Some efficiently solvable problems over integer partition polytopes

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\textbf{ABSTRACT}

The integer partition polytope $P_n$ is the convex hull of all integer partitions of $n$. We provide a novel extended formulation of $P_n$, and use it to show that the extremality, adjacency, and separation problems over $P_n$ can be solved by linear programming without the ellipsoid method.

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1. Introduction

An \textit{integer partition} of $n$ is a nonincreasing sequence of positive integers summing up to $n$. Integer partitions play an important role in a variety of areas of mathematics, in statistical mechanics and theoretical physics, see [1,14] and references therein.

We can and will identify here each partition of $n$ with a nonnegative integer vector $x \in \mathbb{Z}^n_+$, where $x_k$ counts the number of times $k$ appears in the sum. For instance, the partition $1 + 1 + 3 = 5$ corresponds to the vector $x = (2, 0, 1, 0, 0)$. Let $T_n := \{x \in \mathbb{Z}^n_+ : \sum_{k=1}^n kx_k = n\}$ be the set of integer partitions of $n$. The \textit{integer partition polytope} is defined to be $P_n := \text{conv}(T_n)$, the convex hull of all integer partitions of $n$.

The polyhedral approach in the integer partition theory gives rise to many appealing questions. Introducing the polytope of integer partitions revealed the previously unknown geometric structure of the set of partitions of any integer. It demonstrated existence of new classes of partitions, in particular \textit{extreme integer partitions}, which are the vertices of the partition polytopes. Each partition of $n$ is a convex combination of some vertices of $P_n$, thus the set of vertices of $P_n$ forms a basis for the set of partitions of $n$. This engenders a special interest in vertices.

As for every polytope, the other key elements of $P_n$ are its facets. They have been studied in [11] and the vertices were studied in [12,13]. However, no combinatorial characterizations of vertices or facets of $P_n$ are available as yet.

In this regard, the following question arises: what is the computational complexity of the problem

\begin{itemize}
  \item Extremality: for integer partition $x \in T_n$, decide if it is extreme on $P_n$.
  
  It is easy to see that this problem is in co-NP: if $x$ is not extreme then, by Caratheodory's theorem, one can exhibit $2 \leq r \leq n + 1$ affinely independent integer partitions $x^1, \ldots, x^r \in T_n$ such that in the unique solution to $\sum \lambda_i x^i = x, \sum \lambda_i = 1$,
\end{itemize}

where $r = \text{dim}(P_n)$.
$\lambda_i$ are positive. However, the number of all partitions is exponential in $n$ (see [1]), and therefore finding such $x_i$ by exhaustive search takes exponential time.

The main purpose of this paper is to prove that the extremality problem can be decided in polynomial time with the use of linear programming techniques. The lift and project method that we use to solve this problem occurred to be powerful enough to prove polynomial decidability by linear programming of three more problems, which are canonical in combinatorial optimization. These are

- Adjacency: for extreme partitions $x, y \in T_n$, decide if they are adjacent on $P_n$.
- Separation: for $x \in \mathbb{R}^n$, find $h \in \mathbb{R}^n$ with $h^T x > h^T y \forall y \in P_n$ or assert $x \in P_n$.
- Optimization: for $c \in \mathbb{R}^n$, find $x^* \in P_n$ which attains $\min\{c^T x : x \in P_n\}$.

Since $P_n$ is a knapsack type polytope, the optimization problem over $P_n$ can be solved in polynomial time by dynamic programming, see, for example, [6]. This implies that all other problems listed above can also be solved in polynomial time by multiple applications of the ellipsoid method [4]. However, the ellipsoid method is a very heavy tool with a very large running time. In this paper we show that all problems can in fact be solved in polynomial time using linear programming without the ellipsoid method. We prove the following theorem, which summarizes our results about the four problems above in a rounded and symmetric form:

**Theorem 1.1.** The extremality, adjacency, separation, and optimization problems over $P_n$ can all be solved using linear programming without the ellipsoid method.

A key ingredient in the proof of this theorem is the construction of a novel extended formulation for $P_n$, namely, a polynomial time constructible polytope $Q_n$ of size polynomial in $n$ such that $P_n$ is the projection of $Q_n$. Thus, the second main outcome of this paper is the following result which is of interest in its own right.

**Theorem 1.2.** There exists a polynomial time constructible polytope $Q_n$ of size polynomial in $n$ providing an extended formulation for $P_n$, that is, satisfying

$$P_n = \{ x : \exists y \ (x, y) \in Q_n \}.$$

We believe that the advantage of our approach and of the extended formulation in Theorem 1.2 is that, dealing with the set of integer partitions as a polytope and as a projection of another polytope of a polynomial sized description, helps advance the understanding of the geometric structure of the set of partitions, and opens possibilities to apply well-developed polyhedral methods to the study of partitions.

In Sections 2 and 3 we prove Theorems 1.2 and 1.1, respectively. We conclude in Section 4 with some final remarks.

2. **Lifting the integer partition polytope**

In this section we construct a polytope $Q_n$ given by an explicit inequality description of size polynomial in $n$, such that $P_n$ is a projection of $Q_n$. As the first step of this construction we construct a digraph $G_n$ with two distinguished vertices $v_0^n, v_n^n$ such that there is a bijection between integer partitions $x \in T_n$ and $v_0^n - v_n^n$ dipaths in $G_n$. This construction is inspired by $P_n$ being a knapsack type polytope and the connection between the knapsack problem and dynamic programming [6].

The digraph $G_n = (V, E)$ has the vertex set $V = V^0 \cup V^1 \cup \ldots \cup V^n$ with $V^0 := \{ v_0^n \}, V^k := \{ v_k^1, v_k^2, \ldots, v_k^n \}$ for $k = 1, \ldots, n − 1$, and $V^n := \{ v_n^n \}$. There are arcs only between consecutive sets $V^{k−1}, \ V^k$, where an arc $(v_{i−1}^k, v_i^k)$ is included into $E$ if and only if $\frac{i−1}{k} \in \mathbb{Z}_+$. Note that $|V| = \Theta((n^2)$ and $|E| = \Theta(n^3)$. Fig. 1 displays $G_6$ for example.

**Lemma 2.1.** Integer partitions $x \in T_n$ are in bijection with $v_0^n - v_n^n$ dipaths in $G_n$.

**Proof.** Given $x \in T_n$, consider any $k, 1 \leq k \leq n$; let $i(k) := \sum_{r=1}^{k−1} r x_r$, and let $j(k) := \sum_{r=1}^{k} r x_r$; include in the dipath the arc $(v_{i(k)}^k, v_{j(k)}^k)$, which exists in $G_n$ since $\frac{j(k)−i(k)}{k} \in \mathbb{Z}_+$. Since $i(1) = 0$ and $j(n) = n$, this results in a $v_0^n - v_n^n$ dipath. Conversely, given a $v_0^n - v_n^n$ dipath, consider any $k, 1 \leq k \leq n$; let $(v_{i(k)}^k, v_{j(k)}^k)$ be the unique arc from $V^{k−1}$ to $V^k$ on that dipath; set $x_k := \frac{j(k)−i(k)}{k} \in \mathbb{Z}_+$. Then $x \in T_n$ since $i(1) = 0, j(k) = i(k+1)$ for $k = 1, \ldots, n − 1$, and $j(n) = n$, and hence

$$\sum_{k=1}^{n} k x_k = \sum_{k=1}^{n} i(k) − i(1) = (j(n)−i(n)) + \sum_{k=1}^{n−1} i(k + 1) − i(k) = j(n)−i(1) = n. \quad \Box$$

Next we define two polytopes $D_n$ and $Q_n$ with certain properties such that $Q_n$ is a suitable lifting of $D_n$ and $P_n$ is a suitable projection of $Q_n$. This is inspired by the polyhedral methods for dynamic programming in [7].

We begin with $D_n$ which is a polytope with $0 − 1$ vertices standing in bijection with $v_0^n - v_n^n$ dipaths in $G_n$. For this, we assign to each arc $(v_{i−1}^k, v_i^k)$ in $G_n$ a corresponding $0 − 1$ variable $y_{i,j}^k$, and we arrange all the arc variables in a vector

$$y = \left( y_{i,j}^k : k = 1, \ldots, n, \frac{j−i}{k} \in \mathbb{Z}_+ \right) \in \{0, 1\}^{|E|}.$$
We define the polytope

\[ D_n := \{ y : 0 \leq y_{i,j} \leq 1, \sum_{j} y_{r,s} : v_{r}^{-1} \in \delta_{in}(v_{s}) \} - \sum_{t} y_{s,t} : v_{t}^{+1} \in \delta_{out}(v_{s}) \} = \{ y : Ay = a, 0 \leq y \leq 1 \} \]

for a suitable vector \( a \in \mathbb{Z}^{V} \), where \( A \) is the vertex–arc incidence matrix of \( G_n \), and where we use \( \delta_{in}(v_{s}) := \{ v \in V : (v, v_{s}) \in E \} \) and \( \delta_{out}(v_{s}) := \{ v \in V : (v_{s}, v) \in E \} \).

**Lemma 2.2.** Vertices of \( D_n \) are \( 0 \)–\( 1 \) and in bijection with \( v_{0} - v_{n} \) dipaths in \( G_n \).

**Proof.** Since \( A \) is the vertex–arc incidence matrix of a digraph, it is totally unimodular \([9]\) and so the polytope \( D_n \) is integer \([5]\) and its vertices are precisely all \( 0 \)–\( 1 \) vectors \( y \) in \( D_n \). These vectors correspond bijectively to \( v_{0} - v_{n} \) dipaths in \( G_n \), where an arc \((v_{i}^{k-1}, v_{i}^{k})\) is included in the dipath corresponding to \( y \) if and only if \( y_{i,j} = 1 \). \( \square \)

Next we define the polytope

\[ Q_n := \left\{ (x, y) : y \in D_n, x_k = \sum_{j} \left\{ \frac{j-i}{k} y_{i,j} : \frac{j-i}{k} \in \mathbb{Z}_{+} \right\}, k = 1, \ldots, n \right\} \]

\[ = \left\{ (x, y) : (0 A) (x \ y) = a, (I_n - B) (x \ y) = 0, 0 \leq y \leq 1 \right\} \]

for a suitable matrix \( B \). The following lemma characterizes its vertices.

**Lemma 2.3.** The polytopes \( Q_n, D_n \) are affinely isomorphic. For each vertex \((x, y)\) of \( Q_n \) we have that \( y \) is a vertex of \( D_n \) with \( x \in T_n \) the corresponding integer partition.

**Proof.** Clearly, \( x = By \) is uniquely determined by \( y \) and hence \( Q_n \) and \( D_n \) are isomorphic: there is a one-to-one correspondence between the vertices of \( Q_n \) and \( D_n \), and each vertex of \( Q_n \) is the lifting \((x, y) = (By, y)\) of some vertex \( y \) of \( D_n \). Now, consider any such vertex \( y \). By **Lemma 2.2**, it determines a \( v_{0} - v_{n} \) dipath in \( G_n \). Consider any \( 1 \leq k \leq n \). Let \((v_{i(k)}^{k-1}, v_{j(k)}^{k})\) be the unique arc from \( V^{k-1} \) to \( V^{k} \) on that dipath. By **Lemma 2.1**, \n
\[ y_{i,j}^{k} := \begin{cases} 1, & \text{if } i = i(k) \text{ and } j = j(k); \\ 0, & \text{otherwise}. \end{cases} \]

Therefore, by the definition of \( Q_n \),

\[ x_k = \sum_{j} \left\{ \frac{j-i}{k} y_{i,j}^{k} : \frac{j-i}{k} \in \mathbb{Z}_{+} \right\} = \frac{j(k) - i(k)}{k}. \]

So \( x \in T_n \) is the integer partition corresponding via the bijection of **Lemma 2.1** to the \( v_{0} - v_{n} \) dipath which corresponds to \( y \). \( \square \)

We can now prove **Theorem 1.2** showing that \( P_n \) is the projection of \( Q_n \) onto the \( x \) coordinates.
Theorem 1.2. There exists a polynomial time constructible polytope \( Q_n \) of size polynomial in \( n \) providing an extended formulation for \( P_n \), that is, satisfying
\[
P_n = \{ x : \exists y (x, y) \in Q_n \}.
\]

Proof. Let \( L_n := \{ x : \exists y (x, y) \in Q_n \} \) be the projection of \( Q_n \) to \( \mathbb{R}^n \) by erasing the \( y \) coordinates. For \( \text{vert}(L_n) \), the set of vertices of the polytope \( L_n \), by Lemma 2.3, we have \( \text{vert}(L_n) \subseteq T_n \subseteq L_n \), which implies
\[
L_n = \text{conv}(\text{vert}(L_n)) \subseteq \text{conv}(T_n) = P_n \subseteq \text{conv}(L_n) = L_n.
\]

Hence \( P_n = L_n \) is the projection of \( Q_n \).

Now, the polytope \( Q_n \) lives in dimension \( d := O(n^2) \) and is defined by a system of \( m := O(n^3) \) inequalities (including the bounds on the variables) with coefficients of size \( O(n) \) each. One can then easily see from the definitions of the digraph \( G_n \) and the polytopes \( D_n \) and \( Q_n \) that the matrices \( A \) and \( B \) and the vector \( a \) engaged there can be constructed in time polynomial in \( n \). \( \square \)

Before proceeding to the next section, we note that optimization over \( P_n \) can be done in strongly polynomial time as a shortest \( v_0^n - v_n^n \) dipath problem in \( G_n \). For this, given \( c \in \mathbb{R}^n \), assign to each arc \( (v_i, v_j) \) in \( G_n \) the length \( w_{ij}^k := |i - k|/c_k \).

Proposition 2.4. We can optimize over \( P_n \) in strongly polynomial time using \( O(n^3) \) real arithmetic operations by finding a shortest \( v_0^n - v_n^n \) dipath in the digraph \( G_n \).

Proof. By Lemma 2.1, the \( v_0^n - v_n^n \) dipaths are in bijection with integer partitions. The length of the \( v_0^n - v_n^n \) dipath corresponding to an integer partition \( x \in T_n \) is
\[
\sum_{k=1}^{n} |w_{(k, j)(k)}^k| = \sum_{k=1}^{n} \frac{j(k) - i(k)}{k} c_k = \sum_{k=1}^{n} c_k x_k = c^t x.
\]

So, solving the optimization problem \( \min\{ c^t x : x \in T_n \} = \min\{ c^t x : x \in P_n \} \) reduces to finding a shortest \( v_0^n - v_n^n \) dipath in \( G_n \). Since all such dipaths have the same number \( n \) of arcs, adding to all arc lengths the same suitable positive integer if necessary, we may and do assume that all lengths are nonnegative. Then finding a shortest dipath can be done in strongly polynomial time using a number of real arithmetic operations dominated by the number \( O(n^3) \) of arcs of \( G_n \), see, for example, [10]. \( \square \)

While the above algorithm is conceptually simple, we note that, very recently, [3] showed that optimization over \( P_n \) can be done faster yet, using \( O(n^2) \) operations.

3. Extremality, adjacency, and separation

We begin with a sequence of assertions regarding the solution of our algorithmic problems over the projections of polytopes which are given by linear inequalities.

Lemma 3.1. Affinely independent points \( x^0, x^1, \ldots, x^d \) in a polytope \( P \subseteq \mathbb{R}^n \) do not span a q-face of \( P \) if and only if there are points \( u, v \in P \) and \( 1 \leq k \leq n \) such that:
1. \( \frac{1}{q+1}(x^0 + x^1 + \cdots + x^d) = \frac{1}{q+1}(u + v) \);
2. \( (x^i - x^0)^t(u - v) = 0 \), \( i = 1, \ldots, q \);
3. \( u_k < \frac{1}{q+1}(x^0_k + x^1_k + \cdots + x^d_k) \) for some \( 1 \leq k \leq n \).

Proof. The claim being obvious for \( q = n \), so we may and do assume \( 0 \leq q < n \). Let \( b := \frac{1}{q+1}(x^0 + \cdots + x^d) \) and \( F := \text{aff.hull}[x^0, \ldots, x^d] \cap P \). Assume first that \( F \) is a face. Then there is a hyperplane \( H \) supporting \( P \) at \( F \), i.e. with \( P \) on one side of \( H \) and \( H \cap P = F \). Suppose \( u, v \in P \) satisfy (1), (2). Then \( u, v \) lie on the same side of \( H \), which together with \( b \in B \) and \( b = \frac{1}{2}(u + v) \) imply that \( u, v \) must in fact lie on \( H \). Therefore, \( u, v \in \text{aff.hull}[x^0, \ldots, x^d] \) and \( u - v \in \text{lin.hull}[x^1 - x^0, \ldots, x^q - x^0] \). By (2), also \( u - v \in \text{lin.hull}[x^1 - x^0, \ldots, x^q - x^0] \), and hence \( u - v = 0 \). Therefore (1) implies \( b = u \), so (3) does not hold for any \( k \).

Next, assume that \( F \) is not a face. Let \( G \) be a face of \( P \) of minimal affine dimension \( k \), \( q < k \leq n \), that contains \( F \). Consider the polytope \( G \) in the space \( \mathbb{R}^k \), where it actually lies, and prove the claim: every hyperplane \( H \) in \( \mathbb{R}^k \) through \( x^0, x^1, \ldots, x^q \) partitions \( G \) into two parts. Assume, on the contrary, that for some \( H, G \) lies on one side of \( H \). Then \( H \cap G \) is a face of \( G \) and \( F \subseteq H \cap G \). However \( H \cap G \) is also a face of \( P \) and its affine dimension is less than \( k \). This contradicts the minimality of \( G \). So, for every hyperplane \( H \subseteq \mathbb{R}^k \) through \( x^0, x^1, \ldots, x^q \), there are points in \( G \) on both sides of \( H \). Take any such hyperplane \( H \). The claim proved above implies that \( b = \frac{1}{q+1}(x^0 + \cdots + x^d) \), belonging to the relative interior of \( F \) and lying in \( H \), is an interior point in \( G \). Now take any line \( L \) through \( b \) which is orthogonal to \( \text{lin.hull}[x^1 - x^0, \ldots, x^q - x^0] \) in \( \mathbb{R}^k \). Since \( L \) contains an interval in \( G \) in each side of \( H \), we can choose two distinct points \( u, v \in G \) on \( L \) such that \( b = \frac{1}{2}(u + v) \). Then these points satisfy (1), (2). Further, since \( u \neq v \), either \( u_k < b_k \) or \( v_k < b_k \) for some \( k \), and without loss of generality we may assume the former, so (3) also holds. \( \square \)
Lemma 3.2. Let \( P = \{ x : \exists y (x, y) \in Q \} \) be the projection of a polytope \( Q \) given by
\[
Q = \left\{ (x, y) : \begin{bmatrix} G & H \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \leq g \right\}.
\]
Then deciding if affinely independent points \( x^0, x^1, \ldots, x^d \) in \( P \) span a \( q \)-face of \( P \) can be done in polynomial time by linear programming. In particular, the extremality and adjacency problems over \( P \) can be decided in polynomial time by linear programming.

Proof. Suppose \( P \subset \mathbb{R}^n \) and \( Q \subset \mathbb{R}^m \) with \( m \) polynomial in \( n \). For \( k = 1, \ldots, n \), solve the following linear program in two copies of variables \( u, v \in \mathbb{R}^n, y, z \in \mathbb{R}^m \),
\[
\min \left\{ u_k : \frac{1}{2} (u + v) = \frac{1}{q+1} (x^0 + x^1 + \cdots + x^d), (x^i - x^0)^T(u - v) = 0, i = 1, \ldots, q, \begin{bmatrix} u \\ v \end{bmatrix} \in Q, \begin{bmatrix} y \\ z \end{bmatrix} \in Q \right\}.
\]
If any of these programs has optimal value \( u_k^* = \frac{1}{q+1} (x^0_k + x^1_k + \cdots + x^d_k) \), then, by Lemma 3.1, \( x^0, x^1, \ldots, x^d \) do not determine a \( q \)-face of \( P \), whereas if \( u_k^* = \frac{1}{q+1} (x^0_k + x^1_k + \cdots + x^d_k) \) for all \( k \), then \( x^0, x^1, \ldots, x^d \) determine such a face. Extremality and adjacency are solved as the special cases of \( q = 0 \) and \( q = 1 \). \( \square \)

Lemma 3.3. Let \( P = \{ x : \exists y (x, y) \in Q \} \) be the projection of a polytope \( Q \) given by
\[
Q = \left\{ (x, y) : \begin{bmatrix} G & H \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \leq g \right\}.
\]
Then separation over \( P \) can be done in polynomial time by linear programming.

Proof. We have that \( x \in P \) if and only if there exists \( y \) such that \( Hy \leq g - Gx \). By the Farkas lemma in linear programming (see, for example, [9]), either there exists such \( y \) or there exists \( z \geq 0 \) such that \( z^T H = 0 \) and \( z^T (g - Gx) \leq -1 \) (but not both). Using linear programming we can in polynomial time either find such \( y \) and conclude that \( x \in P \) or find such \( z \). Suppose we find such \( z \). For any \( \tilde{x} \in P \) there exists \( \tilde{y} \) such that \( H \tilde{y} \leq g - G\tilde{x} \) and therefore any \( \tilde{z} \geq 0 \) which satisfies \( \tilde{z}^T H = 0 \) must also satisfy \( \tilde{z}^T (g - G\tilde{x}) \geq 0 \). Therefore, for every \( \tilde{x} \in P \) we have
\[
(z^T G)x > z^T g \geq (z^T G)\tilde{x}
\]
so \( h^T := z^T G \) separates \( x \) from \( P \). This solves the separation problem over \( P \). \( \square \)

The simple proof of the following last lemma is included for completeness.

Lemma 3.4. Let \( P = \{ x : \exists y (x, y) \in Q \} \) be the projection of a polytope \( Q \) given by
\[
Q = \left\{ (x, y) : \begin{bmatrix} G & H \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \leq g \right\}.
\]
Then optimization over \( P \) can be done in polynomial time by linear programming.

Proof. We have
\[
\min\{ c^T x : x \in P \} = \min \left\{ (c^T 0^T) : \begin{bmatrix} x \\ y \end{bmatrix} \in Q \right\}.
\]
Therefore, optimization over \( P \) can be done by linear programming over \( Q \). \( \square \)

We are now in position to prove Theorem 1.1.

Theorem 1.1. The extremality, adjacency, separation, and optimization problems over \( P_n \) can all be solved using linear programming without the ellipsoid method.

Proof. By Theorem 1.2, the integer partition polytope satisfies
\[
P_n = \{ x : \exists y (x, y) \in Q_n \},
\]
where \( Q_n \) is a polytope given by an explicit inequality description of size which is polynomial in \( n \). The theorem now follows from Lemmas 3.2–3.4. \( \square \)

We note that linear programming over systems in dimension \( d \) with \( m \) inequalities of bit size \( L \) can be solved, say, by the algorithm of [8], in \( O(m + dL) \) iterations. For the system defining \( Q_n \), these values are \( d = O(n^2) \), \( m = O(n^2) \), and \( L = O(n^2) \) dominated by the number of entries of the defining matrix, and so linear programming over \( Q_n \) can be done using \( O(n^{10.5}) \) iterations. However, this bound is very conservative, since the system defining \( Q_n \) is very sparse, as each variable appears in only 5 inequalities (including the bounds on the variables), and so out of the \( O(n^3) \) entries of the matrix defining \( Q_n \), only \( O(n^2) \) are nonzero. This can be exploited to run much faster in practice, see, for example, [2].
4. Remarks

We conclude with four remarks. First, the solution of the separation problem over $P_n$ implies in particular that we can solve the membership problem of deciding if a given $x$ is in $P_n$. Using the explicit polyhedral description of $Q_n$, the membership problem over $P_n$ reduces to the following feasibility linear program in variables $y$,

$$\{ y : Ay = a, \ By = x, \ 0 \leq y \leq 1 \} \neq \emptyset?$$

The solution of the separation problem over $P_n$ also suggests that a complete inequality description of $P_n$ (possibly with exponentially many inequalities) may be eventually determined. However, as of now, we were unsuccessful in this endeavor. We leave it as a challenge to the interested reader to try and use the new extended formulation of $P_n$ to make progress in this direction. Currently only the subadditive characterization of facets of $P_n$ describes them as vertices of some other polyhedron defined by a simple system of subadditive inequalities and equalities [11], and some properties of the facets through a given vertex [12] are known. We also note that an explicit linear inequality description of $P_n$ can be obtained by applying the Fourier–Motzkin procedure to $Q_n$ to eliminate the $y$ variables [9]. While exponential, it might be useful for small values of $n$.

The second remark is that the solution of the extremality problem over $P_n$ suggests that a combinatorial characterization of extreme partitions might be possible, though presently we cannot suggest any candidate for such a characterization. The definition of a vertex of a polytope as its point, which cannot be represented as a convex combination of its other points, does not look helpful. As of now, such a characterization is known only for points inexpressible as a convex combination of two others, see [12, 13], and nothing is known about characterizing points that cannot be expressed as a convex combination of three, four or other fixed number of points greater than two. The operation of taking convex combination does not seem to be fully understood in a combinatorial way yet.

The third remark refers to restricted integer partitions of $n$, i.e. to the set $T_n(M):=\{x \in \mathbb{Z}^m : \sum_{k=1}^{m} n_k x_k = n\}$ of partitions with all parts in a given subset $M = \{n_1, \ldots, n_m\}$, $1 \leq n_1 < n_2 < \cdots < n_m \leq n$, where the input consists of $n, n_1, \ldots, n_m$ in unary, and the corresponding polytope $P_n(M) := \text{conv}(T_n(M))$. It is not hard to see that a suitable modification $G_n(M)$ of the digraph $G_n$ can be constructed, giving an analog of Lemma 2.1, so that integer partitions $x \in T_n(M)$ are in bijection with $v_0^0 - v_n^0$ dipaths in $G_n(M)$. The vertex set of the digraph $G_n(M)$ is the join of the sets $V^0 = \{v_0^0\}$, $V^q = \{v_0^q, v_1^q, \ldots, v_n^q\}$ for $q \in M$, and $V^n = \{v_n^m\}$. The arcs in $G_n(M)$ connect only consecutive sets $V^{nq-1}, V^{nq}$, and an arc $(v_j^{nq-1}, v_j^{nq})$ exists in $G_n(M)$ if and only if $\frac{j}{nq} \in \mathbb{Z}_+$. Then, in the way similar to above, one can construct the polytopes $D_n(M)$ and $Q_n(M)$ and through the analogs of the subsequent lemmas prove the analogs of Theorems 1.1 and 1.2 over $P_n(M)$. Thus, in the case of restricted partitions of $n$, we also obtain a polynomial sized extended formulation of $P_n(M)$, and the four problems can be also solved in time polynomial in $n$ using linear programming without the ellipsoid method.

The fourth remark is the following. The optimization problem over $P_n$ can be solved quickly by combinatorial algorithms—the one described in Proposition 2.4 and the one provided in [3]. This suggests the challenge of finding efficient combinatorial algorithms also for the extremality, adjacency, and separation problems.

References