Hypergraphic Degree Sequences are Hard

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A $k$-hypergraph on $[n]$ is a subset $H \subseteq \{0,1\}_k^n := \{x \in \{0,1\}^n : \|x\|_1 = k\}$. The degree sequence of $H$ is the vector $d = \sum H := \sum \{x : x \in H\}$. We consider the following decision problem: given $k$ and $d \in \mathbb{Z}_+^n$, is $d$ the degree sequence of some hypergraph $H \subseteq \{0,1\}_k^n$? For $k = 2$ (graphs) the celebrated work of Erdős and Gallai (in their 1960 paper [3]) implies that $d$ is a degree sequence of a graph if and only if the inequalities $\sum_{i=1}^l d_i - \sum_{i=l+1}^n d_i \leq j(l-1)$ hold for $1 \leq j \leq l \leq n$, yielding a polynomial time algorithm. For $k = 3$ the problem was raised over 30 years ago by Colbourn, Kocay and Stinson (Problem 3.1 in their 1986 paper [1]) and was solved by Deza, Levin, Meesum and Onn (in their 2018 paper [2]). Here is the statement and the short proof.

**Theorem:** It is NP-complete to decide if $d \in \mathbb{Z}_+^n$ is the degree sequence of a 3-hypergraph.

**Proof.** The problem is in NP since if $d$ is a degree sequence then a hypergraph $H \subseteq \{0,1\}_3^n$ of cardinality $|H| \leq \binom{n}{3} = O(n^3)$ can be exhibited and $d = \sum H$ verified in polynomial time.

Let $1$ be the all-ones vector. We consider the following three decision problems.

1. Given $a \in \mathbb{Z}_+^n$, $b \in \mathbb{Z}_+$ with $31a = nb$, is there $F \subseteq \{x \in \{0,1\}_3^n : ax = b\}$ with $\sum F = 1$ ?
2. Given $w \in \mathbb{Z}_n$, $c \in \mathbb{Z}_+^n$ with $wc = 0$, is there $G \subseteq \{x \in \{0,1\}_3^n : wx = 0\}$ with $\sum G = c$ ?
3. Given $d \in \mathbb{Z}_+^n$, is there $H \subseteq \{0,1\}_3^n$ with $\sum H = d$ ?

Problem (1) is the so-called $3$-partition problem which is known to be NP-complete [4].

First we reduce (1) to (2). Given $a, b$ with $31a = nb$, let $w := 3a - b1$ and $c := 1$. Then $wc = 0$. Now, for any $x \in \{0,1\}^n_3$ we have $wx = 3ax - b1x = 3(ax - b)$ so $x$ satisfies $ax = b$ if and only if $wx = 0$. So the answer to (1) is YES if and only if the answer to (2) is YES.

Second we reduce (2) to (3). Given $w, c$, with $wc = 0$, define $d := c + \sum S_+$, where $S_\sigma := \{x \in \{0,1\}_3^n : \text{sign}(wx) = \sigma\}$ for $\sigma = -, 0, +$. Suppose there is a $G \subseteq S_0$ with $\sum G = c$. Then $H := G \cup S_+$ satisfies $\sum H = d$. Suppose there is an $H \subseteq \{0,1\}_3^n$ with $\sum H = d$. Then $\sum S_+ = w(c + \sum S_+) = w \sum H = w \sum (H \cap S_+) + w \sum (H \cap S_0) + w \sum (H \cap S_-)$ which implies $H \cap S_- = \emptyset$ and $H \cap S_+ = S_+$. Therefore $G := H \cap S_0$ satisfies $\sum G = \sum H - \sum S_+ = c$. So the answer to (2) is YES if and only if the answer to (3) is YES. \[ \square \]

**References**


