Communication

On the diameter of convex polytopes

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In this communication we prove a theorem on the diameter of convex polytopes, which generalizes a theorem of Naddef [4]. For a recent survey on diameters of polytopes and the related d-step conjecture, the reader can consult [3].

For a polytope $P$ in $\mathbb{R}^n$, we denote by $\text{dim}(P)$ its affine dimension, by $\text{vert}(P)$ its vertex set, by $d_P(u, v)$ the distance between two vertices on its 1-skeleton, and by $\delta(P)$ its diameter. The symmetrization of $P$ is denoted by $P - P = \{x - y: x, y \in P\}$. The linear span of a subset $S$ of $\mathbb{R}^n$ is denoted by $\text{lin}(S)$, and the orthogonal complement of a subspace $L$ by $L^\perp$. For a linear functional $w$ on $\mathbb{R}^n$, denote by $k(P, w)$ the number of distinct values attained by the vertices of $P$ under $w$, minus one, i.e. $k(P, w) = |\left\{ \langle w, v \rangle: v \in \text{vert}(P) \right\}| - 1$, and by $F_-(P, w)$ and $F_+(P, w)$ denote the faces of $P$ containing those vertices attaining the minimum and maximum values under $w$, respectively.

**Theorem.** Let $P$ be a $d$-polytope in $\mathbb{R}^n$, let $\{w_1, \ldots, w_m\}$ be a subset of $\mathbb{R}^n$ whose linear span contains $P - P$, and assume $k(P, w_1) \geq \cdots \geq k(P, w_m)$. Then $\delta(P) \leq \sum_{i=1}^{d} k(P, w_i)$.

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Proof. By induction on $d = \dim(P)$. The bound trivially holds for $d = 0$. Assume then, that $d \geq 1$ and the bound is valid for all dimensions smaller than $d$.

Now, $P - P \subset L = \linearSpan((w_1, \ldots, w_m))$ implies $P - P \subset \sum_{i=1}^m w_i^T$, so $k(P, w_i) \geq 1$. Thus, $F_-(P, w_i)$ and $F_+(P, w_i)$ are distinct proper faces of $P$. Pick two vertices $u$ and $v$ of $P$ such that $\delta(P) = d_p(u, v)$. It is well known that there exists a (possibly empty) path on the 1-skeleton of $P$ from $u$ to some vertex $u_+ \in F_+(P, w_i)$, where the vertices along the path attain strictly increasing values under $w_1$ (see e.g. [1]). Similarly, there exists a strictly increasing path from $v$ to some $v_+ \in F_+(P, w_i)$, and strictly decreasing paths from $u$ and $v$ to some vertices $u_-$ and $v_-$ in $F_-(P, w_i)$, respectively.

Clearly $d_p(u-, u) + d_p(u, u+) \leq k(P, w_i)$ and $d_p(v-, v) + d_p(v, v+) \leq k(P, w_i)$, so

$$\min\{(d_p(u-, u) + d_p(v-, v)), (d_p(u, u+) + d_p(v, v+))\} \leq \frac{(d_p(u-, u) + d_p(v-, v)) + (d_p(u, u+) + d_p(v, v+))}{2} = \frac{(d_p(u-, u) + d_p(v, u_+)) + (d_p(v-, v) + d_p(v, v+))}{2} \leq \frac{k(P, w_i) + k(P, w_i)}{2} = k(P, w_i).$$

Without loss of generality assume $d_p(u-, u) + d_p(v-, v) \leq k(P, w_i)$, and let $F = F_-(P, w_i)$. Since every edge of $F$ is also an edge of $P$, we have

$$\delta(P) = d_p(u, v) \leq d_p(u, u_-) + d_p(u-, u) + d_p(v-, v) \leq k(P, w_i) + \delta(F),$$

so we need to show that $\delta(F) \leq \sum_{i=2}^d k(P, w_i)$. Let $l = \dim(F) \leq d - 1$. If $l = 0$ then $\delta(F) = 0$ and we are done, so assume $l \geq 1$.

Now, $k(F, w_i) = 0$, so $F - F \subset w_i^T$. Thus, letting for $i = 1, \ldots, m - 1$

$$u_i = w_{i+1} - \frac{\langle w_1, w_{i+1} \rangle}{\langle w_1, w_1 \rangle} w_1$$

be the orthogonal projection of $w_{i+1}$ onto $w_1^T$, we have $k(F, u_i) = k(F, w_{i+1}) \leq k(P, w_{i+1})$, and $F - F \subset \linearSpan\{u_1, \ldots, u_{m-1}\}$. Let $\pi$ be a permutation of $\{1, \ldots, m - 1\}$ such that $k(F, u_{\pi(1)}) \geq \cdots \geq k(F, u_{\pi(m-1)})$. By induction,

$$\delta(F) \leq \sum_{i=1}^l k(F, u_{\pi(i)}) \leq \sum_{i=1}^{d-1} k(P, w_{\pi(i+1)}) \leq \sum_{i=2}^d k(P, w_i)$$

which completes the proof. □

Applying the theorem with the standard basis $\{e_1, \ldots, e_n\}$ of $\mathbb{R}^n$, we obtain the following bound on the diameter of an integer polytope.
Corollary. If $V \subseteq \{0, 1, \ldots, k\}^n$ then $\delta(\text{conv}(V)) \leq k \cdot \dim(\text{conv}(V))$.

Finally, we refine the theorem as follows. Let $P$ be a $d$-polytope in $\mathbb{R}^n$, where, to simplify notation, we assume $d \geq 1$, and let $\{w_1, \ldots, w_m\}$ be a subset of $\mathbb{R}^n$ whose linear span contains $L = \text{lin}(P - P)$. Assume that $k(P, w_1) \leq \cdots \leq k(P, w_m)$. For $i = 1, \ldots, m$ let $u_i$ be the orthogonal projection of $w_i$ onto $L$, so $k(P, u_i) = k(P, w_i)$, and let $U = \{u_1, \ldots, u_m\}$, so $L = \text{lin}(U)$. Applying the theorem with any basis of $U$, i.e. a subset $\{u_1, \ldots, u_d\}$ of $U$ which is a basis of $L$, an upper bound $\delta(P) \leq \sum_{i=1}^{d} k(P, u_i)$ is obtained.

Appealing to the well-known validity of the so-called greedy algorithm for finding a minimal weight basis of a matroid ([5,2]), we find that the basis of $U$ yielding the best upper bound is the lexicographically first one, i.e. the basis $\{u_{i_1}, \ldots, u_{i_d}\}$, where $i_1 < \cdots < i_d$ and, for $j = 1, \ldots, d$, the index $i_j$ is the smallest such that $u_{i_j}$ is nonzero and is not contained in $\text{lin}(\{u_r ; r < j\})$.

We thus have the following refined version of the theorem.

Theorem. Let $P$ be a $d$-polytope in $\mathbb{R}^n$ where $d \geq 1$, let $\{w_1, \ldots, w_m\}$ be a subset of $\mathbb{R}^n$ whose linear span contains $P - P$, and assume $k(P, w_1) \leq \cdots \leq k(P, w_m)$. For $i = 1, \ldots, m$, let $u_i$ be the orthogonal projection of $w_i$ onto $\text{lin}(P - P)$, and let $\{u_{i_1}, \ldots, u_{i_d}\}$ be the lexicographically first basis of $\{u_1, \ldots, u_m\}$. Then $\delta(P) \leq \sum_{i=1}^{d} k(P, u_i)$.

References