Finite Connections for Supercritical Bernoulli Bond Percolation in 2D

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Outline

Introduction
  Percolation on $\mathbb{Z}^d$
  Logarithmic Asymptotics of Connectivities
  Sharp Asymptotics of Connectivities

Sketch of Proof
  Setup
  Geometry of Finite Connections
  The Structure of a Cluster
  Asymptotics for No Intersection of Two Decorated RWs

Summary
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Summary
Percolation

- Take $G = (\mathbb{Z}^d, \mathcal{E}^d)$ - the integer lattice with nearest neighbor edges.
- Open each edge with probability $p \in [0, 1]$ independently.
- Let $\mathbb{B}_p$ be the underlying measure.

**Theorem (Broadbent, Hammersley 1957)**

For all $d > 2$, there exists $p_c(d) \in (0, 1)$ such that:

$$\mathbb{B}_p(0 \leftrightarrow \infty) = \begin{cases} 0 & \text{if } p < p_c(d) \\ \theta(p) > 0 & \text{if } p > p_c(d) \end{cases}$$
Basic Picture

Sub-critical density $p < p_c(d)$:
- All clusters (connected components) are finite.
- Radii of clusters have exponentially decaying distributions:
  \[
  \exists \xi_p \in (0, \infty) : \mathbb{P}_p(0 \leftrightarrow \partial B(R)) \approx e^{-\xi_p R}.
  \]
  Russo-Menshikov (86), Barskey-Aizenman (87).

Super-critical density $p > p_c(d)$:
- Unique infinite cluster $\mathbb{B}_p$-a.s.
- Radii of finite clusters have exponentially decaying distributions:
  \[
  \exists \zeta_p \in (0, \infty) : \mathbb{P}_p(\infty \leftrightarrow 0 \leftrightarrow \partial B(R)) \approx e^{-\zeta_p R}.
  \]
  Chayes$^2$-Newman (87), C$^2$-Grimmett-Kesten-Schonmann (89).
Connectivities

The point-to-point connectivity function is defined as:

\[ \tau_p(x, y) = \mathbb{B}_p(x \leftrightarrow y) ; \quad x, y \in \mathbb{Z}^d. \]
Connectivities

The point-to-point connectivity function is defined as:

$$\tau_p(x, y) = \mathbb{B}_p(x \leftrightarrow y) ; \quad x, y \in \mathbb{Z}^d.$$ 

Sub-critical case:
Subadditivity (via FKG of $\mathbb{B}_p$) and exponential decay of cluster radius distribution imply:

**Theorem**

Assume $p < p_c(d)$. Then, for all $x \in \mathbb{R}^d$:

$$\xi_p(x) = \lim_{n \to \infty} -\frac{1}{n} \log \tau_p(0, \lfloor nx \rfloor)$$

is well-defined, convex and homogeneous function that is strictly positive on $\mathbb{R}^d \setminus \{0\}$.

In other words $\xi_p$ is a norm on $\mathbb{R}^d$, called the inverse correlation norm.
Connectivities - cont’d

Super-critical case:
FKG gives a uniform positive lower bound for all $x, y \in \mathbb{Z}^d$:

$$\mathbb{B}_p(x \leftrightarrow y) \geq \mathbb{B}_p(x \leftrightarrow \infty) \mathbb{B}_p(y \leftrightarrow \infty) = \theta^2(p) > 0.$$
Connectivities - cont’d

Super-critical case:
FKG gives a uniform positive lower bound for all \(x, y \in \mathbb{Z}^d:\)

\[
\mathbb{B}_p(x \leftrightarrow y) \geq \mathbb{B}_p(x \leftrightarrow \infty) \mathbb{B}_p(y \leftrightarrow \infty) = \theta^2(p) > 0.
\]

Therefore, define the finite (truncated) connectivity function:

\[
\tau^f_p(x, y) = \mathbb{B}_p(x \xrightarrow{f} y) = \mathbb{B}_p(\infty \leftrightarrow x \leftrightarrow y) ; \quad x, y \in \mathbb{Z}^d.
\]
Connectivities - cont’d

Super-critical case:
FKG gives a uniform positive lower bound for all $x, y \in \mathbb{Z}^d$:

$$B_p(x \leftrightarrow y) \geq B_p(x \leftrightarrow \infty)B_p(y \leftrightarrow \infty) = \theta^2(p) > 0.$$ 

Therefore, define the finite (truncated) connectivity function:

$$\tau_p^f(x, y) = B_p(x \leftrightarrow y) = B_p(\infty \leftrightarrow x \leftrightarrow y) ; \quad x, y \in \mathbb{Z}^d.$$ 

### Theorem

Assume $p \notin \{0, p_c(d), 1\}$. Then, for all $x \in \mathbb{R}^d$:

$$\zeta_p(x) = \lim_{n \to \infty} -\frac{1}{n} \log \tau_p^f(0, [nx])$$

is well-defined, homogeneous and strictly positive on $\mathbb{R}^d \setminus \{0\}$.

This is the finite (truncated) inverse correlation function.
Logarithmic Scale Asymptotics

In other words:

\[ B_p(x \leftrightarrow y) \approx e^{-\xi_p(\theta) \|y-x\|_2} \quad \text{and} \quad B_p(x \leftrightarrow^f y) \approx e^{-\zeta_p(\theta) \|y-x\|_2} \]

for all \( x, y \in \mathbb{Z}^d \) as \( y - x \to \infty \), where \( \theta = (x - y)/\|x - y\|_2 \).
Logarithmic Scale Asymptotics

In other words:

\[ \mathbb{B}_p(x \leftrightarrow y) \approx e^{-\xi_p(\theta)\|y-x\|_2} \quad \text{and} \quad \mathbb{B}_p(x \xleftarrow{f} y) \approx e^{-\zeta_p(\theta)\|y-x\|_2} \]

for all \( x, y \in \mathbb{Z}^d \) as \( y - x \to \infty \), where \( \theta = (x - y)/\|x - y\|_2 \).

Some relations:

- \( \xi_p = \xi_p(e_1) \). \( \zeta_p = \zeta_p(e_1) \).
- If \( d = 2 \), \( p > p_c(2) = \frac{1}{2} \) then \( \zeta_p = 2\xi_{1-p} \).
  
  Chayes-Chayes-Grimmett-Kesten-Schonmann (89).
Logarithmic Scale Asymptotics

In other words:

\[ B_p(x \leftrightarrow y) \approx e^{-\xi_p(\theta)\|y-x\|_2} \quad \text{and} \quad B_p(x \xleftarrow{f} y) \approx e^{-\zeta_p(\theta)\|y-x\|_2} \]

for all \( x, y \in \mathbb{Z}^d \) as \( y - x \to \infty \), where \( \theta = (x - y)/\|x - y\|_2 \).

Some relations:

- \( \xi_p = \xi_p(e_1) \) \cdot \( \zeta_p = \zeta_p(e_1) \).
- If \( d = 2 \), \( p > p_c(2) = \frac{1}{2} \) then \( \zeta_p = 2\xi_{1-p} \).

Chayes-Chayes-Grimmett-Kesten-Schonmann (89).

Want sharp asymptotics:

\[ B_p(x \leftrightarrow y) \sim ? \quad \text{and} \quad B_p(x \xleftarrow{f} y) \sim ? \]
For all $d \geq 2$, $p < p_c(d)$, $x, y \in \mathbb{Z}^d$:

$$\mathbb{B}_p(x \leftrightarrow y) \sim A_p(\theta) \|y - x\|_2^{-(d-1)/2} e^{-\xi_p(\theta)\|y - x\|_2}$$

as $y - x \to \infty$.

- Campanino-Chayes-Chayes (88) ($y - x$ is on the axes).
- Campanino-Ioffe (02) (all $y - x$).

With the Gaussian correction this is called **Ornstein-Zernike Behavior**. After the work of L.Ornstein and F.Zernike.
Sharp Asymptotics - Supercritical Case

\(d \geq 3:\)
For all \(p > p_c(d)\), \(x, y \in \mathbb{Z}^d\), it is expected:

\[
B_p(x \xleftarrow{f} y) \sim \tilde{A}_p(\theta) \|y - x\|_2^{-(d-1)/2} e^{-\zeta_p(\theta)\|y - x\|_2} \text{ as } y - x \to \infty.
\]

Verified for \(p \gg p_c(d)\) and \(y - x\) on axes. Braga-Procacci-Sanchis (04).
Sharp Asymptotics - Supercritical Case

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For all \( p > p_c(d) \), \( x, y \in \mathbb{Z}^d \), it is expected:

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Verified for \( p \gg p_c(d) \) and \( y - x \) on axes. Braga-Procacci-Sanchis (04).

\[ d = 2: \]
Sharp Asymptotics - Supercritical Case

\(d \geq 3:\) For all \(p > p_c(d), x, y \in \mathbb{Z}^d,\) it is expected:

\[
\mathbb{B}_p(x \xleftarrow{\mathcal{F}} y) \sim \tilde{A}_p(\theta) \|y - x\|_2^{-(d-1)/2} e^{-\zeta_p(\theta) \|y - x\|_2} \text{ as } y - x \to \infty.
\]

Verified for \(p \gg p_c(d)\) and \(y - x\) on axes. Braga-Procacci-Sanchis (04).

\(d = 2:\)

\[?\]
A Related Model - Nearest-Neighbor Ising
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Exactly solvable in \( d = 2 \) (Onsager (44)).

Explicit formulas for (truncated) \( k \)-point correlation functions at all temperatures (Wu, McCoy, Tracy, Potts, Ward, Montroll (’70)).

**Theorem (Cheng-Wu, Wu)**

If \( \beta < \beta_c(2) \) then

\[
\langle \sigma_x; \sigma_y \rangle_\beta \triangleq \langle \sigma_x \sigma_y \rangle_\beta \sim A_\beta(\theta) \|y - x\|_2^{-1/2} e^{-\xi_\beta(\theta) \|y - x\|_2}
\]

and if \( \beta > \beta_c(2) \) then:

\[
\langle \sigma_x; \sigma_y \rangle_\beta^T \triangleq \langle \sigma_x \sigma_y \rangle_\beta - \langle \sigma_x \rangle_\beta \langle \sigma_y \rangle_\beta \sim \tilde{A}_\beta(\theta) \|y - x\|_2^{-2} e^{-\zeta_\beta(\theta) \|y - x\|_2}
\]

for all \( x, y \in \mathbb{Z}^2 \) as \( y - x \to \infty \).

No OZ Behavior in \( d = 2 \) below the critical temperature!
Sharp Asymptotics - Supercritical Case

\( d \geq 3 \):
For all \( p > p_c(d) \), \( x, y \in \mathbb{Z}^d \), it is expected:

\[
\mathbb{B}_p(x \leftrightarrow y) \sim \widetilde{A}_p(\theta) \|y - x\|^{-(d-1)/2} e^{-\zeta_p(\theta)\|y-x\|_2} \quad \text{as } y - x \to \infty.
\]

Verified for \( p \gg p_c(d) \) and \( y - x \) on axes. Braga-Procacci-Sanchis (04).

\( d = 2 \):

?
Sharp Asymptotics - Supercritical Case

$d \geq 3$:
For all $p > p_c(d)$, $x, y \in \mathbb{Z}^d$, it is expected:

$$\mathbb{B}_p(x \xleftarrow{f} y) \sim A_p(\theta) \|y - x\|_2^{-(d-1)/2} e^{-\zeta_p(\theta) \|y-x\|_2} \quad \text{as } y - x \to \infty.$$ 

Verified for $p \gg p_c(d)$ and $y - x$ on axes. Braga-Procacci-Sanchis (04).

$d = 2$:

**Theorem (Campanino, Ioffe, L. (09))**

For all $p > p_c(2) = 1/2$, $x, y \in \mathbb{Z}^2$:

$$\mathbb{B}_p(x \xleftarrow{f} y) \sim \tilde{A}_p(\theta) \|y - x\|_2^{-2} e^{-\zeta_p(\theta) \|y-x\|_2} \quad \text{as } y - x \to \infty.$$
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Summary
Dual lattice.

- In $d = 2$ there is an isomorphic dual $\mathbb{Z}^2_*$.
- Set: $b^*$ is open $\iff b$ is close.
- The dual model is Percolation with $p^* = 1 - p$.
- We assume $p < p_c(2)$ and find $\mathbb{B}_p(x^* \leftrightarrow y^*)$.
- However, we’ll express this event mainly using direct bonds.
- For simplicity: $x^* = 0^*$, $y^* = 0^* + (N, 0) \equiv N^*$. 
Setup

Dual lattice.

- In $d = 2$ there is an isomorphic dual $\mathbb{Z}^2_*$.
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- However, we’ll express this event mainly using direct bonds.
- For simplicity: $x^* = 0^*$, $y^* = 0^* + (N, 0) \triangleq N^*$.

A bit of notation.

- $\text{Cl}_{m,r}(x, y)$ - The (possibly empty) cluster that contains $x, y$ and uses only edges in the strip $[m, r] \times \mathbb{Z}$.
Geometry of Finite Connections

\[ \{0^* \xrightarrow{f} x_N^* \} = \{0^* \leftrightarrow x_N^* \} \cap \{ \exists \text{ direct loop } \gamma_N \text{ around } 0^* \text{ and } x_N^* \} . \]
Decomposition of Finite Connection Event

Idea:
Find a geometric decomposition which is both unique and the sets of bonds that define each pieces are disjoint.
Decomposition of Finite Connection Event

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Find a geometric decomposition which is both unique and the sets of bonds that define each pieces are disjoint.

In our case:
Let $c_N$ be the inner most loop that contains $Cl(0^*, N^*)$.

Cut along the left-most line $H_m$ and right-most line $H_{N-r}$ that intersect $c_N$ exactly twice:

$$c_N \cap H_m = \{x, v\}, \quad c_N \cap H_{N-r} = \{u, y\}$$

Get 3 pieces: $L([v, x]), A([v, x], [u, y]), R([u, y])$. 
Decomposition - cont’d

\[ B_p \left( 0^* \leftrightarrow x_N^* \right) \]
Introduction

Sketch of Proof

Summary

Decomposition - cont’d

\[
\mathbb{B}_p \left( 0^* \xrightarrow{f} x_N^* \right) = \sum_{x,v,y,u} \mathbb{B}_p \left( \mathcal{I}([v, x], [u, y]) \right) + \mathbb{B}_p \left( \mathcal{I}(\emptyset) \right)
\]
Decomposition - cont’d

\[
\mathbb{B}_p \left( 0^* \xrightarrow{f} x_N^* \right) = \sum_{x,v,y,u} \mathbb{B}_p \left( \mathcal{I}([v, x], [u, y]) \right) + \mathbb{B}_p \left( \mathcal{I}(\emptyset) \right)
\]

\[
\mathbb{B}_p \left( \mathcal{L}([v, x]) \right) \mathbb{B}_p \left( \mathcal{A}([v, x], [u, y]) \right) \mathbb{B}_p \left( \mathcal{R}([u, y]) \right) + \mathbb{B}_p \left( \mathcal{I}(\emptyset) \right)
\]
**Decomposition - cont’d**

In fact:

\[
\mathbb{B}_p \left( 0^* \xrightarrow{f} x_N^* \right) \sim \sum_{|x|, |v| \lesssim \log N} \mathbb{B}_p (\mathcal{L}([v, x])) \mathbb{B}_p (\mathcal{A}([v, x], [u, y])) \mathbb{B}_p (\mathcal{R}([u, y]))
\]

and, modulo the exponential decay, the asymptotics will come from

\[
\mathbb{B}_p (\mathcal{A}([v, x], [u, y]))
\]
Two Disjoint Boundary Clusters

\[ A([v, x], [u, y]) = \{ \ldots \}, \]

\[ x \leftrightarrow y, \ v \leftrightarrow u, \]

\[ \text{Cl}_{m,N-r}(v, u) \cap \text{Cl}_{m,N-r}(x, y) = \emptyset \]
Two Disjoint Boundary Clusters

\[ A([v, x], [u, y]) = \{ \ldots , \]
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\[ A([v, x], [u, y]) = \{ \ldots , \]
\[ x \leftrightarrow y, v \leftrightarrow u, \]
\[ \text{Cl}_{m,N-r}(v, u) \cap \gamma^{up}(\text{Cl}_{m,N-r}(x, y)) = \emptyset \} \]
Two Disjoint Boundary Clusters

\[ A([v, x], [u, y]) = \{ \ldots, x \leftrightarrow y, v \leftrightarrow u, \quad Cl_{m,N-r}(v, u) \cap Cl_{m,N-r}(x, y) = \emptyset \} \]

\[ A([v, x], [u, y]) = \{ \ldots, x \leftrightarrow y, v \leftrightarrow u, \quad Cl_{m,N-r}(v, u) \cap \gamma_{up}(Cl_{m,N-r}(x, y)) = \emptyset \} \]

Exploration of \( Cl_{m,N-r}(v, u) \) and \( \gamma_{up}(Cl_{m,N-r}(x, y)) \) uses different bonds.
Two Disjoint Boundary Clusters

\[ A([v, x], [u, y]) = \{ \ldots \}, \]
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Exploration of \( \text{Cl}_{m,N-r}(v, u) \)
and \( \gamma^{up}(\text{Cl}_{m,N-r}(x, y)) \) uses
different bonds.

\[ \implies \text{We can sample the clusters independently}: \]
\[ \mathbb{B}_p(A(\ldots)) = \bigotimes \mathbb{B}_p(A(\ldots)) \]
The Structure of One Cluster

We would like to have some geometric decomposition of a cluster Cl(x, y).
The Structure of One Cluster

We would like to have some geometric decomposition of a cluster $\mathcal{C}(x, y)$.

Assume $x = 0$, $y = (N, 0)$. 
The Structure of One Cluster

We would like to have some geometric decomposition of a cluster \( \text{Cl}(x, y) \).

Assume \( x = 0, y = (N, 0) \).

Definition: \( \mathcal{H}_m \) is an \( \alpha \)-cone-cut-line and \( z \in \mathcal{H}_m \) is an \( \alpha \)-cone-cut-point of \( \text{Cl}(x, y) \) if

\[
\text{Cl}(x, y) \subseteq (z - C_\alpha) \cup (z + C_\alpha).
\]

where: \( C_\alpha = \{ x = (t, x) : |x| \leq \alpha t \} \).
The Structure of One Cluster

We would like to have some geometric decomposition of a cluster \( \text{Cl}(x, y) \).

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**Definition:** \( \mathcal{H}_m \) is an \( \alpha \)-cone-cut-line and \( z \in \mathcal{H}_m \) is an \( \alpha \)-cone-cut-point of \( \text{Cl}(x, y) \) if

\[
\text{Cl}(x, y) \subseteq (z - C_\alpha) \cup (z + C_\alpha).
\]

where: \( C_\alpha = \{x = (t, x) : |x| \leq \alpha t \} \).

There is a well-defined irreducible decomposition of \( \text{Cl}(x, y) \) along cone-cut-lines:
Cluster Decomposition - An Illustration

\[ x = 0 \quad y = (\mathbb{N}, 0) \]
Cluster Decomposition - An Illustration
Cluster Decomposition - An Illustration

\[ x = 0 \quad y = (N, 0) \]
Cluster Decomposition - An Illustration
Cluster Decomposition - An Illustration
Cluster Decomposition - cont’d

Let \( \mathcal{F} \) be the set of all pieces that can appear between any two succeeding cone-cut-points in any decomposition.
Cluster Decomposition - cont’d

Let $\mathcal{F}$ be the set of all pieces that can appear between any two succeeding cone-cut-points in any decomposition.

A piece $\Gamma \in \mathcal{F}$ comes with an offset vector $\sigma = \sigma(\Gamma)$, which is the difference between its left and right cut-points.
Cluster Decomposition - cont’d

Let $\mathcal{F}$ be the set of all pieces that can appear between any two succeeding cone-cut-points in any decomposition.

A piece $\Gamma \in \mathcal{F}$ comes with an offset vector $\sigma = \sigma(\Gamma)$, which is the difference between its left and right cut-points.

Similarly, define $\mathcal{F}_b$ and $\mathcal{F}_f$ for all possible initial and final pieces.
Cluster Decomposition - cont’d

Then,

\[ B_p(\text{Cl}(x, y) \neq \emptyset) = B_p(\text{no cone-cut-lines}) + \sum_{\Gamma_b} B_p(\{\Gamma_b\}) B_p(\{\Gamma_1\}) \cdots B_p(\{\Gamma_n\}) B_p(\{\Gamma_f\}) \]

where the sum is over all:

- \( \Gamma_b \in F_b \),
- \( \Gamma_i \in F \) for \( i = 1, \ldots, n \) and all \( n \),
- \( \Gamma_f \in F_f \),

such that: \( y = x + \sigma_b + \sigma_1 + \cdots + \sigma_n + \sigma_f \).
Cluster Decomposition - cont’d

Then,

\[ \mathbb{B}_p(\text{Cl}(x, y) \neq \emptyset) = \mathbb{B}_p(\text{no cone-cut-lines}) + \sum_{\Gamma} \mathbb{B}_p(\{\Gamma_b\})\mathbb{B}_p(\{\Gamma_1\}) \cdots \mathbb{B}_p(\{\Gamma_n\})\mathbb{B}_p(\{\Gamma_f\}) \]

where the sum is over all:

- \( \Gamma_b \in \mathcal{F}_b \),
- \( \Gamma_i \in \mathcal{F} \) for \( i = 1, \ldots, n \) and all \( n \),
- \( \Gamma_f \in \mathcal{F}_f \),

such that: \( y = x + \sigma_b + \sigma_1 + \cdots + \sigma_n + \sigma_f \).

In fact,

\[ \mathbb{B}_p(\text{Cl}(x, y) \neq \emptyset) \sim \sum_{\Gamma} \mathbb{B}_p(\{\Gamma_b\})\mathbb{B}_p(\{\Gamma_1\}) \cdots \mathbb{B}_p(\{\Gamma_n\})\mathbb{B}_p(\{\Gamma_f\}) \]
Growing a cluster

We can grow $\text{Cl}(x, y)$ iteratively:

1. Draw $\Gamma_b \in \mathcal{F}_b$ w.p. $\mathbb{B}_p(\{\Gamma_b\})$.
   Draw $\Gamma_f \in \mathcal{F}_f$ w.p. $\mathbb{B}_p(\{\Gamma_f\})$.

2. Set $C_0 = \emptyset$, $S_0 = 0$.

3. At each step $m$:
   3.1 Draw $\Gamma_m \in \mathcal{F}$ w.p. $\mathbb{B}_p(\{\Gamma_m\})$.
   3.2 $C_m = C_{m-1} \lor \Gamma_m$; $S_m = S_{m-1} + \sigma_m$.
   3.3 $C_{mbf}^m = \{x\} \lor \Gamma_b \lor C_m \lor \Gamma_f$.
   $S_{mbf}^m = x + \sigma_b + S_m + \sigma_f$. 
Growing a cluster

We can grow $\text{CI}(x, y)$ iteratively:

1. Draw $\Gamma_b \in F_b$ w.p. $B_p(\{\Gamma_b\})$.
   Draw $\Gamma_f \in F_f$ w.p. $B_p(\{\Gamma_f\})$.

2. Set $C_0 = \emptyset$, $S_0 = 0$.

3. At each step $m$:
   3.1 Draw $\Gamma_m \in F$ w.p. $B_p(\{\Gamma_m\})$.
   3.2 $C_m = C_{m-1} \lor \Gamma_m$ ; $S_m = S_{m-1} + \sigma_m$.
   3.3 $C_{bf} = \{x\} \lor \Gamma_b \lor C_m \lor \Gamma_f$.
      $S_{bf} = x + \sigma_b + S_m + \sigma_f$.

If $\tilde{P}_x$ is the measure for this decorated RW, then

$$B_p(\text{CI}(x, y) \neq \emptyset, \exists \text{cone-cut-lines}) = \tilde{P}_x(\exists n \text{ s.t. } S_{bf}^n = y).$$
Growing a Cluster - Illustration
Growing a Cluster - Illustration

\[ x = 0 \quad y = (N, 0) \]
Growing a Cluster - Illustration

$\begin{aligned}
\sigma_b & \quad \sigma_1 \\
\Gamma_b & \quad \Gamma_1 \\
\sigma_2 & \quad \sigma_f \\
S_{2f}^{bf} & \quad C_2^{bf} \\
0 & \quad (N, 0)
\end{aligned}$
Growing a Cluster - Illustration

\[
x = 0 \\
y = (N, 0)
\]
Growing a Cluster - Illustration
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Growing a Cluster - Illustration

\[ x = 0 \quad y = (N, 0) \]
Normalizing Steps

\[ \Gamma \] are not \textbf{properly} defined: \[ \sum_{\Gamma \in \mathcal{F}} \mathbb{B}_p(\{\Gamma\}) < 1. \]
i.e. the random walk can \textbf{die}.
Normalizing Steps

\( \Gamma \) are not properly defined: \( \sum_{\Gamma \in \mathcal{F}} \mathbb{B}_p(\{\Gamma\}) < 1 \).
i.e. the random walk can die.

Tilt to make proper:

\[
\mathbb{P}(\Gamma = \gamma) = e^{K\langle \sigma(\gamma), e_1 \rangle} \tilde{\mathbb{P}}(\Gamma = \gamma).
\]
\[
\mathbb{P}(\Gamma_{b,f} = \gamma) = k_1 e^{K\langle \sigma(\gamma), e_1 \rangle} \tilde{\mathbb{P}}(\Gamma_{b,f} = \gamma).
\]
Normalizing Steps

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i.e. the random walk can die.

Tilt to make proper:

\[
\begin{align*}
\mathbb{P}(\Gamma = \gamma) &= e^{K \langle \sigma(\gamma), e_1 \rangle} \mathbb{P}(\Gamma = \gamma). \\
\mathbb{P}(\Gamma_{b,f} = \gamma) &= k_1 e^{K \langle \sigma(\gamma), e_1 \rangle} \mathbb{P}(\Gamma_{b,f} = \gamma).
\end{align*}
\]

\( K = \xi_p(e_1) \) !!
Normalizing Steps

\(\Gamma_\cdot\) are not properly defined: \(\sum_{\Gamma \in \mathcal{F}_\cdot} \mathbb{B}_p(\{\Gamma\}) < 1\).
i.e. the random walk can die.

Tilt to make proper:

\[
\begin{align*}
\mathbb{P}(\Gamma_\cdot = \gamma) &= e^{K\langle \sigma(\gamma), e_1 \rangle} \tilde{\mathbb{P}}(\Gamma_\cdot = \gamma), \\
\mathbb{P}(\Gamma_{b,f} = \gamma) &= k_1 e^{K\langle \sigma(\gamma), e_1 \rangle} \tilde{\mathbb{P}}(\Gamma_{b,f} = \gamma).
\end{align*}
\]

\(K = \xi_p(e_1)!!\)

Then:

\[
\mathbb{B}_p(\text{Cl}(x, y) \neq \emptyset) \sim Ke^{-\xi_p(e_1)\langle y-x, e_1 \rangle} \mathbb{P}_x(\exists n \text{ s.t } S_n^{bf} = y).
\]
Back to the Two Boundary Clusters

Thus,

$$\bigotimes \mathbb{B}_p(\mathcal{A}([v, x], [u, y])) \sim K e^{-2\xi_p(e_1)\langle y-x, e_1 \rangle} \times$$

$$\bigotimes \mathbb{P}_{x, v} \left( \exists n_1, n_2 \text{ s.t } S_{n_1}^{bf, 1} = y, S_{n_2}^{bf, 2} = u \text{ and } C_{n_1}^{bf, 1} \cap C_{n_2}^{bf, 2} = \emptyset \right),$$

where under $\bigotimes \mathbb{P}_{x, v}$ two decorated RWs: $C_{\bullet}^{bf, 1}$, $C_{\bullet}^{bf, 1}$ evolve independently starting from $(x, v)$.

The prefactor will come, therefore, from the asymptotics of the probability of the RW event above.
Asymptotics of No Intersection

Let $N = \langle y - x, e_1 \rangle = \langle u - v, e_1 \rangle$. Then

**Lemma**

As $N \to \infty$, uniformly in $x, y, u, v \in D(N)$:

$$\otimes \mathbb{P}_{x,v} \left( \exists n_1, n_2 \text{ s.t } S_{n_1}^{bf,1} = y, S_{n_2}^{bf,2} = u \text{ and } C_{n_1}^{bf,1} \cap C_{n_2}^{bf,2} = \emptyset \right)$$

$$\sim U(x - v)U(y - u) \frac{1}{N^2}.$$ 

$U(\cdot)$ has polynomial growth.
Proof of Intersection Asymptotics

1. One (unbiased) RW staying positive:

\[ \Pr_0(S_n = y, S_k > 0 \forall k) \sim \frac{U(y)}{n} \Pr_0(S_n = y) ; \ y \ll n^{-1/2} \]

Combinatorial proof by Alili-Doney (97).
Proof of Intersection Asymptotics

1. One (unbiased) RW staying positive:

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Combinatorial proof by Alili-Doney (97).

2. Starting from \( x > 0 \):

\[ \mathbb{P}_x(S_n = y, \ S_k > 0 \ \forall k) \sim \frac{U(x)U(y)}{n} \mathbb{P}_x(S_n = y) \quad ; \ x, y \ll n^{-1/2} \]
Proof of Intersection Asymptotics

1. One (unbiased) RW staying positive:

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3. Two independent RWs not intersecting:

\[ \bigotimes P_{x,v}(S^1_n = y, S^2_n = u, S^1_k > S^2_k \forall k) \sim \frac{U(x-v)U(y-u)}{n} P_x(S_n = y)P_v(S_n = u) \quad ; \quad x, v, y, u \ll n^{-1/2} \]
Proof of Intersection Asymptotics - cont’d

4 Add random time steps:

\[ \bigotimes \mathbb{P}_{(0,x),(0,v)} \left( \exists n_1, n_2 \text{ s.t. } S^1_{n_1} = (N, y), S^2_{n_2} = (N, u), S^1 \cap S^2 = \emptyset \right) \]

\[ \sim \frac{U(x - v)U(y - u)}{N} \frac{1}{\sqrt{N}} \frac{1}{\sqrt{N}} \]

; \ x, v, y, u \ll N^{-1/2}
Proof of Intersection Asymptotics - cont’d

4 Add random time steps:

\[ \otimes \mathbb{P}_{(0,x),(0,v)} \left( \exists n_1, n_2 \text{ s.t. } S^1_{n_1} = (N, y), S^2_{n_2} = (N, u), S^1_{n_1} \cap S^2_{n_2} = \emptyset \right) \]

\[ \sim \frac{U(x - v)U(y - u)}{N} \frac{1}{\sqrt{N}} \frac{1}{\sqrt{N}} \]

; \quad x, v, y, u \ll N^{-1/2}

5 By entropic repulsion:

\[ \mathbb{P}_{(0,x),(0,v)} \left( C^1_{n_1} \cap C^2_{n_2} = \emptyset \mid S^1_{n_1} \cap S^2_{n_2} = \emptyset, \ldots \right) \rightarrow \text{const} \]
Proof of Intersection Asymptotics - cont’d

4 Add random time steps:

\[ \otimes P_{(0,x),(0,v)} \left( \exists n_1, n_2 \text{ s.t } S_{n_1}^1 = (N, y), S_{n_2}^2 = (N, u), S_1^1 \cap S_2^2 = \emptyset \right) \]
\[ \sim \frac{U(x - v)U(y - u)}{N} \frac{1}{\sqrt{N}} \frac{1}{\sqrt{N}} \quad ; \quad x, v, y, u \ll N^{-1/2} \]

5 By entropic repulsion:

\[ \mathbb{P}_{(0,x),(0,v)} \left( C_{n_1}^1 \cap C_{n_2}^2 = \emptyset \mid S_1^1 \cap S_2^2 = \emptyset, \ldots \right) \rightarrow \text{const} \]

6 Initial and final piece only change the constant:

\[ \otimes P_{(0,x),(0,v)} \left( \exists n_1, n_2 \text{ s.t } S_{n_1}^{1,bf} = (N, y), S_{n_2}^{2,bf} = (N, u), C_{n_1}^{1,bf} \cap C_{n_2}^{2,bf} = \emptyset \right) \]
\[ \sim \frac{U(x - v)U(y - u)}{N^2} \quad ; \quad x, v, y, u \ll N^{-1/2} \]
Outline

Introduction
Percolation on $\mathbb{Z}^d$
Logarithmic Asymptotics of Connectivities
Sharp Asymptotics of Connectivities

Sketch of Proof
Setup
Geometry of Finite Connections
The Structure of a Cluster
Asymptotics for No Intersection of Two Decorated RWs

Summary
Open Questions

- Do the same for 2D Random Cluster (FK) model ($q \neq 2$).
- Supercritical percolation on $\mathbb{Z}^d$ for $d \geq 3$
  Show OZ behavior for finite connectivities in all directions and all $p > p_c(d)$. 
Thank You

Thank you.