Lattice-Free Polytopes and Their Diameter *

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Abstract

A convex polytope in real Euclidean space is *lattice-free* if it intersects some lattice in space exactly in its vertex set. Lattice-free polytopes form a large and computationally hard class, and arise in many combinatorial and algorithmic contexts.

In this article, affine and combinatorial properties of such polytopes are studied. First, bounds on some invariants, such as the diameter and layer-number, are given. It is shown that the diameter of a d-dimensional lattice-free polytope is $O(d^3)$. A bound of $O(nd+d^3)$ on the diameter of a d-polytope with n facets is deduced for a large class of integer polytopes. Second, Delaunay polytopes and $[0,1]$-polytopes, which form major subclasses of lattice-free polytopes, are considered. It is shown that, up to affine equivalence, for any $d \geq 3$ there are infinitely many d-dimensional lattice-free polytopes but only finitely many Delaunay and $[0,1]$-polytopes. Combinatorial-types of lattice-free polytopes are discussed, and the inclusion relations among the subclasses above are examined. It is shown that the classes of combinatorial-types of Delaunay polytopes and $[0,1]$-polytopes are mutually incomparable starting in dimension 6, and that both are strictly contained in the class of combinatorial-types of all lattice-free polytopes.

keywords: lattice polytope, lattice free, Delaunay polytope, $(0,1)$-polytope, diameter, Hirsch conjecture, width

1 Introduction

A convex polytope in real Euclidean space $\mathbb{R}^d$ is *lattice-free* if it intersects some lattice in $\mathbb{R}^d$ exactly in its vertex set. Lattice-free polytopes form a large class of rational polytopes, and arise in many combinatorial and algorithmic contexts. In this article we study affine and combinatorial properties of lattice-free polytopes. We give bounds, shared by all such polytopes, on some of their combinatorial and affine invariants. We also study important subclasses of lattice-free polytopes, and examine the inclusion relations among these subclasses.

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Before describing the content of the article in more detail, we mention a few situations where lattice-free polytopes appear. A first example is provided by $[0,1]$-polytopes - those with all vertices having all coordinates $\{0,1\}$-valued. Such polytopes arise when polyhedral methods are applied to combinatorial optimization problems (see, e.g., [13] for more information on this subject). They also provide a link between convex polytopes and finite set systems, a recent example being the resolution of Borsuk's problem [17]. Another example is given by complexes of maximal lattice free bodies and test sets for integer programming [1]. With a finite set $\mathcal{A} = \{a_0, \cdots, a_n\} \subset \mathbb{R}^d$ is associated the collection $\Delta(\mathcal{A})$ of all integer-free polytopes of the form

$$conv\{x \in \mathbb{Z}^d: \; a_i x \leq r_i, \quad i = 0, \cdots, n\},$$

obtained as $r_0, \cdots, r_n$ range over all real numbers. The points $x \in \mathbb{Z}^d$ for which $\{0, x\}$ is contained in a polytope of $\Delta(\mathcal{A})$, constitute a finite *test set*, which enables one to deduce global optimality from local optimality for each of the integer programs

$$min\{a_0 x: \; x \in \mathbb{Z}^d, \quad a_i x \leq s_i, \quad i = 1, \cdots, n\},$$

where $s_1, \cdots, s_n$ are any real numbers. For details see [1]. A third example comes from lattice tilings of Euclidean space. A *Delaunay* polytope is a lattice-free polytope whose vertices lie on a sphere. Unlike the collection $\Delta(\mathcal{A})$ above, whose polytopes may have intersecting interiors and is an extremely complex object, the collection of all Delaunay polytopes of a given lattice forms a tiling of Euclidean space since any two of its members intersect in a common face (see, e.g., [14] for more information).

Lattice-free polytopes are, computationally, hard objects. Given a polytope with vertices in a lattice, deciding whether it is lattice-free (which is essentially integer programming) is NP-complete, though polynomial time solvable in fixed dimension (see [28]). Analogously, counting the number of lattice points contained in such a polytope is $\#P$-complete, but polynomial time solvable when the dimension is fixed, an algorithm having been found only very recently [3]. In fact, the decision and counting questions are hard even for simplices, and a good characterization of lattice-free simplices exists only for dimension at most three [26].

In the next section we fix some terminology and give an example of a classical lattice-free polytope - the *Schläfli polytope* - which will be of use throughout the article. Next, in Section 3 we consider the *layer-number* and *diameter* of lattice-free polytopes. The layer-number, an affine invariant of a polytope, is the smallest number of layers formed by its vertices when ordered under a linear functional. We show (Theorem 3.7) that the layer number of a d-dimensional lattice-free polytope is $O(d^2)$, and that, while it is known to be 1 in dimensions $d \leq 3$, it can be larger in dimension 6 and higher. The exact growth rate remains unsettled. We proceed to prove (Theorem 3.10) that the (combinatorial) diameter of a d-dimensional lattice-free polytope is $O(d^3)$, and deduce (Corollary 3.13) a polynomial upper bound $O(nd + d^3)$ on the diameter, in terms of the dimension d and number of facets n, for a large class of rational polytopes. This is of interest in connection with the well known Hirsch conjecture on the diameter of convex polytopes (see [20]).

In Section 4 we turn to discuss affine properties of the class of lattice-free polytopes and two important subclasses of it: the class of Delaunay polytopes and the class of $[0,1]$-polytopes (and more generally, $[0,k]$-polytopes for any fixed k). We establish a few properties of such polytopes (Proposition 4.5 being of interest in its own right) which, together with a fundamental theorem of Voronoi [30], yields Theorem 4.8. It asserts that for each dimension d there are finitely many affine-types of Delaunay-polytopes and $[0,1]$-polytopes, but (for $d \geq 3$) infinitely many affine-types of lattice-free polytopes. In Section 5 we proceed to discuss the combinatorial-types of the class of lattice-free polytopes and its subclasses of Delaunay polytopes and $[0,1]$-polytopes. Theorem 5.6 establishes the inclusion relations among these type-classes in each dimension. In particular, it follows from the theorem that these three classes pairwise differ in infinitely many combinatorial-types. One important ingredient in the proof is provided by the classification of three and four dimensional Delaunay polytopes in [9]. Another ingredient is a combinatorial invariant which we introduce - the *width* of a polytope (Definition 5.2), defined to be the smallest layer-number over all possible embeddings of the polytope. Like for the layer-number, the rate of growth, as a function of the dimension, of the largest width of a lattice-free polytope, is an open and seemingly hard question.

2 Lattice-free polytopes

Here we briefly discuss some basics of lattices and polytopes, and give some definitions and examples. As references for the geometry of lattices we suggest [14] and the survey article [12], and for convex polytopes [15]. Let $\mathbb{R}^d$ be Euclidean space equipped with the standard basis $\{e_1, \cdots, e_d\}$ and inner product $\langle e_i, e_j \rangle = \delta_{i,j}$. Thus, $\langle e_i, \cdot \rangle$ is the i-th coordinate function on $\mathbb{R}^d$ and an element $\sum_{i=1}^d x_i e_i \in \mathbb{R}^d$ is also the d-tuple $(x_1, \cdots, x_d)$ of its coordinates. A *lattice* in $\mathbb{R}^d$ is a $\mathbb{Z}$-submodule of $\mathbb{R}^d$ generated by a set of $\mathbb{R}$-linearly independent vectors, hence, in particular, is a free $\mathbb{Z}$-module. The *integer lattice* is the lattice $\mathbb{Z}^d$ consisting of all *integer points* in $\mathbb{R}^d$, i.e., points all of which coordinates are integers. A *polytope* will always mean a convex polytope in some affine space $\mathbb{R}^n$. We denote by $vert(P)$ the vertex set of a polytope P and by $\dim(P)$ its affine dimension. If $\dim(P) = d$ then P is a *d-polytope*, and if P is a d-polytope in $\mathbb{R}^d$ then it is *full dimensional.* Two polytopes $P \subset \mathbb{R}^n$ and $Q \subset \mathbb{R}^m$ are *affinely equivalent* if there is an affine transformation $\mathbb{R}^n \longrightarrow \mathbb{R}^m$ which maps P bijectively onto Q. Given a lattice $\Lambda \subset \mathbb{R}^d$, a polytope $P \subset \mathbb{R}^d$ is a *Λ-polytope* if all its vertices are in Λ, and is *Λ-free* if, moreover, it contains no lattice points other than its vertices, i.e., if $P \cap \Lambda = vert(P)$. A $\mathbb{Z}^d$-polytope (respectively, $\mathbb{Z}^d$-free polytope) will be called *integer* (respectively, integer-free).

Example 2.1 *Regular integer polytopes* (see [5] for terms used in this example). of all the regular polytopes, only the following admit a regular integer embedding ([25]): the regular d-dimensional simplex α_d, cross polytope β_d, and cube γ_d for all d, and, in addition, the regular hexagon and the 4-dimensional so-called 24-cell. Not all of the above admit a *full dimensional* regular integer embedding: for example, the d-simplex has a regular embedding in $\mathbb{Z}^d$ if and only if $d+1$ is an odd square, the sum of two odd squares, or a multiple of four.

Definition 2.2 A *lattice-free* polytope is one which is Λ-free for some lattice Λ.

Given two sets $S_1 \subset \mathbb{R}^{n_1}$ and $S_2 \subset \mathbb{R}^{n_2}$, their product

$$S_1 \times S_2 = \{(x_1, x_2) : \ x_1 \in S_1, \ x_2 \in S_2\} \subset \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \cong \mathbb{R}^{n_1+n_2}$$

is a polytope if S_1 and S_2 are and is a lattice if S_1 and S_2 are. The affine dimension of $S_1 \times S_2$ equals the sum of the affine dimensions of S_1 and S_2. If $\Lambda_i \subset \mathbb{R}^{n_i}$ is a lattice and P_i is a Λ_i-free polytope ($i = 1, 2$), then $P_1 \times P_2$ is $(\Lambda_1 \times \Lambda_2)$-free. Also, any face of a lattice-free polytope is lattice-free. We record this below.

Observation 2.3 *The class of lattice-free polytopes is closed under taking products and taking faces.*

We now give an example of an important lattice-free polytope, which will be used later on. It had been studied by several authors in connection with the 27 lines on the cubic surface. For a more detailed account of this polytope, see [4] and [5], and also [7].

Example 2.4 *Schläfli polytope.* Let P_{SCH} be the 6-dimensional *Schläfli polytope* (2_{21} in Coxeter notation [5]) in $\mathbb{R}^8$, with the 27 vertex set

$$\{e_i + e_j : \; i = 1, 2, \;\; 3 \le j \le 8\} \cup \{\frac{1}{2}\sum_{i=1}^{8} e_i - (e_j + e_k) : \; 3 \le j < k \le 8\} \subset \mathbb{R}^8.$$

It is an E_8-free polytope (moreover, it is a Delaunay polytope of E_8 - see next section), where E_8 is the root lattice having the $\mathbb{Z}$-basis

$$\{e_i - e_{i-1} : \; 3 \le i \le 8\} \cup \{e_1 + e_2, \; \frac{1}{2}(e_1 + \sum_{i=6}^{8} e_i - \sum_{i=2}^{5} e_i)\} \subset \mathbb{R}^8.$$

Any $\mathbb{Z}$-submodule of $\mathbb{Z}^d$ is free (see e.g. [16, Chapter 2, Theorem 1.6]), and is moreover a lattice, since any $\mathbb{Z}$-basis for it is also $\mathbb{R}$-linearly independent. Thus, more generally, any $\mathbb{Z}$-submodule of any lattice is a lattice. In particular, the intersection $L \cap \Lambda$ of a lattice $\Lambda \subset \mathbb{R}^n$ and an $\mathbb{R}$-linear subspace $L \subset \mathbb{R}^n$ is a lattice. Thus, if P is a d-dimensional Λ-polytope then there is an isomorphism of affine spaces $\text{aff}(P) \longrightarrow \mathbb{R}^d$ which maps $\Lambda \cap \text{aff}(P)$ bijectively onto $\mathbb{Z}^d$. In particular, P is mapped bijectively onto an integer polytope which is integer-free if and only if P is Λ-free.

Proposition 2.5 *Any lattice-free polytope is affinely equivalent to a full dimensional integer-free polytope.*

Remark. With each integer d-polytope there are associated two integer vectors $h = (h_0, \cdots, h_d)$ and $h^* = (h_0^*, \cdots, h_d^*)$. The first is determined by the so-called *f-vector* of P, whose components are the numbers of faces of P of various dimensions; the second is determined by the Ehrhart polynomial of P (see e.g. [29]). While $h_0 = h_0^*$ always, the polytope is integer-free if and only if $h_1 = h_1^*$ as well.

3 Layer-number and diameter of lattice-free polytopes

Let $\text{lin}(P - P)$ denote the $\mathbb{R}$-linear span of $P - P = \{x - y : \; x, y \in P\}$, that is, the linear subspace parallel to the affine hull of P. We use $F \le P$ to denote that F is a face of P, possibly the empty face. For a polytope $P \subset \mathbb{R}^n$ and $a \in \mathbb{R}^n$ let

$$F(P, a) = \{x \in P : \; \langle a, y \rangle \le \langle a, x \rangle \;\text{ for all }\; y \in P\}$$

be the face of P in direction a, and let

$$w(P,a) = |\{\langle a, v\rangle : \ v \in vert(P)\}| - 1$$

be the number of layers formed by the partition of the vertices of P according to their values under $\langle a, \cdot\rangle$.

Definition 3.1 The *layer-number* $w(P)$ of a polytope P is defined to be zero if $\dim(P) = 0$ and otherwise by

$$w(P) = min\{w(P,a) : \ a \in \operatorname{lin}(P-P) \setminus \{0\}\}.$$

It is invariant under affine equivalence.

Remark. A related, but different, notion of layer-number was considered in [10] in the context of $\mathbb{Z}$-equivalence. It is, though, of little relevance to our study here, and will not be discussed.

In low dimensions, the following result of Scarf [27] about the layer-number is available.

Proposition 3.2 *For $d \leq 3$, all d-dimensional lattice-free polytopes satisfy $w(P) \leq 1$.*

We now show that in higher dimensions the layer-number of a lattice-free polytope may exceed 1.

Let $v(P)$ denote the number of vertices of a polytope P.

Definition 3.3 The *index* of a polytope P is the ratio

$$i(P) = \frac{v(P)}{max\{v(F) : \ F < P\}}$$

of the number of vertices of P to the maximal number of vertices of proper face of P (it is infinite when P is 0-dimensional).

Proposition 3.4 *If the layer-number of a polytope P satisfies $w(P) = 1$, then there are two proper faces of P that partition the vertex set of P, and the index of P satisfies $i(P) \leq 2$.*

Proof. Pick $a \in \operatorname{lin}(P-P)$ such that $w(P,a) = w(P) = 1$. Then each vertex of P is in exactly one of the two proper faces $F(P,a)$ and $F(P,-a)$. One of these faces contains at least half the number of vertices of P, so the index of P is at most two. □

Proposition 3.5 *The layer-number of the Schläfli polytope satisfies $w(P_{\mathrm{SCH}}) = 2$.*

Proof. The vector $e_1 - e_2$ lies in $\operatorname{lin}(P_{\mathrm{SCH}} - P_{\mathrm{SCH}})$, so $w(P_{\mathrm{SCH}}) \leq w(P_{\mathrm{SCH}}, e_1 - e_2) = 2$. Now, it is known (see [4, Section 4]), and can be verified by direct computation, that all proper faces of P_{SCH} are either simplices of dimension less than or equal to 5, or 5-dimensional cross-polytopes. Thus, the maximum number of vertices in a proper face of P_{SCH} is 10, and so $i(P_{\mathrm{SCH}}) = \frac{27}{10} > 2$. Then, by Proposition 3.4, we have $w(P_{\mathrm{SCH}}) \neq 1$. The claim follows. □

We need the following result of Kannan and Lovász [19].

Proposition 3.6 *There exists a constant c with the following property: for any convex body K in* $\mathbb{R}^d$ *containing no integer points, there exists a non-zero element* $a \in \mathbb{Z}^d$ *such that*

$$max\{\langle a, x\rangle : \ x \in K\} - min\{\langle a, x\rangle : \ x \in K\} \leq cd^2.$$

We now establish the following theorem.

Theorem 3.7 *There exists a constant c such that the layer-number of any d-dimensional lattice-free polytope P satisfies* $w(P) \leq cd^2$. *While* $w(P) \leq 1$ *for any lattice-free polytope P of dimension* $d \leq 3$, *for all* $k \geq 1$ *there is a lattice-free polytope P of dimension* $d = 6k$ *satisfying* $w(P) \geq 2$.

Proof. For the first statement, consider any lattice-free polytope P of dimension $d \geq 1$. Since $w(P)$ is an affine invariant, by Proposition 2.5 we may assume that $P \subset \mathbb{R}^d$ is a full dimensional integer-free polytope. Let c be the constant guaranteed in Proposition 3.6 and choose λ in the range

$$\frac{cd^2}{\lfloor cd^2 \rfloor + 1} < \lambda < 1.$$

Let v be any point in the interior of P and let K be obtained by shrinking P by factor λ about v, that is

$$K = \lambda P + (1-\lambda)v = \{\lambda x + (1-\lambda)v : \ x \in P\}.$$

Then $K \subset int(P)$ so that K is a convex body containing no integer points. Thus, by Proposition 3.6 there exists a non-zero $a \in \mathbb{Z}^d$ such that

$$max\{\langle a, x\rangle : \ x \in K\} - min\{\langle a, x\rangle : \ x \in K\} \leq cd^2.$$

Let x_0 be an element of P achieving $min\{\langle a, x\rangle : \ x \in P\}$ and let x_1 be an element achieving $max\{\langle a, x\rangle : \ x \in P\}$. Then $max\{\langle a, x\rangle : \ x \in K\} \geq \langle a, (\lambda x_1 + (1-\lambda)v)\rangle$ and $min\{\langle a, x\rangle : \ x \in K\} \leq \langle a, (\lambda x_0 + (1-\lambda)v)\rangle$, so we have

$$\begin{aligned} z &:= max\{\langle a, x\rangle : \ x \in P\} - min\{\langle a, x\rangle : \ x \in P\} = \langle a, x_1\rangle - \langle a, x_0\rangle \\ &= \frac{1}{\lambda}(\langle a, (\lambda x_1 + (1-\lambda)v\rangle - \langle a, (\lambda x_0 + (1-\lambda)v\rangle) \\ &\leq \frac{1}{\lambda}(max\{\langle a, x\rangle : \ x \in K\} - min\{\langle a, x\rangle : \ x \in K\}) \leq \frac{1}{\lambda}cd^2 < \lfloor cd^2 \rfloor + 1. \end{aligned}$$

Now, the maximum and minimum over P are attained at vertices, which are integer points, and $a \in \mathbb{Z}^d$, so z is an integer, and so in fact $z \leq \lfloor cd^2 \rfloor \leq cd^2$. Moreover, for any two vertices u, v of P, if $\langle a, v\rangle > \langle a, u\rangle$ then in fact $\langle a, v\rangle \geq \langle a, u\rangle + 1$, and so

$$w(P) \leq w(P, a) = |\{\langle a, v\rangle : \ v \in vert(P)\}| - 1 \leq z \leq cd^2$$

and we are done.

We now turn to the second statement. The claim about dimensions $d \leq 3$ being provided by Proposition 3.2, it remains to construct lattice-free polytopes with layer-number exceeding 1 as claimed. First we show that the index of the product of two polytopes satisfies

$$i(P_1 \times P_2) = min\{i(P_1), \ i(P_2)\}.$$

Indeed, the faces of the product $P_1 \times P_2$ are exactly the products $F_1 \times F_2$ where F_i is a face of P_i $(i = 1, 2)$. For $i = 1, 2$, let F_i be a proper face of P_i with maximum number of vertices. Then either $F_1 \times P_2$ or $P_1 \times F_2$ is a proper face of $P_1 \times P_2$ with maximum number of vertices. Thus, as claimed,

$$\begin{aligned} i(P_1 \times P_2) &= \frac{v(P_1 \times P_2)}{max\{v(F_1 \times P_2),\ v(P_1 \times F_2)\}} \\ &= min\{\frac{v(P_1)v(P_2)}{v(F_1)v(P_2)},\ \frac{v(P_1)v(P_2)}{v(P_1)v(F_2)}\} = min\{i(P_1),\ i(P_2)\}. \end{aligned}$$

Now, the Schläfli polytope is a 6-dimensional lattice-free polytope with $i(P_{\text{SCH}}) > 2$. By Observation 2.3 and the claim above, for any $k \geq 1$ the k-fold product

$$P_k = P_{\text{SCH}} \times \cdots \times P_{\text{SCH}}$$

of the Schläfli polytope is a $6k$-dimensional lattice-free polytope of index $i(P_k) = i(P_{\text{SCH}}) > 2$. By Proposition 3.4, $w(P_k) \geq 2$ and the statement is proved. □

Let $f(d)$ be the maximum value of $w(P)$ over all d-dimensional lattice-free polytopes P. Theorem 3.7 asserts that $f(d) \leq cd^2$ for some constant c, that $f(d) \leq 1$ for $d \leq 3$, and that $f(d) \geq 2$ for all $d = 6k$ $(k \geq 1)$. The exact value of $f(d)$ is unknown for all $d \geq 4$, and in particular it is unknown whether $f(4) > 1$ and $f(5) > 1$. Of most interest is the following question.

Question 3.8 *How fast does $f(d)$ grow with d?*

We now turn to discuss the diameter of lattice-free polytopes. Denote by $d(u, v)$ the distance between two vertices u and v of P on its 1-skeleton, i.e. the smallest number of edges in a walk from u to v. The *diameter* of P is

$$\delta(P) = max\{d(u, v) :\ u, v \in vert(P)\}.$$

The following result is from [21]. We include the proof for completeness.

Proposition 3.9 *Let P be a polytope of positive dimension. Then*

$$\delta(P) \leq w(P) + max\{\delta(F) :\ F < P\}.$$

Proof. Pick $a \in \text{lin}(P - P)$ satisfying $w(P) = w(P, a)$, and let $F' = F(P, -a)$ and $F'' = F(P, a)$ be the two distinct proper faces of P in directions $-a$ and a, respectively. Pick two vertices u and v of P such that $\delta(P) = d(u, v)$. There exists a (possibly empty) path on the 1-skeleton of P from u to some vertex $u' \in F'$, where the vertices along the path attain strictly decreasing values under $\langle a, \cdot \rangle$. Similarly, there exists a strictly decreasing path from v to some $v' \in F'$, and strictly increasing paths from u and v to some vertices u'' and v'' in F'', respectively. Now

$$\begin{aligned} min\{d(u, u') + d(v', v),\ d(u, u'') + d(v'', v)\} &\leq \frac{1}{2}(d(u, u') + d(v', v) + d(u, u'') + d(v'', v)) \\ &\leq \frac{1}{2}(w(P, a) + w(P, a)) = w(P, a), \end{aligned}$$

since $d(u', u) + d(u, u'') \leq w(P, a)$ and $d(v', v) + d(v, v'') \leq w(P, a)$. Assuming then that, say, $d(u, u') + d(v', v) \leq w(P, a)$ and noting that every edge of F' is also an edge of P, we find

$$\begin{aligned} \delta(P) &= d(u, v) \leq d(u, u') + d(u', v') + d(v', v) \\ &\leq w(P, a) + \delta(F') \leq w(P) + max\{\delta(F) : F < P\}. \quad \square \end{aligned}$$

We now prove that the diameter of any lattice-free polytope is bounded by a low degree polynomial in its dimension.

Theorem 3.10 *There exists a constant c such that the diameter of any d-dimensional lattice-free polytope P satisfies $\delta(P) \leq cd^3$.*

Proof. We use induction on the dimension d of P. If $d = 0$ then $\delta(P) = 0$ as well. Assume then that $d \geq 1$. Any proper face F of P is also lattice-free and has smaller dimension than d, so by induction $\delta(F) \leq c(d-1)^3$. By Proposition 3.9 and Theorem 3.7 we then obtain

$$\delta(P) \leq w(P) + max\{\delta(F) : F < P\} \leq cd^2 + c(d-1)^3 \leq cd^3. \quad \square$$

It is known (see [23] and [21]) that the diameter of a d-dimensional $[0,1]$-polytope (see next section) is at most d. As we shall see, $[0,1]$-polytopes form a strict subclass of lattice-free polytopes, and so the following question remains.

Question 3.11 *Can the diameter of a lattice-free polytope exceed its dimension?*

Let $\Delta(n, d)$ be the largest diameter of any d-polytope with at most n facets. The *Hirsch conjecture* is that $\Delta(n, d) = n - d$, but it is as yet unknown whether $\Delta(n, d)$ is bounded from above by any polynomial in n and d (see [20] for more information). Existing upper bounds on $\Delta(n, d)$ include $n2^{d-3}$ [22] and n^{logd+2} [18]. The first of these can be combined with the results above to obtain the following more general bound for lattice polytopes.

Theorem 3.12 *There exists a constant c for which the following holds: the diameter of any d-dimensional lattice polytope P with no more than n facets and the property that for some $k \leq d$ no face of P of dimension larger than k contains a lattice point in its relative interior, satisfies*

$$\delta(P) \leq cd^2(d-k) + (n - (d-k))2^k.$$

Proof. First note that any facet F of a polytope P has fewer facets than P since each facet of F is the intersection of F and another facet of P. We use induction on $d - k$. If $d - k = 0$ then the bound is $n2^d$ and is guaranteed by [22]. Assume then that $d - k \geq 1$. Any proper face F of P has at most $n - 1$ facets. By induction and using Proposition 3.9 and Theorem 3.7 we thus obtain

$$\delta(P) \leq cd^2 + (c(d-1)^2(d-1-k) + (n-1-(d-1-k))2^k) \leq cd^2(d-k) + (n-(d-k))2^k. \quad \square$$

Applying this theorem with $k = \log(d)$, we obtain a polynomial bound on the diameter for a larger class of lattice polytopes.

Corollary 3.13 *There exists a constant c for which the following holds: the diameter of any d-dimensional lattice polytope P with no more than n facets and the property that no face of P of dimension larger than $\log(d)$ contains a lattice point in its relative interior, satisfies*

$$\delta(P) \leq nd + cd^3.$$

4 Affine-types of lattice-free polytopes

In this section we consider affine-types of the class of lattice-free polytopes and its following two important subclasses: the class of Delaunay-polytopes, consisting of lattice-free polytopes which are embeddable on an empty sphere; and the class of $[0,1]$-polytopes, consisting of lattice-free polytopes which are embeddable on the unit cube. Our main concern is the determination of whether or not the number of affinely nonequivalent polytopes in each of the three classes is finite. By the *affine-type* of a polytope we shall mean its equivalence class under affine equivalence. We start with the class of Delaunay polytopes. By a *d-ball* in $\mathbb{R}^n$ we mean the intersection of a full dimensional closed ball with a d-dimensional affine subspace of $\mathbb{R}^n$. Given a lattice $\Lambda \subset \mathbb{R}^n$, a d-ball in $\mathbb{R}^n$ is a Λ-*hole* if it contains $d+1$ affinely independent lattice points in its relative boundary (in its affine hull), whereas its relative interior contains none. A *Delaunay* Λ-*polytope* is the convex hull of the lattice points in some Λ-hole. In particular, a Delaunay Λ-polytope is Λ-free.

Definition 4.1 A *Delaunay polytope* is one which is a Delaunay Λ-polytope for some lattice Λ.

Examples of Delaunay polytopes in suitable lattices include, for all d, the d-dimensional simplex, cross-polytope, and cube. Of special interest to us is the Schläfli polytope.

Example 4.2 *Schläfli polytope.* Let $E_8 \subset \mathbb{R}^8$ be the root lattice having the $\mathbb{Z}$-basis as in Example 2.4, so that the Schläfli polytope P_{SCH} is E_8-free. Let

$$A = \{x = (x_1, \cdots, x_8) \in \mathbb{R}^8 : \; x_1 + x_2 = 1, \; \; \frac{1}{2}\sum_{i=1}^{8} x_i = 1\},$$

and let $B = \{x \in A : \; ||x|| \leq \sqrt{2}\}$. Then A is a 6-dimensional affine subspace of $\mathbb{R}^8$ and B is a 6-ball, and it is easy to verify that the vertex set of P_{SCH} lies on the relative boundary of B in A. It can be verified that, moreover, $B \cap E_8 = vert(P_{\mathrm{SCH}})$ (for more details see [7]). Thus, the Schläfli polytope P_{SCH} is a Delaunay polytope.

We mention that Delaunay polytopes are closely related to *Voronoi polytopes* [30]: given a full dimensional lattice $\Lambda \subset \mathbb{R}^d$, the Λ-Voronoi polytope of $x \in \Lambda$ is the set of all points in $\mathbb{R}^d$ the distance of which from x is no larger than from any other lattice point. The vertices of the Λ-Voronoi polytope of x are precisely the centers of the full dimensional Λ-holes that contain x. As is well known, Both the collection of all Λ-Voronoi polytopes and the collection of all full dimensional Delaunay Λ-polytopes form tilings of $\mathbb{R}^d$. The finiteness of the number of affinely nonequivalent Dalaunay polytopes was established by Voronoi [30]. An accessible proof can be found in [6].

Proposition 4.3 *For any d there are finitely many affine-types of d-dimensional Delaunay polytopes.*

Next, we consider the class of $[0,1]$-polytopes, and more generally, the class of $[0,k]$-polytopes consisting of polytopes which are embeddable on a fixed grid of size k. Here we loose no generality by assuming the lattice to be the integer lattice, and we let $\{0, \cdots, k\}^n$ be the set of integer points in $\mathbb{Z}^n$ all coordinates of which (with respect to the standard basis) are in $\{0, 1, \cdots, k\}$.

Definition 4.4 A $[0,k]$-*polytope* is one whose vertices lie in $\{0,\cdots,k\}^n \subset \mathbb{Z}^n$ for some n.

Of course, any rational polytope is homothetic to a $[0,k]$-polytope for some k, but for a fixed k, the $[0,k]$-polytopes form a proper subclass. Note also that the layer-number of a $[0,k]$-polytope is at most k. Clearly, $[0,1]$-polytopes are integer-free.

The definition of a $[0,k]$-polytope is not invariant under affine transformations, so the following question arises: does any d-dimensional $[0,k]$-polytope in $\mathbb{R}^n$ have a full dimensional affinely equivalent $[0,k]$-polytope in $\mathbb{R}^d$? Proposition 4.5 below provides a positive answer. The argument is essentially the same as the one used in [24] for a special case regarding 1-skeletons; the second author thanks W. Pulleyblank for drawing his attention to this fact.

Proposition 4.5 *Any* $[0,k]$*-polytope is affinely equivalent to a full dimensional* $[0,k]$*-polytope.*

Proof. We prove by induction on $n-d$ that any d-dimensional $[0,k]$-polytope $P \subset \mathbb{R}^n$ has an affinely equivalent full dimensional $[0,k]$-polytope in $\mathbb{R}^d$. The statement holding trivially for $n-d=0$, assume that $n-d \geq 1$ and let $P \subset \mathbb{R}^n$ be a d-dimensional $[0,k]$-polytope. Let $a=(a_1,\cdots,a_n) \in \mathbb{R}^n$ be a nonzero element in the orthogonal complement of $\mathrm{lin}(P-P)$. Without loss of generality assume that $a_n \neq 0$, and consider the projection

$$\pi : \mathbb{R}^n \longrightarrow \mathbb{R}^{n-1} : \ (x_1,\cdots,x_n) \mapsto (x_1,\cdots,x_{n-1}).$$

We now show that π is injective on P. Suppose $x,y \in P$ have the same image under π. Then

$$\mathbf{0} = \pi(x) - \pi(y) = \sum_{i=1}^{n-1}(x_i - y_i)e_i,$$

so $x_i - y_i = 0$ for $i = 1,\cdots,n-1$. Since $a \in (\mathrm{lin}(P-P))^{\perp}$, we get

$$0 = \langle a, x-y \rangle = \sum_{i=1}^{n} a_i(x_i - y_i) = a_n(x_n - y_n).$$

Since $a_n \neq 0$, we find that $x_n - y_n = 0$ as well, and so $x = y$. Thus, $\pi(P) \subset \mathbb{R}^{n-1}$ is affinely equivalent to P, and is clearly a $[0,k]$-polytope. By induction, $\pi(P)$, and hence P, has an affinely equivalent full dimensional $[0,k]$-polytope in $\mathbb{R}^d$, and we are done. □

The following analogue of Voronoi's theorem (Proposition 4.3) can be deduced, as pointed out by a referee, from the upper bounds of Konyagin-Sevastyanov and Bárány-Vershik on the number of nonequivalent lattice polytopes of fixed volume [2]. However, Proposition 4.5 above provides the following simple proof for this fact.

Proposition 4.6 *For any d there are finitely many affine-types of d-dimensional* $[0,k]$*-polytopes.*

Proof. By the above Proposition, any d-dimensional $[0,k]$-polytope is affinely equivalent to one with vertices in $\{0,\cdots,k\}^d$, and there are only finitely many such. □

Turning now to the class of all lattice-free polytopes, it is not too hard to determine all affine types of lattice-free polytopes in each dimension $d \leq 2$: for example, the affine types that occur among 2-dimensional lattice-free polytopes are the triangle and the parallelogram (both admitting both Dalaunay and $[0,1]$-embeddings). However, in higher dimensions there turn out to be infinitely many affine types of lattice-free polytopes, as we now show. To this end, we need to discuss the *spread* of a simplicial point configuration [11]. Given an affinely independent set of points $S = \{v_0, \cdots, v_d\} \subset \mathbb{R}^d$, let $V(S)$ denote the (positive) volume of the simplex conv(S), given, up to a normalizing constant, by the absolute value of the determinant of $[v_1 - v_0, \cdots, v_d - v_0]$. For a finite set $T \subset \mathbb{R}^d$ of points in general position, i.e., no $d+1$ of which are affinely dependent, define (see [11]) the *spread* $D(T)$ of T as the ratio

$$D(T) = \frac{max_R V(R)}{min_S V(S)},$$

where R and S range over all $(d+1)$-subsets of T. The spread is invariant under nonsingular affine transformations on $\mathbb{R}^d$, since the ratio $\frac{V(\phi(S))}{V(S)}$ is the same for any d-simplex $S \subset \mathbb{R}^d$.

Proposition 4.7 *For any $d \geq 3$ there are infinitely many affine-types of d-dimensional lattice-free polytopes.*

Proof. Consider first the case $d = 3$. For $n \geq 1$, let

$$S_n = \{(0,0,0), (1,0,0), (0,1,0), (0,0,1), (1,1,n)\} \subset \mathbb{R}^3.$$

Then S_n is a set of points in general position and $D(S_n) = n$, and $P_n = \text{conv}(S_n)$ is a 3-dimensional integer-free polytope. Since $D(S_m) \neq D(S_n)$ for all $1 \leq m < n$, no nonsingular affine transformation takes S_m to S_n, and so P_m and P_n are affinely nonequivalent. Now, define $P_{d,n}$ for all $d \geq 3$ and $n \geq 1$ recursively by putting $P_{3,n} = P_n$ and setting $P_{d+1,n} = \text{conv}(P_{d,n} \cup \{e_{d+1}\}) \subset \mathbb{R}^{d+1}$, a pyramid over $P_{d,n}$. By induction on d, we claim that all $P_{d,n}$ are integer-free and that $P_{d,m}$ and $P_{d,n}$ are affinely nonequivalent for all $1 \leq m < n$. Indeed, $P_{d+1,n}$ is integer-free since $P_{d,n}$ is; also, an affine bijection of $P_{d+1,m}$ onto $P_{d+1,n}$ must take the face $P_{d,m}$ of $P_{d+1,m}$ bijectively onto the face $P_{d,n}$ of $P_{d+1,n}$, which is impossible by induction. This completes the proof. □

We collect Propositions 4.3, 4.6, and 4.7 in the following theorem.

Theorem 4.8 *For each dimension d there are finitely many affine-types of Delaunay-polytopes and $[0,1]$-polytopes, but (for $d \geq 3$) infinitely many affine-types of lattice-free polytopes.*

5 Combinatorial-types of lattice-free polytopes

We turn now to discuss the combinatorial-types of the class of lattice-free polytopes and its subclasses of Delaunay polytopes and $[0,1]$-polytopes. The *combinatorial-type* of a polytope $P \subset \mathbb{R}^d$ is its equivalence class $[P]$ under isomorphism of face lattices (cf. [15]); equivalently, two polytopes Q and P are of the same combinatorial-type if and only if there exists a bijection $\phi : vert(P) \longrightarrow vert(Q)$ between their vertex sets such that conv(F) is a face of P if and only if conv$(\phi(F))$ is a face of Q for all $F \subseteq vert(P)$. If two polytopes are affinely equivalent then they are also of the same combinatorial-type. We will use Δ, Σ, and $[P]$ to denote combinatorial-types. An *embedding* of a combinatorial-type Δ is any polytope P of that type $[P] = \Delta$. In contrast with Proposition 4.7, the following statement holds for combinatorial-types of lattice-free polytoes.

Proposition 5.1 *For any d there are finitely many combinatorial-types of d-dimensional lattice-free polytopes.*

Proof. The statement follows from the following well known fact from [8]: the number of vertices of a d-dimensional lattice-free polytope can not exceed 2^d. Indeed, if a lattice-polytope P, which we may assume is full dimensional and integer (Proposition 2.5) has more vertices, then two of them agree in parity in each of their coordinates, so their midpoint is a non-vertex integer point in P. □

If $\mathcal{C}$ is a class of combinatorial-types of polytopes, we write $\mathcal{C}_d$ for its subclass of types of d-dimensional polytopes so that $\mathcal{C} = \cup_{d \geq 0} \mathcal{C}_d$. Denote by $\mathcal{F} = \cup_{d \geq 0} \mathcal{F}_d$ the class of all combinatorial-types of lattice-free polytopes, and by $\mathcal{D} = \cup_{d \geq 0} \mathcal{D}_d$ (respectively, $\mathcal{I} = \cup_{d \geq 0} \mathcal{I}_d$) the class of all combinatorial-types of Delaunay polytopes (respectively, $[0,1]$-polytopes). Delaunay polytopes and $[0,1]$-polytopes are lattice-free, so $\mathcal{D}_d \cup \mathcal{I}_d \subseteq \mathcal{F}_d$ for all d; our main concern here is to provide more precise information about the interelations among $\mathcal{F}_d, \mathcal{D}_d$, and $\mathcal{I}_d$ for all d. We need to introduce the following combinatorial invariant of polytopes.

Definition 5.2 The *width* $W(P)$ of a polytope P is the smallest value of the layer-number $w(Q)$ (Definition 3.1) over all polytopes Q of the same combinatorial-type $[Q] = [P]$.

Proposition 5.3 *For any $k \geq 1$, the k-fold product $P_k = P_{\text{SCH}} \times \cdots \times P_{\text{SCH}}$ of the Schläfli polytope satisfies $W(P_k) \geq 2$.*

Proof. The index of P_k satisfies $i(P_k) > 2$ (see proof of Theorem 3.7). Let P be a polytope of the same combinatorial-type as P_k achieving $W(P_k) = W(P) = w(P)$. Since the index is a combinatorial invariant, P has index $i(P) = i(P_k) > 2$ as well, and so, by Proposition 3.4, we have $w(P) \geq 2$. The claim follows. □

In analogy to the discussion about the layer-number, let $g(d)$ be the maximum value of $W(P)$ over all d-dimensional lattice-free polytopes P. Note that $g(d) \leq f(d)$ for all d, where $f(d)$ is the function discussed in Question 3.8. Thus, $g(d) \leq cd^2$ for some constant c, $g(d) \leq 1$ for $d \leq 3$, and $g(d) \geq 2$ for all $d = 6k$, $k \geq 1$ (Proposition 5.3), but the exact value of $g(d)$ is unknown for all $d \geq 4$.

Question 5.4 *How fast does $g(d)$ grow with d?*

By a *face* of a combinatorial-type Δ we mean any combinatorial-type Σ of any face of any polytope having combinatorial-type Δ. The *product* $\Delta_1 \times \Delta_2$ of two combinatorial-types is the combinatorial-type of $P_1 \times P_2$, where P_i is any embedding of Δ_i $(i = 1, 2)$. It is well defined since, as mentioned before, the faces of $P_1 \times P_2$ are the products $F_1 \times F_2$ where F_i is a face of P_i $(i = 1, 2)$, so $[P_1 \times P_2]$ depends only on $[P_1]$ and $[P_2]$. A class $\mathcal{C}$ is closed under *taking products* if $\Delta_1 \times \Delta_2$ is in $\mathcal{C}$ whenever Δ_1 and Δ_2 are, and is closed under *taking faces* if any face Σ of $\Delta \in \mathcal{C}$ is also in $\mathcal{C}$. It is easy to see that, in analogy with Observation 2.3, the product of two Delaunay polytopes (respectively, $[0,1]$-polytopes) is again a Delaunay polytope (respectively, $[0,1]$-polytope). Thus, each of the classes $\mathcal{F}, \mathcal{D}$, and $\mathcal{I}$ of combinatorial-types of lattice-free polytopes, Delaunay polytopes, and $[0,1]$-polytopes, respectively, is closed under taking products and under taking faces.

Proposition 5.5 *Let $\mathcal{C} = \cup_{d\geq 0}\mathcal{C}_d$ be a class of combinatorial-types which is closed under taking products, and let $\mathcal{M} = \cup_{d\geq 0}\mathcal{M}_d$ be a class of combinatorial-types which is closed under taking faces. Assume further that the combinatorial type of the segment belongs to $\mathcal{C}$. If $\mathcal{C}_{d_0} \not\subseteq \mathcal{M}_{d_0}$ for some d_0, then $\mathcal{C}_d \not\subseteq \mathcal{M}_d$ for all $d \geq d_0$.*

Proof. Let Σ denote the segment, so that $\Sigma \in \mathcal{C}_1$. Pick any $\Delta \in \mathcal{C}_{d_0} \setminus \mathcal{M}_{d_0}$. Given $d > d_0$, let $\Delta' = \Delta \times \Sigma \times \cdots \times \Sigma$ be the product of Δ with the $(d-d_0)$-fold product of Σ. Then $\Delta' \in \mathcal{C}_d$. But Δ is face of Δ' which is not in $\mathcal{M}$, so Δ' cannot be in $\mathcal{M}$ either. $\square$

We now prove the theorem about the interelations among the classes $\mathcal{F}_d$, $\mathcal{D}_d$ and $\mathcal{I}_d$ in each dimension d. The proof makes use of the complete lists of types in $\mathcal{D}_3$ and $\mathcal{D}_4$ provided by Theorems 5.1 and 6.2, respectively, of [9]. The only relation unsettled by our theorem is whether or not $\mathcal{D}_5 \subset \mathcal{I}_5$. In particular, it follows from our theorem that the classes $\mathcal{F}$, $\mathcal{D}$, and $\mathcal{I}$ pairwise differ in infinitely many combinatorial-types.

Theorem 5.6 *The following interelations among the classes of combinatorial-types of lattice-free polytopes, Delaunay polytopes, and $[0,1]$-polytopes, hold:*

$$
\begin{array}{lll}
(1) & \mathcal{D}_d = \mathcal{I}_d = \mathcal{F}_d & (d \leq 2) \\
(2) & \mathcal{D}_d \cup \mathcal{I}_d \neq \mathcal{F}_d & (d \geq 3) \\
(3) & \mathcal{I}_d \not\subseteq \mathcal{D}_d & (d \geq 3) \\
(4) & \mathcal{D}_d \subset \mathcal{I}_d & (d = 3, 4) \\
(5) & \mathcal{D}_d \not\subseteq \mathcal{I}_d & (d \geq 6).
\end{array}
$$

Proof.
(1). It is easily seen that in dimensions at most two all three classes consist of the following combinatorial-types: the point, the segment, the triangle, and the quadrangle.
(2). Let $U = \{(0,0,0),(1,0,0),(0,1,0),(1,1,0)\}$ and $V = \{(0,0,1),(1,0,1),(2,1,1)\}$ be two sets of integer points in $\mathbb{R}^3$. Now conv(U) is an integer-free square in the hyperplane $\{x : x_3 = 0\}$ and conv(V) is an integer-free triangle in the hyperplane $\{x : x_3 = 1\}$, so $P = \text{conv}(U \cup V)$ is an integer-free polytope, hence $[P] \in \mathcal{F}_3$. By Theorem 5.1 in [9], no 3-dimensional Delaunay polytope has 7 vertices as does P, so $[P] \notin \mathcal{D}_3$. Now, P has a quadrangular facet conv(U) and a triangular facet conv(V) which are disjoint. It is not hard to see that any $[0,1]$-polytope in $\mathbb{R}^3$ with this property is affinely equivalent to $Q = \text{conv}(U \cup \{(0,0,1),(1,0,1),(1,1,1)\})$. It is then easy to verify that Q has 3 quadrangular facets whereas P has only 2, so P and Q are not of the same combinatorial-type. Hence $[P] \notin \mathcal{I}_3$ as well. The claim follows from Proposition 5.5 with $\mathcal{C}_d = \mathcal{F}_d$, $\mathcal{M}_d = \mathcal{D}_d \cup \mathcal{I}_d$, and $d_0 = 3$.
(3). The $[0,1]$-polytope Q in (2) has 7-vertices, so $[Q] \in \mathcal{I}_3 \setminus \mathcal{D}_3$, and the claim follows again from Proposition 5.5 with $\mathcal{C}_d = \mathcal{I}_d$, $\mathcal{M}_d = \mathcal{D}_d$, and $d_0 = 3$.
(4). Theorems 5.1 and 6.2 in [9] describe all five combinatorial-types in $\mathcal{D}_3$ and all nineteen types in $\mathcal{D}_4$. We have verified that each of these types has a $[0,1]$-embedding. The details are omitted.
(5). The layer-number, hence the width, of any $[0,1]$-polytope is at most 1. We conclude from Proposition 5.3 that the combinatorial-type of the Schläfli polytope P_{SCH} satisfies $[P_{\text{SCH}}] \in \mathcal{D}_6 \setminus \mathcal{I}_6$. The claim then follows from Proposition 5.5 with $\mathcal{C}_d = \mathcal{D}_d$, $\mathcal{M}_d = \mathcal{I}_d$, and $d_0 = 6$. $\square$

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