Extreme and Large Values of the DGFF

Zürich 4/9/2018

Based on joint work with:

M. Biskup (UCLA)  L. Hartung (NYU)

D. Yeo
(Technion)

A. Curtoes
(Zürich)

S. Saglietti
(Technion)
The Discrete Gaussian Free Field (DGFF)

For $D \subseteq \mathbb{R}^d$ "nice" (open, bounded, non-empty, ...), e.g., $D = (0,1)^2$

$D_n := n \mathbb{Z} \subset \mathbb{Z}^d$ (careful!), e.g., $D_n = (0,n)^2$

DGFF on $D_n$ is $h_n = (h_n(x) : x \in D_n) \overset{d}{=} N(0, C_n)$

where $C_n$ is the discrete Green function on $D_n$ (with o.b.c.)

Known:

$G_n(x,y) = g \log N - g \log \left(1 - \frac{|x-y|}{n}\right) + O(1)$

$g = 2/\pi \quad \forall x,y \in D_n \text{ away from } \partial D_n$

$\mathbb{E} h_n^2(x) = g \log N + O(1) \quad \mathbb{E} \left( h_n(x) - h_n(y) \right)^2 = 2g \log |x-y| + O(1)$
Typical and Large Values

Typically, \(|h_n(x)| = O(\sqrt{\log n})\).

What are "large" values?

1st Moment Calculation

\[
\mathbb{E}\left| \{x \in \mathcal{O}_n; h_n(x) \geq u \} \right| \approx |\mathcal{O}_n| \cdot \Pi(\log(x)) \geq u \sim c \cdot N^2 \cdot \frac{1}{\sqrt{2\pi u}} \cdot e^{-\frac{u^2}{2}} \sim \frac{1}{\sqrt{2\pi u}} \cdot e^{-\frac{u^2}{2}}
\]

Take,

\(u = \mu_n(x) = \lambda(2\log \log n)\). Then,

\[
\mathbb{E}\left| -u \right| \leq c_\lambda \left( \log n \right)^{-\frac{1}{2}} N^{2\lambda - 1}
\]
\[ \log \left| \{ x \in D_n : h_n(x) \geq m_N(x) \} \right| \sim \log k_n(\lambda) \]

in prob as \( N \to \infty \)

\( \lambda \in [0,1] \)

(LB builds on method by Bolthausen - Deuschel - Giacomin '01)

\[ (x, h_n(x)) : h_n(x) \geq m_N(x) \text{ with } \lambda = 0.2 \]

\( h_n \sim \text{Geo}(D_n) \)

\( h_n \overset{iid}{\sim} N(0, g \log N) \)
\[ \mathcal{B} \quad (\text{supp} - L) \]

Set: \( q_n^\lambda := \frac{1}{k_n(\lambda)} \sum_{x \in \mathbb{Z}_n} \delta_{(x_n, h_n(x) - m_n(x))} \quad \lambda \in (0, 1) \)

Then \( q_n^\lambda \xrightarrow{N \to \infty} q_\infty^\lambda \) where

\[ q_\infty^\lambda (dx \, dt) = Z_0^\lambda (dx) \otimes e^{-\sqrt{2\pi} \lambda t} \, dt \rightharpoonup \text{LQG at } r = 2\lambda \]

Moreover,
\[
\left| \left\{ x \in \mathbb{Z}_n : h_n(x) \geq m_n(x) \right\} \right| \xrightarrow{N \to \infty} \frac{1}{k_n(\lambda)} Z_0^\lambda (0)
\]

Rem.: Argument can be used to prove Kesten–Stigum-type–thm for supercritical BBM with negative drift and killing at 0. (Joint work with S. Saglietti)
**Extreme Values**

Set \( M_n := \max_{x \in D_n} h_n(x) \)

Davidov \( \Rightarrow \) \( M_n \sim 2\sqrt{\log n} \)

**Breslin - Ding - Zeitouni '13**

\( M_n - m_n \overset{d}{\Rightarrow} M_\infty \), where \( m_n := 2\sqrt{\log n - \frac{3}{2} \sqrt{\log \log \log n}} \)

Proof is based on earlier results by BOZ and others. In particular

**Theorem (0-2 '12):** \( \frac{8}{N h_n} \overset{d}{\rightarrow} 0 \)

Fix any \( \eta > 0 \), \( \forall \epsilon > 0 \), Then \( \forall t \in \mathbb{R} \)

\[
\Pr \left( \exists x : h_n(x) \geq m_n + t, \ |x - y| \in \left[ \frac{N}{n}, \frac{n}{N} \right] \right) \overset{d}{\rightarrow} 0
\]

\( \{ x : h_n(x) \geq m_n + t \} \)
Then (Bishop - L '15):

Set: \( \hat{\mathbf{z}}_n := \sum_{x \in \mathcal{N}_n} \delta_{(x/\mathcal{N}_n^3, h(x))} - \mu_n, \quad (h(x+y) - h(x))_{y \perp < r_n} \)

\[ \times \bigcap \{ \delta h_n(x) \leq h(x+y) \quad \forall y \text{ s.t } |y| < r_n \} \]

for some \( (\mathcal{N}_n)_{n \geq 1} \), with \( \mathcal{N}_n \wedge (n/r_n) \to \infty \).

Then,

\[ \hat{\mathbf{z}}_n \overset{n \to \infty}{\to} \hat{\mathbf{z}}_\infty \sim \text{APP} \left( \mathcal{Z}_0(dx) \otimes e^{-\sqrt{2\pi \tau} dt} \otimes \nu(dx) \right) \]

where:

\( \mathcal{Z}_0 \) - LQG on \( D \) at criticality \( \kappa = 2 \)

\( \nu \) - Deterministic dist on \( (\mathbb{R}_-)^2 \).
Extreme Values of $h_n$ for Large $N$ (Schematic Picture)

$t_n, h_n \sim \exp(|Z_0| e^{-\sqrt{2\pi} t} dt) + m_n$

$x, x, \ldots \sim \tilde{Z}_0 = Z_0 / |Z_0|

w, w, \ldots \sim \nu$

$\left[ Z_0(x) - \log M \right] (r=2)$

$t \rightarrow e^{-\sqrt{2\pi} \left(t - \frac{1}{\sqrt{m}} \log |Z_0|\right)}$
Fixer Description of the extreme level sets

Questions:

w.h.p \( \Rightarrow N \rightarrow \infty \):

Q1: Asymptotic size of a "real" level set ?

Q2: Typical height of local max whose clusters contribute to the level set ?

Q3: Typical "shape" of such contributing clusters ?

Key need input: properties of \( V ! \)

(\( \tilde{E}_0 \) is explicit modulo \( V (\text{and } \mathbb{Z}_0) \))

Disclaimer: Results of following results are written only for BMM/BMV.
$$\text{Recall: } \hat{h}_n = \sum_{x \in \mathbb{Z}_n} \delta(\hat{V}_n - h_n(x), (h_n(z) - h_n(x))_{|z| < r_n})$$

$$\text{Def. } I_n = \sum_{x \in \mathbb{Z}_n} \delta(h_n(x) - h_n(x), \mathbb{E}_n \sum_{|z| < r_n} \delta(t - wz))$$

Also, $$I_n(B) : = \sum_{(t, w) \in B} \left(\sum_{|z| < r_n} \delta(t - wz)\right) A_B(t, w)$$

for $$B \subseteq \mathbb{R} \times (\mathbb{R}_+)^2$$
\[ \mathbb{L}_n (\mathcal{E}, \infty) \sim C_\mathcal{E} \mathcal{Z}_0 (0) S e^{\sqrt{2\pi} S} \quad \text{in prob as } n \to \infty, S \to \infty. \]

Compare with \[ \mathbb{L}_n (\mathcal{E}, \infty) \times \mathbb{R}^2 \sim C \mathcal{Z}_0 (0) e^{\sqrt{2\pi} S}. \]

A similar result holds for BBM (Conjectured by Brunet-Derrida '10).

Then (C-H-L) \hfill (Q2)

**Def.** \( L(-t) := [-t, \infty) \times \mathbb{R}^2 \). Then

\[ \mathbb{L}_n (\mathcal{E}, \infty) \mid L(-t) \quad \mathbb{L}_n (\mathcal{E}, \infty) \sim \frac{t}{S} \quad \text{in prob as } n \to \infty, S \to \infty, t \wedge (S-t) \to \infty \]

Contribution to level set \( m_{n} - s \) from clusters with local max at \( m_{n} - t \) is uniformly \( \sim C_\mathcal{E} \mathcal{Z}_0 (0) e^{\sqrt{2\pi} S} \quad \forall t \leq s \).
**Def:** A cluster \( u \in R \times \mathbb{Z}^2 \) will be called \((u, \delta)\)-fat \((u, \delta > 0)\),

\[
\left| \left\{ y : \left| \frac{\log |y|}{u^2} \right| \leq \frac{\epsilon}{2}, \log \left| \frac{y}{u^2} \right| \leq -u \right\} \right| \geq e^{-ue^{\sqrt{\pi}u}}
\]

**Set:**

\[
F_\delta(s) := \left\{ (-t, w) : t = s, w \text{ is } (s-t, \delta)\text{-Fat} \right\}
\]

**Thm (C-H-L):**

\[
\forall \epsilon > 0, \exists \delta > 0 \text{ s.t. } \lim_{n \to \infty} \frac{\hat{\nu}_n((-s, \infty) \cup F_\delta(-s))}{\hat{\nu}_n((s, \infty); L(-t))} \geq 1 - \epsilon \text{ but } \frac{\hat{\nu}_n(L(-t) \cap F_\delta(s))}{\hat{\nu}_n(L(-t)))} < \epsilon
\]
Most of the contribution to level set \( m_N - s \) from clusters around local max at height \( m_N - t \), comes from a small (\( \leq (s-t)^{5/4} \)) fraction of clusters, which are fat, namely their contribution is typically large (\( \geq (s-t)^{5/4} \)). A dual description is given by:

**Theorem (C-H-L):**

For \( s > 0 \), let \( X \in D_0 \) be chosen uniformly from \( \{ x : h_N(x) \geq m_N - s \} \). Set \( Y : = \arg \max \{ h_N(x+y) : |y| < r_N \} \). Then

\[
\left( \frac{\log |Y|}{s^2}, \frac{h_N(x+y) - h_N(x)}{s} \right) \xrightarrow{s \to 0} (T, 1-\frac{3}{2})
\]

where \((T, \frac{3}{2})\) is the time and height of global minimum of BM starting from 1 at time 0 and conditioned to stay indefinitely positive.
Infinite Volume

DGFF cannot be defined on $\mathbb{Z}^2$ ($G_{\mathbb{Z}^2}(\infty) = \infty$), but can be defined on $\mathbb{Z}^2 \setminus \{0\}$. Denote by $h^0$.

Q: How does $h^0$ grow away from 0?

\[ \lim_{|x| \to \infty} \frac{h^0(x) + 2\sqrt{2} \log |x|}{\log \log |x|} = -1 \]

\[ \lim_{|x| \to \infty} \frac{h^0(x) - 2\sqrt{2} \log |x|}{\log \log |x|} = 1 \]

(Law of iterated logarithm for $h^0$)

\[ L \cdot \text{Yao 18+} \]
\( h^0 \text{ conditioned to stay positive} \)

Denlop et al., '91:

- \( h^0 \mid \{ h^0(y) \gg 1 \} \sim \mathcal{N}(3, \infty) \).

- \( \mathbb{E} h^+(x) \propto \log |x| \).

Then \((L - \infty) \Rightarrow (\text{Entropic Repulsion for } h^+)\):

- \( \mathbb{E} h^+(x) = 2 \sqrt{g \log |x|} + \sqrt{2g (\log |x|)} \) \text{ (1+o(1)) as } \ x \to \infty. \)

- \( \lim_{x \to \infty} \frac{h^+(x)}{\sqrt{2g \log |x| / (\log \log |x|)^{1+\varepsilon}}} = 0 \), \( \lim_{x \to \infty} \frac{h^+(x)}{\sqrt{2g \log |x| / (\log \log |x|)}} = 0 \) for \( \varepsilon > 0 \).

- \( \lim_{x \to \infty} \frac{h^+(x) - 4 \sqrt{g \log |x|}}{\sqrt{2g \log |x| \log (\log |x|)}} = 1 \) \text{ w.p. 1.}
Ideas From the Proofs

3 key ideas:

1. Truncation via VB on global maximum

- Need to understand prob of events of the form

\[ \sum h_n(x) - m_n \leq dt, \quad (h_n(x+y) - h_n(x) : |y| < R) \in A \]

- This will be input for various 1st and 2nd moment calc.

- Usually, for moments to reflect the typical order of the quantity in question, one adds a truncation event. We use:

\[ \sum \max h_n(y) : y \leq 0, \leq m_n + n^3 \]

- By tightness of centered max, the prob of latter \( T_1 \) as \( n \rightarrow \infty \)

Hence for statements w.h.p., this restriction is harmless.
Concentric decomposition / Decorated RW representation

Set: \( \mathcal{R}_n = 0_{2^n} = (-2^n, 2^n)^2 \cap \mathbb{Z}^2 \quad (0 = (-1,1)^2) \)

\( N = 2^n \)

The Gibbs-Markov property implies:

\[
h_n \mid \mathcal{R}_n \setminus \mathcal{R}_{n-1} \quad \cong \quad \mathcal{P}_n \oplus h'_n
\]

where:

\( \mathcal{P}_n = \mathbb{E}(h_n \mid \mathcal{R}_n) : y \in 2\mathcal{R}_n \cup 2\mathcal{R}_{n-1} \)

\( h'_n \cong \text{OGFF} (\mathcal{R}_n \setminus \mathcal{R}_{n-1}) \quad \sqrt{\text{indep}} \)
Sol: \[ S_k = \mathbb{E}(h_n(0) \mid h_n(y), y \in \mathbb{R}_{k-1}), \quad S_0 = h_n(0). \]

**Facts:**
- \( (S_k : k = 0, \ldots, n) \) - RN with \((almost)\) uniform centered Gaussian steps, starting from 0.
- \( \max \left( \left| T_n(y) - S_k \right| : y \in \mathbb{R}_n \setminus \mathbb{R}_{k-1} \right) \) exp tight in \( kn \).
- \( h_n \), \( \ldots \), \( h_0 \) indep and indep of \( T_n \)'s and \( S_k \)'s.
- \( \max \left( h_k(x) : x \in \mathbb{R}_n \setminus \mathbb{R}_{k-1} \right) - m_k \) is exp tight in \( n \).

**Then:**
\[ h_n \mid \mathbb{R}_n \setminus \mathbb{R}_{k-1} \doteq S_k + \Delta_k \odot h'_k, \quad \max \Delta_k \text{ exp tight (in bulk)} \]
and
\[ \max \left( h_n(y) : y \in \mathbb{R}_n \setminus \mathbb{R}_{k-1} \right) \doteq S_k + \varepsilon_k + m_{2k}, \quad \varepsilon_k \text{ exp tight r.v.} \]
Therefore:
\[ \left\{ h_n(0) - m_n e^t, (h_n(y) - h_n(0)) \right\}_{k \geq r} \in A, \quad \forall x < h_n(y) \leq m_n + u \right\} \\
\text{can be recast as:} \\
\left\{ S_0 - m_n e^t, (S_k - S_0 + \Delta x \Theta l_k') \right\}_{k \geq r} \in A, \quad S_k - \varepsilon_k + m_n \leq m_n + u \quad \forall k \right\} \\
\text{The prob of this is therefore} \\
P \left( S_k \leq m_n - m_n \varepsilon_k + u \quad \forall k, \quad (S_k - S_0 + \Delta x \Theta l_k') \in A \mid S_0 = m_n + t \right) \\
\times P(S_0 - m_n e^t) \\
\text{But } m_n - m_n \varepsilon_k = m_n \frac{n^i - k}{n + 1} + o(\log(k \lambda^i n^i)). \text{ So the 1st prob is} \\
P(S_k \leq m_n \frac{n^i - k}{n + 1} - \varepsilon_k + u; \quad \forall k = n, ..., 0, \quad A_r \mid S_0 = m_n + t)
3) Sharp asymptotics for decoerated RW conditioned to stay below a line - entropic repulsion

By tilting \( S_k \rightarrow S_k + \frac{u-n}{n+1} m_n \)
the local prob is equal to

\[
P \left( S_k + \bar{\xi}_k \equiv u \text{ } \forall k=0, \ldots, n, \text{ } A_n \mid S_0=\bar{t}, \text{ } S_{n+1}=0 \right)
\]

Asymptotics for prob of RW to stay below \( u \) are known, but:

- \( S \) doesn't have uniform steps \( \checkmark \) (almost uniform)
- \( \bar{\xi}_i \)'s are not indep of \( S \). \( \checkmark \) (dependence decays fast)
- \( \bar{\xi}_i \)'s can be arbitrary large. \( \checkmark \) (but exp. tight)
- \( \bar{\xi}_i \)'s will grow logarithmically fast \( \checkmark \) (U+L bounds available, same order)
But we need sharp asymptotics!

Solution - Entropic Repulsion. LB & UB are enough to show:

\[ P( \exists k \in [m, n-m+1]: S_k \geq -k^E \mid S_k + \tilde{\Xi}_k \leq u \ \forall k, \ S_0 = t, \ S_{n+1} = 0 ) \xrightarrow{\rightarrow} 0 \quad \text{unif in } n \]

Since the same is also true without the \( \tilde{\Xi}_k \), we get:

\[ P( S_k + \tilde{\Xi}_k = u \ \forall k \mid S_{n+1} = 0 ) \sim P( S_k + \tilde{\Xi}_k = u \ \forall k < m, k > n-m+1 \mid S_0 = t ) \]

Thus, one can use asymptotics for RW conditioned to stay below \( u \) in the bulk, by conditioning on \((S_k, \tilde{\Xi}_k, S_m, \ldots, \tilde{\Xi}_{n-m+1}, k=0, \ldots, m-1)\) and then take expectation.
Thank you!!