Hypergraphic Degree Sequences are Hard

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A *k-hypergraph on* $[n]$ is a subset $H \subseteq \{0,1\}^n_k := \{x \in \{0,1\}^n : \|x\|_1 = k\}$. The *degree sequence of* H is the vector $d = \sum H := \sum\{x : x \in H\}$. We consider the following decision problem: given k and $d \in \mathbb{Z}^n_+$, is d the degree sequence of some hypergraph $H \subseteq \{0,1\}^n_k$? For $k = 2$, that is, for graphs, the celebrated work of Erdős and Gallai [3, 1960] implies that d is a degree sequence of a graph if and only if $\sum d_i$ is even and, permuting d so that $d_1 \geq \cdots \geq d_n$, the inequalities $\sum_{i=1}^j d_i - \sum_{i=l+1}^n d_i \leq j(l-1)$ hold for $1 \leq j \leq l \leq n$, yielding a polynomial time algorithm. For $k = 3$ the problem has been open ever since, was formally raised over 30 years ago by Colbourn, Kocay, and Stinson [1, 1986, Problem 3.1], and was recently solved by Deza, Levin, Meesum, and Onn [2, 2018]. Here is the statement and its short proof.

Theorem: It is NP-complete to decide if $d \in \mathbb{Z}^n_+$ is the degree sequence of a 3-hypergraph.

Proof. The problem is in NP since if d is a degree sequences then a hypergraph $H \subseteq \{0,1\}^n_3$ of cardinality $|H| \leq \binom{n}{3} = O(n^3)$ can be exhibited and $d = \sum H$ verified in polynomial time.

We consider the following three decision problems where $\mathbf{1}$ is the all-ones vector.

(1) Given $a \in \mathbb{Z}^n_+$, $b \in \mathbb{Z}_+$ with $3\mathbf{1}a = nb$, is there $F \subseteq \{x \in \{0,1\}^n_3 : ax = b\}$ with $\sum F = \mathbf{1}$?
(2) Given $w \in \mathbb{Z}^n$, $c \in \mathbb{Z}^n_+$ with $wc = 0$, is there $G \subseteq \{x \in \{0,1\}^n_3 : wx = 0\}$ with $\sum G = c$?
(3) Given $d \in \mathbb{Z}^n_+$, is there $H \subseteq \{0,1\}^n_3$ with $\sum H = d$?

Problem (1) is the so-called 3*-partition problem* which is known to be NP-complete [4].

First we reduce (1) to (2). Given a, b with $3\mathbf{1}a = nb$, let $w := 3a - b\mathbf{1}$ and $c := \mathbf{1}$. Then $wc = 0$. Now, for any $x \in \{0,1\}^n_3$ we have $wx = 3ax - b\mathbf{1}x = 3(ax - b)$ so x satisfies $ax = b$ if and only if $wx = 0$. So the answer to (1) is YES if and only if the answer to (2) is YES.

Second we reduce (2) to (3). Given w, c, with $wc = 0$, define $d := c + \sum S_+$, where $S_\sigma := \{x \in \{0,1\}^n_3 : \mathrm{sign}(wx) = \sigma\}$ for $\sigma = -, 0, +$. Suppose there is a $G \subseteq S_0$ with $\sum G = c$. Then $H := G \cup S_+$ satisfies $\sum H = d$. Suppose there is an $H \subseteq \{0,1\}^n_3$ with $\sum H = d$. Then

$$w \sum S_+ \;=\; w(c + \sum S_+) \;=\; w \sum H \;=\; w \sum (H \cap S_-) + w \sum (H \cap S_0) + w \sum (H \cap S_+)$$

which implies $H \cap S_- = \emptyset$ and $H \cap S_+ = S_+$. Therefore $G := H \cap S_0$ satisfies $\sum G = \sum H - \sum S_+ = c$. So the answer to (2) is YES if and only if the answer to (3) is YES. □

References

[1] Colbourn, C.J., W.L. Kocay, W.L., Stinson, D.R.: Some NP-complete problems for hypergraph degree sequences. Discrete Applied Mathematics 14:239–254 (1986)

[2] Deza, A., Levin, A., Meesum, S.M., Onn, S.: Optimization over degree sequences. SIAM Journal on Discrete Mathematics 32:2067–2079 (2018)

[3] Erdős, P., Gallai, T.: Graphs with prescribed degrees of vertices (in Hungarian). Matematikai Lopak 11:264–274 (1960)

[4] Garey, M.R., Johnson, D.S.: Computers and Intractability. Freeman (1979)