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Generating uniform random vectors over a simplex with implications to the volume of a certain polytope and to multivariate extremes

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Abstract A uniform random vector over a simplex is generated. An explicit expression for the first moment of its largest spacing is derived. The result is used in a proposed diagnostic tool which examines the validity of random number generators. It is then shown that the first moment of the largest uniform spacing is related to the dependence measure of random vectors following any extreme value distribution. The main result is proved by a geometric proof as well as by a probabilistic one.

Keywords Convex polytope · Multivariate extreme value distribution · Pickands dependence function · Simplex · Triangulation · Uniform spacings

1 Introduction

Let

$$\Omega_k := \left\{ \mathbf{v} = (v_1, v_2, \ldots, v_k) : v_i \geq 0, \sum_{i=1}^{k} v_i = 1 \right\}$$

be the standard unit-simplex in $\mathbb{R}^k$ $(k \geq 2)$. Recall that a *simplex* is an iterated pyramid, that is, a polytope whose number of vertices exceeds its dimension by one, such as a triangle and the standard pyramid. Consider the real function $A_0 : \Omega_k \mapsto [k^{-1}, 1]$ defined by

$$A_0(\mathbf{v}) := \max_{1 \leq i \leq k} v_i, \quad \mathbf{v} \in \Omega_k.$$

Dedicated to Reuven Rubinstein on his seventieth birthday.

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The main purpose of this paper is to prove that the volume under A_0 is given by

$$\mathrm{vol}(A_0) := \int_{\Omega_k} A_0(\mathbf{v})d\mathbf{v} = \frac{k^{1/2}}{k!}\sum_{i=1}^{k}\frac{1}{i} =: \frac{k^{1/2}}{k!}\zeta_1(k). \tag{1}$$

Why should one be interested in $\mathrm{vol}(A_0)$? In Sect. 2 we show its relevance to the generation of uniform random variables and in particular to the generation of uniform random vectors over a simplex. A diagnostic tool for the validity of a random number generator is proposed. In Sect. 3 we show the connection between $\mathrm{vol}(A_0)$ and multivariate extreme value distributions. The proof of (1) is given in two versions—a geometric one in Sect. 4 and a probabilistic proof in Sect. 5. Along the way, we discover another interesting connection between $\mathrm{vol}(A_0)$ and the volume of a certain convex polytope P_k (defined by (3)), namely

$$\mathrm{vol}(A_0) = k^{1/2}k!\,\mathrm{vol}(P_k).$$

Remark 1 Volume computations (in fact volume approximations) of convex bodies of dimension n are in general quite complex. Most of the published algorithms involve multiphase Monte-Carlo techniques, based on random walks in $\mathbb{R}^n$. In a recent survey, Vempala (2005) lists the improvements in complexity from n^{23} in 1991 to n^4 in 2003. In the present paper we were able to compute the volume under A_0 and get an exact expression for all $k \geq 2$.

2 Generating a uniform random vectors over Ω_k

Suppose one wants to generate a random vector, uniformly distributed over Ω_k. Rubinstein and Kroese (2007), Algorithm 2.5.3, suggest the following:

Algorithm 1

1. *Generate* $k-1$ *independent random variables* $U_1, \dots, U_{k-1}$ *from* $U(0, 1)$.
2. *Sort the* $\{U_i\}$ *into the order statistics*

$$U_{(1)} \leq U_{(2)} \leq \cdots \leq U_{(k-1)}$$

   and let $U_{(0)} = 0$ *and* $U_{(k)} = 1$.
3. *Define the spacings*

$$V_i = U_{(i)} - U_{(i-1)}, \quad i = 1, \dots, k$$

   and return the vector $\mathbf{V} = (V_1, \dots, V_k)$.

For the proof that $\mathbf{V}$ is indeed uniformly distributed over the simplex Ω_k see David (1981), pp. 99–100. A more efficient algorithm, which does not require sorting (the major time-consuming step) is given by Rubinstein and Melamed (1998) as Algorithm 2.7.1.

Algorithm 2

1. *Generate* k *independent unit-exponential random variables* $Y_1, \dots, Y_k$ *and compute* $T_k = \Sigma_1^k Y_i$.
2. *Define* $E_i = Y_i/T_k$ *and return* $\mathbf{E} = (E_1, \dots, E_k)$.

The vectors **E** and **V** are identically distributed. The reason is that the Y_i can be thought of as the times between successive events of a homogeneous Poisson process. It is well known (see for instance Karlin and Taylor 1975, p. 126 or Epstein and Weissman 2008, p. 17) that the conditional joint distribution of

$$(Y_1, Y_1 + Y_2, \dots, Y_1 + \cdots + Y_{k-1}),$$

given T_k, is the same as the joint distribution of $k-1$ order statistics from $U(0, T_k)$.

Obviously, there is a direct connection between $\mathrm{vol}(A_0)$ and the random vector **V**. If we let $V_{(k)} = \max V_i = A_0(\mathbf{V})$ be the largest spacing, then

$$EV_{(k)} = \frac{\int_{\Omega_k} A_0(\mathbf{v})d\mathbf{v}}{\int_{\Omega_k} d\mathbf{v}} = k^{-1/2}(k-1)!\mathrm{vol}(A_0).$$

By (1),

$$EV_{(k)} = \frac{1}{k}\sum_{i=1}^{k}\frac{1}{i} = \frac{1}{k}\zeta_1(k). \tag{2}$$

This and $V_{(k)}$ are useful when one wants to check the validity of a certain random number generator. Suppose we generate $N = m \times (k-1)$ uniform on [0, 1] random variables. For each block (sample) of size $k-1$, we compute $V_{(k)}$. Having m independent and identically distributed (i.i.d.) replications of $V_{(k)}$, one can compare their sample-mean with $EV_{(k)}$ and test the significance of the difference. A more detailed diagnostic would be to plot the $\{V_{(k)}\}$ against their block number (from 1 to m). As in control charts, we also plot the "target" value $EV_{(k)}$ and an upper and a lower "control limits". If the random number generator is valid, the m points should scatter around the target value, within the control limits. We take the control limits from Devroye (1982), who proved that

$$\lim\inf(kV_{(k)} - \log k + \log_3 k) = -\log 2 \quad \text{a.s.},$$

and

$$\lim\sup(kV_{(k)} - \log k)/(2\log_2 k) = 1 \quad \text{a.s.}$$

Here, $\log_j$ is the j times iterated logarithm.

Example 1 As an example, we generated $N = 10000 = 100 \times 100$ uniform random numbers in S-Plus. Thus we have 100 independent replications of $V_{(101)}$, which are plotted in Fig. 1 as described above. Although the lower and upper limits are based on the asymptotic formulas (to improve the approximation for finite $k = 101$, we replaced $\log k$ by $\zeta_1(k)$), there is only one outlier out of 100 samples. Using (5), it turns out that the upper and lower limits correspond, respectively, to the upper and lower second percentiles (more precisely, to 1.988% and 2.054%). Hence, even the most trustful random number generator may produce, on the average, two upper and two lower outliers. The sample-mean in this example is .05091, while $EV_{(101)} = .05146$. With standard deviation of .01164, the difference is negligible.

Example 2 Here we generated $N = 10000$ random variables from the *autoregressive* model. That is, U_1 is uniform on (0, 1) and for $i = 2, 3, \dots, 10000$,

$$U_i = 0.1U_{i-1} + 0.9E_i,$$

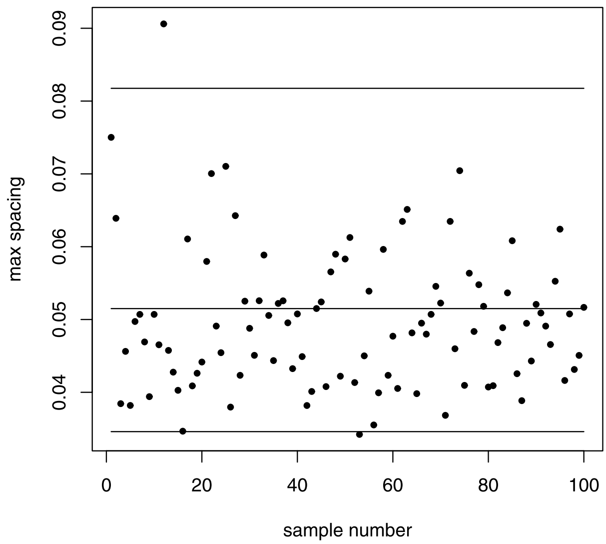

Fig. 1 Control Chart for $V_{(k)}$ ($k = 101$), i.i.d., S-Plus generated

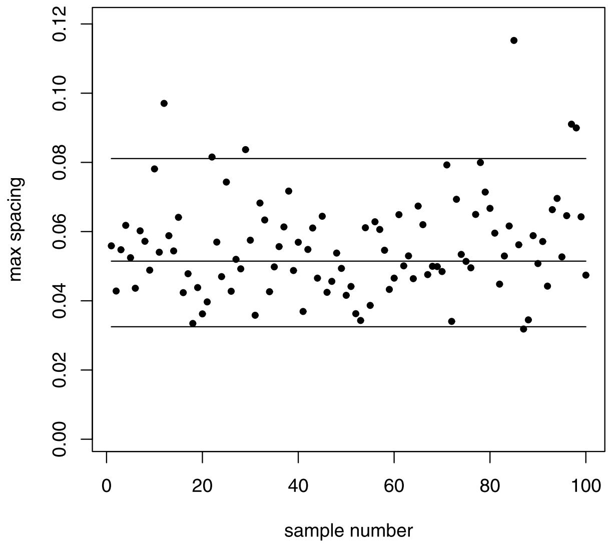

Fig. 2 Control Chart for $V_{(k)}$ ($k = 101$), autoregressive model, S-Plus generated

where the $\{E_i\}$ are i.i.d. uniform on (0, 1), generated by S-Plus. Note, the autocorrelation of lag 1 is 0.1 and we expect a different behavior of the $V_{(k)}$. We repeated the same procedure as before and the results are shown in Fig. 2. Compared with the previous case, we observe a general upward shift, 6 upper outliers and sample-mean of 0.05790, which is significantly larger than expected if the model was i.i.d. uniform.

The authors have experimented with a variety of examples. It seems that this diagnostic tool is quite sensitive to deviations from the i.i.d. uniform model, but it would be worthwhile to do more research. In particular, given a sample of size N, what is the most effective partition into m sub-samples of size k each?

3 Multivariate extreme value distributions

Another interesting connection is between $\mathrm{vol}(A_0)$ and *multivariate extremes*. Suppose that $\mathbf{X} = (X_1, X_2, \dots, X_k)$ is a random vector in $\mathbb{R}^k$, having a *multivariate extreme value distribution* $\mathbf{G}$ with margins $G_i(x) = P\{X_i \le x\}$. If we let

$$\mathbf{B}(\mathbf{x}) := \frac{-\log \mathbf{G}(\mathbf{x})}{\sum_1^k -\log G_i(x_i)},$$

then obviously

$$\mathbf{G}(\mathbf{x}) = \exp\left\{\mathbf{B}(\mathbf{x}) \sum_1^k \log G_i(x_i)\right\}.$$

Pickands (1981) showed that $\mathbf{B}(\mathbf{x}) = A(\mathbf{v})(\mathbf{v} \in \Omega_k)$, where $v_i = \log G_i(x_i)/\sum_1^k \log G_j(x_j)$. Further, A is convex and satisfies

$$\frac{1}{k} \le A_0(\mathbf{v}) \le A(\mathbf{v}) \le 1, \quad \mathbf{v} \in \Omega_k.$$

For a detailed account of the subject and the properties of the *Pickands dependence function* A, see Beirlant et al. (2004). Since we are only interested in the dependence among the $\{X_i\}$, without any loss of generality we can choose a convenient set of margins $\{G_i\}$. It is customary in the extreme value literature to choose the Fréchet distribution function $G_i(x) = \exp(-1/x)(x > 0)$ for all $i = 1, 2, \dots, k$. Thus, our $\mathbf{G}$ has the form

$$\mathbf{G}(\mathbf{x}) = \exp\left(-A(\mathbf{v}) \sum_1^k x_i^{-1}\right), \quad x_i > 0, \mathbf{v} \in \Omega_k,$$

where $v_i = x_i^{-1}/\sum_1^k x_j^{-1}$. If the $\{X_i\}$ are independent, then $A \equiv 1$ and if they are completely dependent, then $A \equiv A_0$. Thus, a natural *coefficient of dependence* for the random vector $\mathbf{X}$ is

$$\tau = \frac{\int_{\Omega_k} (1 - A(\mathbf{v}))d\mathbf{v}}{\int_{\Omega_k} (1 - A_0(\mathbf{v}))d\mathbf{v}} = \frac{k^{1/2}/(k-1)! - \mathrm{vol}(A)}{k^{1/2}/(k-1)! - \mathrm{vol}(A_0)}.$$

Clearly $0 \le \tau \le 1$, $\tau = 0, 1$ correspond to independence and complete dependence, respectively. Now, the numerator of τ is case-specific, while the denominator depends on k only. Thus, it will be useful to have an explicit expression for the latter.

Weissman (2008) gives several examples with their τ-values, including the *logistic model*. In the latter case,

$$A(\mathbf{v}) = \left(\sum_{i=1}^k v_i^{1/\alpha}\right)^\alpha, \quad \mathbf{v} \in \Omega_k, 1 < \alpha \le 1.$$

Clearly, $\alpha = 1$ corresponds to total independence and $\alpha \downarrow 0$ corresponds to complete dependence. The case $k = 2$ is shown in Fig. 3. For each $\alpha = 0, .25, .50, .75$ and 1, the function $A(v, 1 - v)$ is plotted as a function of $v \in [0, 1]$. For each α, the value of τ is 4 times the volume (area) between A and 1.

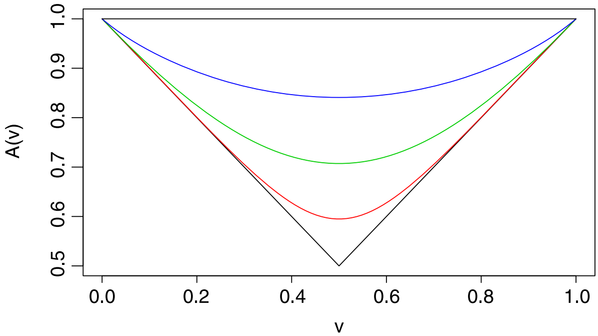

Fig. 3 Pickands dependence function for the logistic model ($\alpha = 0, 0.25, 0.50, 0.75, 1$)

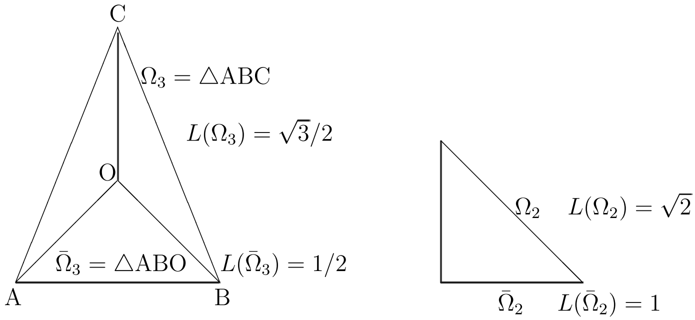

Fig. 4 The Lebesgue measures of Ω_k vs. $\bar{\Omega}_k$

4 A geometric proof of (1)

In what follows it will be convenient to use the full-dimensional simplex

$$\bar{\Omega}_k := \left\{ \mathbf{v} = (v_1, v_2, \dots, v_{k-1}) : v_i \geq 0, \sum_{i=1}^{k-1} v_i \leq 1 \right\},$$

that is, the projection of Ω_k into $\mathbb{R}^{k-1}$. In this case, $v_k = 1 - \sum_{i=1}^{k-1} v_i$. Note that while $L(\bar{\Omega}_k)$, the Lebesgue measure of $\bar{\Omega}_k$, is equal to $1/(k-1)!$, one has $L(\Omega_k) = k^{1/2} L(\bar{\Omega}_k) = k^{1/2}/(k-1)!$. The reason is this: As we project Ω_k onto $\bar{\Omega}_k$, the vertex $(0, \dots, 0, 1)$ goes to $(0, \dots, 0)$, all other vertices are unaffected. The altitude of Ω_k (i.e., the distance between $(1/(k-1), \dots, 1/(k-1), 0)$ and $(0, \dots, 0, 1)$) is $(k/(k-1))^{1/2}$, which is $k^{1/2}$ times the altitude of $\bar{\Omega}_k$. This point is illustrated in Fig. 4. Goodman and O'Rourke (2004, p. 374), give the volume and surface area of some standard polytopes.

We now show how to evaluate the integral $\text{vol}(A_0) = \int_{\Omega_k} A_0(\mathbf{v}) d\mathbf{v}$.

Some simple reductions The standard simplex Ω_k is the union $\Omega_k = \bigcup_\sigma \Omega_\sigma$ over all $k!$ permutations $\sigma : \{1, \dots, k\} \longrightarrow \{1, \dots, k\}$, where the simplex corresponding to σ is

$$\Omega_\sigma := \left\{ (v_1, \dots, v_k) : 0 \leq v_{\sigma(k)} \leq v_{\sigma(k-1)} \leq \dots \leq v_{\sigma(2)} \leq v_{\sigma(1)}, \sum_{i=1}^{k} v_i = 1 \right\}.$$

Clearly, the desired integral is the sum of the integrals over these simplices,

$$\mathrm{vol}(A_0) = \sum_{\sigma} \int_{\Omega_\sigma} A_0(\mathbf{v}) d\mathbf{v}.$$

By symmetry, it is enough to evaluate one of these integrals, say the one corresponding to the identity permutation $\sigma = e$. But over Ω_e we have $A_0(\mathbf{v}) = \max_i(v_i) = v_1$, and therefore, we obtain

$$\mathrm{vol}(A_0) = k! \int_{\Omega_e} A_0(\mathbf{v}) d\mathbf{v} = k! \int_{\Omega_e} v_1 d\mathbf{v}.$$

Now,

$$\Omega_e = \left\{ (v_1, v_2, \ldots, v_k) : 0 \le v_k \le v_{k-1} \le \cdots \le v_2 \le v_1, \sum_{i=1}^{k} v_i = 1 \right\}$$

so, writing $v_k = 1 - \sum_{i=1}^{k-1} v_i$, we can evaluate $\int_{\Omega_e} v_1 d\mathbf{v}$ by integrating v_1 over

$$\bar{\Omega}_e = \left\{ (v_1, v_2, \ldots, v_{k-1}) : 0 \le 1 - \sum_{i=1}^{k-1} v_i \le v_{k-1} \le \cdots \le v_2 \le v_1 \right\}$$

(and multiplying by $k^{1/2}$). This reduces to computing the volume of the following convex polytope in $\mathbb{R}^k$:

$$P_k = \left\{ (v_0, v_1, \ldots, v_{k-1}) : 0 \le 1 - \sum_{i=1}^{k-1} v_i \le v_{k-1} \le \cdots \le v_1, 0 \le v_0 \le v_1 \right\}, \qquad (3)$$

since

$$\int_{\bar{\Omega}_e} v_1 d\mathbf{v} = \int_{\bar{\Omega}_e} \int_0^{v_1} dv_0 d\mathbf{v} = \mathrm{vol}(P_k).$$

Triangulating P_k and computing its volume In order to compute the volume $\mathrm{vol}(P_k)$, we determine a triangulation of the polytope P_k, compute the volumes of the simplices in that triangulation, and add them up. For this, we need first to determine the vertices of P_k.

Lemma 1 *The polytope P_k has $2k$ vertices $\mathbf{a}^i, \mathbf{b}^i$, $i = 1, \ldots, k$, as follows*:

$$a_j^i = \begin{cases} 0, & j = 0 \\ \frac{1}{i,} & 1 \le j \le i \\ 0, & i < j < k \end{cases} \qquad b_j^i = \begin{cases} \frac{1}{i,} & j = 0 \\ \frac{1}{i,} & 1 \le j \le i \; (1 \le i \le k, 0 \le j \le k-1) \\ 0, & i < j < k \end{cases}$$

Proof The vertices are obtained as the elements of P_k satisfying with equality k linearly independent inequalities from the system of $k+2$ inequalities defining P_k. It is easy to see that in each such choice, precisely one of the inequalities $0 \le v_0$ or $v_0 \le v_1$ must hold with equality. For each, there are k choices of forcing equality on k out of the $k+1$ remaining inequalities, and each choice gives exactly one of the vectors appearing in the above claimed list. □

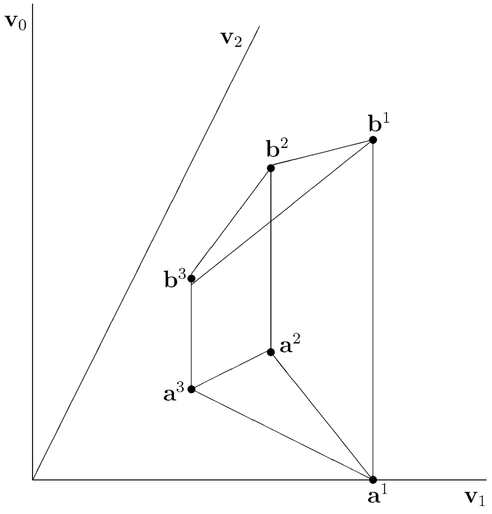

Fig. 5 The convex polytope P_3

Note that for each i, the vectors $\mathbf{a}^i$ and $\mathbf{b}^i$ agree on all coordinates except for the 0th coordinate. For instance, for $k = 3$ these vectors are:

$$\mathbf{a}^1 = (0, 1, 0), \qquad \mathbf{a}^2 = \left(0, \frac{1}{2}, \frac{1}{2}\right), \qquad \mathbf{a}^3 = \left(0, \frac{1}{3}, \frac{1}{3}\right),$$

$$\mathbf{b}^1 = (1, 1, 0), \qquad \mathbf{b}^2 = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right), \qquad \mathbf{b}^3 = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}).$$

For $k = 4$ these vectors are:

$$\mathbf{a}^1 = (0, 1, 0, 0), \quad \mathbf{a}^2 = \left(0, \frac{1}{2}, \frac{1}{2}, 0\right), \quad \mathbf{a}^3 = \left(0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right), \quad \mathbf{a}^4 = \left(0, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right),$$

$$\mathbf{b}^1 = (1, 1, 0, 0), \quad \mathbf{b}^2 = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0\right), \quad \mathbf{b}^3 = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right), \quad \mathbf{b}^4 = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right).$$

The case $k = 3$ is illustrated in Fig. 5.

Lemma 2 *For $s = 1, \dots, k$, the following polytope*

$$\Delta_s := \operatorname{conv}\{\mathbf{a}^1, \dots, \mathbf{a}^s, \mathbf{b}^s, \dots, \mathbf{b}^k\},$$

is a k-dimensional simplex whose volume is

$$\operatorname{vol}(\Delta_s) = \frac{1}{(k!)^2 s}.$$

Here, conv stands for the convex hull.

Proof The convex hull $\Delta := \operatorname{conv}\{\mathbf{v}^0, \mathbf{v}^1, \dots, \mathbf{v}^k\}$ of $k + 1$ points in $\mathbb{R}^k$ is a simplex if and only if the determinant $\delta := \det(\mathbf{v}^1 - \mathbf{v}^0, \dots, \mathbf{v}^k - \mathbf{v}^0)$ is nonzero, in which case its volume is given by $\mathrm{vol}(\Delta) = \frac{1}{k!}|\delta|$. So, for $s = 1, \ldots, k$, we shall compute the following determinant,

$$\delta_s := \det(\mathbf{a}^1 - \mathbf{a}^s, \ldots, \mathbf{a}^{s-1} - \mathbf{a}^s, \mathbf{b}^s - \mathbf{a}^s, \mathbf{b}^{s+1} - \mathbf{a}^s, \ldots, \mathbf{b}^k - \mathbf{a}^s).$$

For any vector $\mathbf{v} = (v_0, v_1, \ldots, v_{k-1})$ in $\mathbb{R}^k$, let $\bar{\mathbf{v}} := (v_1, \ldots, v_{k-1})$ be its projection to $\mathbb{R}^{k-1}$ obtained by erasing the 0th coordinate. Then, for all i, we have $\bar{\mathbf{b}}^i = \bar{\mathbf{a}}^i$. Thus, expanding the determinant δ_s on the column $\mathbf{b}^s - \mathbf{a}^s = (\frac{1}{s}, 0, \ldots, 0)$, we find that $|\delta_s| = \frac{1}{s}|\mu|$, where

$$\mu := \det(\bar{\mathbf{a}}^1 - \bar{\mathbf{a}}^s, \bar{\mathbf{a}}^2 - \bar{\mathbf{a}}^s, \ldots, \bar{\mathbf{a}}^{s-1} - \bar{\mathbf{a}}^s, \bar{\mathbf{a}}^{s+1} - \bar{\mathbf{a}}^s, \ldots, \bar{\mathbf{a}}^{k-1} - \bar{\mathbf{a}}^s, \bar{\mathbf{a}}^k - \bar{\mathbf{a}}^s).$$

Subtracting the first column from every other column, flipping its sign, and permuting, we get

$$\mu = (-1)^{s-1} \det(\bar{\mathbf{a}}^2 - \bar{\mathbf{a}}^1, \bar{\mathbf{a}}^3 - \bar{\mathbf{a}}^1, \ldots, \bar{\mathbf{a}}^{k-1} - \bar{\mathbf{a}}^1, \bar{\mathbf{a}}^k - \bar{\mathbf{a}}^1).$$

Using the multilinearity of the determinant, μ can be expanded as a sum of 2^{k-1} determinantal terms. Among these, each term in which $\bar{\mathbf{a}}^1$ occurs more than once vanishes, and so does each term in which both $\bar{\mathbf{a}}^{k-1}$ and $\bar{\mathbf{a}}^k$ occur, being a scalar multiple of one another (see definition of the $\mathbf{a}^i$ in Lemma 1). It is easy to see, then, that only two terms remain, giving

$$\mu = (-1)^s \left(\det(\bar{\mathbf{a}}^2, \bar{\mathbf{a}}^3, \ldots, \bar{\mathbf{a}}^{k-2}, \bar{\mathbf{a}}^{k-1}, \bar{\mathbf{a}}^1) + \det(\bar{\mathbf{a}}^2, \bar{\mathbf{a}}^3, \ldots, \bar{\mathbf{a}}^{k-2}, \bar{\mathbf{a}}^1, \bar{\mathbf{a}}^k)\right).$$

These two remaining terms are easy to compute, since by permuting $\mathbf{a}^1$ to be the first column in each, the corresponding matrices become upper triangular. Since $\bar{\mathbf{a}}^i_i = \frac{1}{i}$, we finally obtain

$$\begin{aligned} |\mu| &= \det(\bar{\mathbf{a}}^1, \bar{\mathbf{a}}^2, \ldots, \bar{\mathbf{a}}^{k-2}, \bar{\mathbf{a}}^{k-1}) - \det(\bar{\mathbf{a}}^1, \bar{\mathbf{a}}^2, \ldots, \bar{\mathbf{a}}^{k-2}, \bar{\mathbf{a}}^k) \\ &= \frac{1}{1 \cdot 2 \cdots (k-2) \cdot (k-1)} - \frac{1}{1 \cdot 2 \cdots (k-2) \cdot k} = \frac{1}{k!}. \end{aligned}$$

Summing up, we get, as claimed, for $s = 1, \ldots, k$,

$$\mathrm{vol}(\Delta_s) = \frac{1}{k!}|\delta_s| = \frac{1}{k!}\frac{1}{s}|\mu| = \frac{1}{(k!)^2 s}.$$

□

Lemma 3 *The k simplices $\Delta_1, \ldots, \Delta_k$ form a triangulation of P_k.*

For example, when $k = 3$,

$$\begin{aligned} \Delta_1 &= \mathrm{conv}\{\mathbf{a}^1, \mathbf{b}^1, \mathbf{b}^2, \mathbf{b}^3\} \\ \Delta_2 &= \mathrm{conv}\{\mathbf{a}^1, \mathbf{a}^2, \mathbf{b}^2, \mathbf{b}^3\} \\ \Delta_3 &= \mathrm{conv}\{\mathbf{a}^1, \mathbf{a}^2, \mathbf{a}^3, \mathbf{b}^3\}. \end{aligned}$$

Proof We show by induction on s that $\Delta_1, \ldots, \Delta_s$ form a triangulation of the polytope

$$Q_s := \mathrm{conv}\left(\bigcup_{i=1}^{s} \Delta_i\right) = \mathrm{conv}\{\mathbf{a}^1, \ldots, \mathbf{a}^s, \mathbf{b}^1, \ldots, \mathbf{b}^k\}.$$

For $s = 1$ this is surely true since $Q_1 = \Delta_1$. Suppose now the claim holds for Q_s. Consider the next point $\mathbf{a}^{s+1}$ to be added so as to form $Q_{s+1} = \mathrm{conv}(Q_s \cup \{\mathbf{a}^{s+1}\})$. We claim that $F_s := \mathrm{conv}\{\mathbf{a}^1, \ldots, \mathbf{a}^s, \mathbf{b}^{s+1}, \ldots, \mathbf{b}^k\}$ is a facet of Q_s, and that it is the only facet of Q_s for which $\mathbf{a}^{s+1}$ is *beyond* the hyperplane H_s spanned by F_s, that is, H_s separates $\mathbf{a}^{s+1}$ from Q_s. For more details on the so-called *beneath-beyond method* for the inductive construction of polytopes see Sect. 5.2 of Grübaum (2003).

To prove the claim, it suffices to show, for each vertex $\mathbf{v}$ of Q_s which does not lie in F_s, that the open line segment between $\mathbf{a}^{s+1}$ and $\mathbf{v}$ intersects F_s. Now, the vertices of Q_s which are not in F_s are $\mathbf{b}^1, \ldots, \mathbf{b}^s$, so consider any of the $\mathbf{b}^i$ with $i \le s$. By the definition of the $\mathbf{a}^j$ and $\mathbf{b}^j$ (see Lemma 1), we have $i(\mathbf{b}^i - \mathbf{a}^i) = (s+1)(\mathbf{b}^{s+1} - \mathbf{a}^{s+1})$, which by suitable manipulation gives

$$\frac{s+1}{s+1+i} \cdot \mathbf{a}^{s+1} + \frac{i}{s+1+i} \cdot \mathbf{b}^i = \frac{s+1}{s+1+i} \cdot \mathbf{b}^{s+1} + \frac{i}{s+1+i} \cdot \mathbf{a}^i.$$

The left hand side of this equation is in the open segment between $\mathbf{a}^{s+1}$ and $\mathbf{b}^i$ whereas the right hand side is in F_s since so are $\mathbf{a}^i$ and $\mathbf{b}^{s+1}$, proving the claim.

Since F_s is a (simplicial) facet of Q_s and the only one for which $\mathbf{a}^{s+1}$ is beyond the hyperplane H_s spanned by F_s, a triangulation for $Q_{s+1} = \mathrm{conv}(Q_s \cup \{\mathbf{a}^{s+1}\})$ is obtained from the triangulation $\Delta_1, \ldots, \Delta_s$ of Q_s by adding the single new simplex $\mathrm{conv}(F_s \cup \{\mathbf{a}^{s+1}\}) = \Delta_{s+1}$. This completes the induction. Since $P_k = Q_k$, it follows that $\Delta_1, \ldots, \Delta_k$ is a triangulation of P_k and the lemma follows. □

Combining the above statements implies

$$k! \int_{\bar{\Omega}_e} v_1 d\mathbf{v} = k! \cdot \mathrm{vol}(P_k) = k! \sum_{s=1}^{k} \mathrm{vol}(\Delta_s)$$

$$= k! \sum_{s=1}^{k} \frac{1}{s(k!)^2} = \frac{1}{k!}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k}\right) = \frac{1}{k!}\zeta_1(k).$$

This completes the geometric proof. □

5 A probabilistic proof of (1)

Let $Y_1, Y_2, \ldots, Y_k$ be independent unit-exponential random variables and let $T = \sum_1^k Y_j$ be their sum. Then, by Sect. 1,

$$\left(\frac{Y_1}{T}, \frac{Y_2}{T}, \ldots, \frac{Y_k}{T}\right) \stackrel{d}{=} (V_1, V_2, \ldots, V_k),$$

both random vectors are uniform over Ω_k. For the probabilistic proof we need two Lemmas.

Lemma 4 *The random vector* $(\frac{Y_1}{T}, \frac{Y_2}{T}, \ldots, \frac{Y_k}{T})$ *and* T *are independent.*

Proof Let $E_i = Y_i/T$ and consider the transformation from $(Y_1, Y_2, \ldots, Y_k)$ to $(E_1, E_2, \ldots, E_{k-1}, T)$. The inverse transformation is given by

$$Y_1 = TE_1$$

$$Y_2 = TE_2$$

$$\vdots$$
$$Y_{k-1} = TE_{k-1}$$
$$Y_k = T(1 - E_1 - \cdots - E_{k-1}).$$

It is straight forward to show that the Jacobian of the transformation is equal to T^{k-1}. It follows that the joint density function is given by

$$f_{E_1,\ldots,E_{k-1},T}(e_1,\ldots,e_{k-1},t) = \mathbf{1}\{\mathbf{e} \in \bar{\Omega}_k\} \cdot \exp(-t)t^{k-1}\mathbf{1}\{t > 0\},$$

where $\mathbf{e} = (e_1, e_2, \ldots, e_{k-1})$. □

Lemma 5 *For any two random variables U and W with finite first moments* (*and* $EW \neq 0$),

$$E\frac{U}{W} = \frac{EU}{EW} \quad \Leftrightarrow \quad \mathrm{COV}\left(W, \frac{U}{W}\right) = 0.$$

Proof

$$\mathrm{COV}\left(W, \frac{U}{W}\right) = 0 \quad \Leftrightarrow \quad EU = EW \cdot E\frac{U}{W} \quad \Leftrightarrow \quad E\frac{U}{W} = \frac{EU}{EW}. \qquad \square$$

Combining the last two Lemmas,

$$EV_{(k)} = E\frac{\max Y_j}{T} = \frac{E\max Y_j}{ET} = \frac{1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k}}{k}. \tag{4}$$

In view of (2), the probabilistic proof of (1) is complete. □

Remark 2 Equation (4) can be derived by integrating (on [0, 1])

$$P\{V_{(k)} > x\} = \sum_{j=1}^{k} (-1)^{j-1} \binom{k}{j} (1 - jx)_+^{k-1}. \tag{5}$$

According to Darling (1953), this expression dates back to W.A. Whitworth in 1897. However, we have not found any reference to (4) in the entire statistical and probabilistic literature.

Remark 3 The argument which led us to (4) can be used to compute other moments of $V_{(k)}$. For instance, if $\zeta_2(k) = \sum_1^k j^{-2}$, then

$$EV_{(k)}^2 = \frac{E(\max Y_j)^2}{ET^2} = \frac{\mathrm{Var}(\max Y_j) + \zeta_1^2(k)}{\mathrm{Var}(T) + k^2} = \frac{\zeta_2(k) + \zeta_1^2(k)}{2k + k^2}.$$

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