynomial bound
Graver Norm Bounds

IP: \[ \max \{ wx : Ax = b, \ l \leq x \leq u, \ x \in \mathbb{Z}^n \} \]

Parametric input: \(a, d\) where \(a = |A|_\infty\) and \(d = \min\{td(A), td(A^T)\}\)
**Graver Norm Bounds**

IP: \[ \max \{ w x : A x = b, \; l \leq x \leq u, \; x \in \mathbb{Z}^n \} \]

Parametric input: \( a, d \) where \( a = |A|_\infty \) and \( d = \min\{ \text{td}(A), \text{td}(A^T) \} \)

- **Primal case**: \( d = \text{td}(A) \)
- **Dual case**: \( d = \text{td}(A^T) \)

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**Graver Norm Bounds**

IP: \[ \max \{ wx : Ax = b, \; l \leq x \leq u, \; x \in \mathbb{Z}^n \} \]

Parametric input: \(a,d\) where \(a = |A|_\infty\) and \(d = \min\{td(A), \; td(A^T)\}\)

**Primal case:** \(d = td(A)\)  \hspace{1cm} **Dual case:** \(d = td(A^T)\)

**Lemma (primal):** With \(d = td(A)\) every \(h \in G(A)\) satisfies \(|h|_\infty \leq g_\infty(a,d)\)
**Graver Norm Bounds**

**IP:** \[ \max \{ wx : Ax = b, \ l \leq x \leq u, \ x \in \mathbb{Z}^n \} \]

Parametric input: \( a, d \) where \( a = |A|_\infty \) and \( d = \min \{ \text{td}(A), \text{td}(A^T) \} \)

Primal case: \( d = \text{td}(A) \) \quad Dual case: \( d = \text{td}(A^T) \)

**Lemma (primal):** With \( d = \text{td}(A) \) every \( h \in G(A) \) satisfies \( |h|_\infty \leq g_\infty(a,d) \)

**Lemma (dual):** With \( d = \text{td}(A^T) \) every \( h \in G(A) \) satisfies \( |h|_1 \leq g_1(a,d) \)
Graver Norm Bounds

\[ \text{IP: } \max \left\{ wx : Ax = b, \ l \leq x \leq u, \ x \in \mathbb{Z}^n \right\} \]

Parametric input: \( a,d \) where \( a = |A|_\infty \) and \( d = \min\{ \text{td}(A), \text{td}(A^T) \} \)

Primal case: \( d = \text{td}(A) \) \quad Dual case: \( d = \text{td}(A^T) \)

Lemma (primal): With \( d = \text{td}(A) \) every \( h \in G(A) \) satisfies \( |h|_\infty \leq g_\infty(a,d) \)

\[ \] 
Lemma (dual): With \( d = \text{td}(A^T) \) every \( h \in G(A) \) satisfies \( |h|_1 \leq g_1(a,d) \)

Extend and improve results of Aschenbrenner-Hemmecke and Chen-Marx

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Finding Graver-Best Steps: Primal Case

Let $x$ be a feasible point of the integer program

$$\text{IP}: \quad \max \left\{ wx : Ax = b, \ l \leq x \leq u, \ x \in \mathbb{Z}^n \right\}$$
Finding Graver-Best Steps: Primal Case

Let $x$ be a feasible point of the integer program

$$\text{IP: } \max \{ wx : Ax = b, \ l \leq x \leq u, \ x \in \mathbb{Z}^n \}$$

$a = |A|_{\infty}$ and $d = \text{td}(A)$ so $|h|_{\infty} \leq g = g_{\infty}(a, d)$ for all $h \in G(A)$
Finding Graver-Best Steps: Primal Case

Let $x$ be a feasible point of the integer program

$$\text{IP: } \max \{ wx : Ax = b, \ l \leq x \leq u, \ x \in \mathbb{Z}^n \}$$

$a = |A|_\infty$ and $d = \text{td}(A)$ so $|h|_\infty \leq g = g_\infty(a,d)$ for all $h \in G(A)$

1. Compute a set of $O(gn)$ potential positive integer step sizes $c$
Finding Graver-Best Steps: Primal Case

Let \( x \) be a feasible point of the integer program

\[
\text{IP:} \quad \max \{ \ w x : A x = b, \ l \leq x \leq u, \ x \in \mathbb{Z}^n \} \]

\( a=|A|_{\infty} \) and \( d=td(A) \) so \( |h|_{\infty} \leq g=g_{\infty}(a,d) \) for all \( h \in G(A) \)

1. Compute a set of \( O(gn) \) potential positive integer step sizes \( c \)

2. For each size \( c \) solve in time \( g^{O(d)}n \) the program below recursively on decomposition of matrices with bounded primal tree-depth

\[
\max \{ \ wch : A h = 0, \ l \leq x+ch \leq u, \ |h|_{\infty} \leq g, \ h \in \mathbb{Z}^n \} \]
Finding Graver-Best Steps: Primal Case

Let \( x \) be a feasible point of the integer program

\[
\text{IP: } \max \{ wx : Ax = b, \ l \leq x \leq u, \ x \in \mathbb{Z}^n \}
\]

\( a=|A|_\infty \) and \( d=td(A) \) so \( |h|_\infty \leq g=g_\infty(a,d) \) for all \( h \in G(A) \)

1. Compute a set of \( O(gn) \) potential positive integer step sizes \( c \)

2. For each size \( c \) solve in time \( g^{O(d)}n \) the program below recursively on decomposition of matrices with bounded primal tree-depth

\[
\max \{ wch : Ah = 0, \ l \leq x+ch \leq u, \ |h|_\infty \leq g, \ h \in \mathbb{Z}^n \}
\]

3. Return the best \( ch \) over all these programs as a Graver-best step
Finding Graver-Best Steps: Dual Case

Let $x$ be a feasible point of the integer program

$$\text{IP: } \max \{ wx : Ax = b, \ l \leq x \leq u, \ x \in \mathbb{Z}^n \}$$
Finding Graver-Best Steps: Dual Case

Let $x$ be a feasible point of the integer program

$$\text{IP: } \max \{ wx : Ax = b, \ l \leq x \leq u, \ x \in \mathbb{Z}^n \}$$

$a = |A|_{\infty}$ and $d = \text{td}(A^T)$ so $|h|_1 \leq g = g_1(a, d)$ for all $h \in G(A)$
Finding Graver-Best Steps: Dual Case

Let $x$ be a feasible point of the integer program

$$\text{IP: } \max \{ wx : Ax = b, \ l \leq x \leq u, \ x \in \mathbb{Z}^n \}$$

$a = |A|_{\infty}$ and $d = \text{td}(A^T)$ so $|h|_1 \leq g = g_1(a,d)$ for all $h \in G(A)$

1. Compute a set of $O(gn)$ potential positive integer step sizes $c$
Finding Graver-Best Steps: Dual Case

Let $x$ be a feasible point of the integer program

\[
\text{IP: } \quad \max \{ wx : Ax = b, \; l \leq x \leq u, \; x \in \mathbb{Z}^n \}
\]

$a = |A|_\infty$ and $d = td(A^T)$ so $|h|_1 \leq g = g_1(a,d)$ for all $h \in G(A)$

1. Compute a set of $O(gn)$ potential positive integer step sizes $c$

2. For each $c$ solve in time $(ag)^{O(d)n}$ the program below recursively on decomposition of matrices with bounded dual tree-depth

\[
\max \{ wch : Ah = 0, \; l \leq x + ch \leq u, \; |h|_1 \leq g, \; h \in \mathbb{Z}^n \}
\]
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Finding Graver-Best Steps: Dual Case

Let $x$ be a feasible point of the integer program

$$\text{IP: } \max \{ wx : Ax = b, \ l \leq x \leq u, \ x \in \mathbb{Z}^n \}$$

For each $c$ solve in time $(ag)^{O(d)n}$ the program below recursively on decomposition of matrices with bounded dual tree-depth

$$\max \{ wch : Ah = 0, \ l \leq x + ch \leq u, \ |h|_1 \leq g , h \in \mathbb{Z}^n \}$$

Return the best $ch$ over all these programs as a Graver-best step

1. Compute a set of $O(gn)$ potential positive integer step sizes $c$

2. For each $c$ solve in time $(ag)^{O(d)n}$ the program below recursively on decomposition of matrices with bounded dual tree-depth

3. Return the best $ch$ over all these programs as a Graver-best step
Concluding the Proof

**Theorem** (Koutecký, Levin, Onn): Integer Programming is solvable in parameter tractable and strongly polynomial time $f(a,d)\text{poly}(n)$

where $a=|A|_\infty$ and $d=\min\{\text{td}(A), \text{td}(A^T)\}$

**Proof**: Each of $\text{poly}(n)$ Graver-best steps is found in $f(a,d)\text{poly}(n)$ time
References

A Parameterized Strongly Polynomial Algorithm for Block Structured Integer Programs

Martin Koutecký, Asaf Levin, Shmuel Onn, ICALP 2018

An Algorithmic Theory of Integer Programming in Variable Dimension

Background in my Book Available Online on my Homepage