Selection Games and Deterministic Lotteries

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Abstract

The design of deterministic and fair mechanisms for selection among a set of self-motivated agents based solely on these agents' input is a major challenge for electronic commerce. These mechanisms are a special case of zero-sum games where the only possible outcomes are selections of a single agent among the set of agents. We assume the lack of an external coordinator, and therefore we focus on mechanisms which have a solution where the agents play weakly dominant strategies. Our first major result shows that dominated strategies could be added to any selection mechanism, so that the resulting mechanism becomes quasi-symmetric. For fairness, we require the mechanism to be non-imposing; that is, the mechanism should allow any agent to be selected in such a solution. We first show that such mechanisms do not exist when there are two or three agents in the system. However, surprisingly, we show that such mechanisms exist when there are four or more agents. Moreover, in our second major result, we show that there exist selection mechanisms that implement any distribution over the agents, when the agents play mixed dominant strategies. These results also have significance for distributed computing, ranking systems, and social choice.

1 Introduction

The design of deterministic and fair mechanisms for selection among a set of self-motivated agents based solely on these agents' input is a major challenge for electronic commerce. For example, a common marketing tactic is to conduct lotteries among customers. However, legal constraints limit the use of such lotteries, leading businesses to apply alternative means for fair distribution of prizes. Therefore, it may be of great potential interest to provide a deterministic and fair mechanism which will mimic the outcome generated by such lottery. Naturally, such legal constraints imply that the mechanism should rely only on the agents' inputs. Moreover, whenever a group of self-motivated participants need to randomly select a candidate in a setting in which there is no external coordinator and any agent can listen to the others, we face a similar problem.

In this paper, we aim to design selection mechanisms that yield a deterministic selection of winners, by distributing the central randomization device among the self-interested agents, applying game-theoretic techniques to ensure adherence of the agents to protocol. This serves as an alternative to cryptographic techniques in the context of leader election.

In order to capture the agents' incentives, we define the notion of selection games. These games are a special case of zero-sum games where the only possible outcomes are selections of a single agent among the set of agents. We assume that the agents will play (a mixture of) weakly dominant strategies if such strategies exist. We therefore focus on selection games where a desired outcome is attained when all agents play such strategies. Our aim in the design of such mechanisms is that the mechanisms will be fair and will implement a desired probability distribution on the outcomes.

Fairness in this context is captured by the notion of quasi-symmetry. Quasi-symmetry means that all agents have the same strategy set, and that the outcome depends only on the multiset of strategies played, and not on the identity of the players. In fact, in a quasi-symmetric game all agents have exactly the same influence on the outcome. Note however, that the agents' incentive structure may have a major effect on the strategies actually selected, and thus on the practical influence of each agent in the game.

Our first main result shows that any selection game can be extended to a quasi-symmetric game preserving all agents' dominant strategies. Hence, if we are able to implement some desired outcome in dominant strategies, we can also do so fairly, in the sense of quasi-symmetry. We then show, in our second main result, that in fact *any* distribution over at least four agents may be implemented under mixed dominant strategies. These two results together offer a practical means of conducting arbitrary deterministic lotteries in a fair manner. Alternatively, this result can be viewed as providing means for leader election for an arbitrary distribution.

These results are applicable to several types of popular online lotteries. For example, businesses conduct sweepstakes where the chances of winning are proportional to some participation indicator (such as points, products purchased, etc.). Our general distribution result directly applies in this context. Another example is in the selection of ads to be shown based on agents' bids[13]. If we wish that the selection will be random based on the relative value of bids, then our machinery becomes highly relevant.

Several additional results are also obtained in this paper. We show that, surprisingly, in the special case where there are only two or three agents, *no* selection game exists where all agents have nonzero probability of being elected. Formally, this is captured by the fact no non-imposing selection games exist for two or three agents. We also discuss the notion of dummy players — players which do not influence the outcome when playing their dominant strategies. We show that there exist non-imposing selection mechanisms without dummy players for all $n \geq 4$.

As mentioned above, our selection games setting is related to the leader election problem in distributed computing[10]. In particular, the study of the so-called *cheater's edge*[3] can be applied to our domain. That study focuses on limiting the probability of a failed agent being elected under a Byzantine failure model. Specifically, that paper presents a protocol $\mathcal{P}_0$ that guarantees that under at most one fault, the faulty agent will be elected with probability of at most $\frac{1}{n}$, which is the minimal attainable probability. This can be seen as a special case of our second main result, as this protocol implements the uniform distribution.

Another line of research, initiated by Ben-Or and Linial[5], deals with the problem of collective coin flipping. In this context, the idea is to try and minimize the influence of agents and coalitions on the result of the lottery in the face of Byzantine failures. Some powerful mechanisms showing a relatively small amount of influence have been introduced (see [9] for a discussion of such results). However, these approaches do not utilize the incentive structure of the agents, and thus agents must be given some level of influence, which may be abused to ensure self-selection.

This problem of influence has been addressed by using cryptography in the form of one-way functions in order to circumvent the problem of open communication. This approach however is not secure in the information-theoretic sense, as agents with an unlimited computational power are able to reverse the one-way function and respond. On the positive side, such protocols are secure even in the face of coalitions. In our discussion of selection games with solutions in strong Nash equilibrium, we demonstrate such a secure mechanism for the case where communication is private.

In section 2 we define the selection games setting and solution concepts in this setting. In section 3 we prove our first major result with regard to the existence of quasi-symmetric selection games. In sections 4 and 5, we prove the existence of non-imposing selection games and selection games for general distributions respectively. Section 6 discusses the existence of selection games where all agents are active, and section 7 discusses the strong Nash equilibrium solution concept. Finally, in section 8 we discuss the impact of this work on related fields.

2 Selection Games

In order to begin our discussion of selection games, we must first formally define the notion of a selection game as a special case of a zero-sum normal form game:

Definition 2.1. A *selection game* is a tuple $G = (N, S, v)$, where $N = \{1, \ldots, n\}$ is a set of players ($n > 1$), $S = (S_1, S_2, \ldots S_n)$ is a vector of strategy sets $S_i = \{s_i^1, \ldots s_i^{m_i}\}$ for each player, and v is a function $v : \mathbb{S} \mapsto N$ ($\mathbb{S} = S_1 \times S_2 \times \cdots \times S_n$) that maps every strategy profile $s \in \mathbb{S}$ to a winner $v(s) \in N$.

A selection game can be mapped to a zero-sum normal form game with the utility function

$$u_i(s) = \begin{cases} 1 & v(s) = i \\ 0 & \text{Otherwise.} \end{cases}$$

This definition means that all the results that apply to general zero-sum games apply to selection games as well. Specifically, classical games such as *Matching Pennies* or even *Chess* (assuming no ties) are in fact two-player selection games.

In Game Theory, whenever a type of game is discussed we try to define solution concepts for that type of game. Such a solution concept may be pure, mixed, or correlated:

Definition 2.2. A *(pure) solution concept for selection games* is a function $\mathcal{C} : \mathbb{G} \mapsto \wp(\mathbb{S})$ that maps every selection game $G \in \mathbb{G}$ to a set of strategy profiles in that game.

A *mixed solution concept for selection games* is a function $\mathcal{C} : \mathbb{G} \mapsto \wp(\Delta(S_1) \times \Delta(S_2) \times \cdots \times \Delta(S_n))$ that maps every selection game to a set of mixed strategy profiles in that game.

A *correlated solution concept for selection games* is a function $\mathcal{C} : \mathbb{G} \mapsto \wp(\Delta(\mathbb{S}))$ that maps every selection game to a set of probability distributions over strategy profiles in that game.

Note that any pure solution concept can be mapped to an equivalent mixed solution concept, and every mixed solution concept can be mapped to an equivalent correlated solution concept.

We can now present several solution concepts in weakly dominant strategies for normal-form games:

Definition 2.3. Let $G = (N, S, v)$ be a selection game and let $i \in N$ be some agent. The weakly dominant strategy set for i in G, denoted by $D_G(i)$, is the set of all strategies $s_i \in S_i$ such that for all strategy profiles $s' \in \mathbb{S}$: $v(s'_i, s'_{-i}) = i \Rightarrow v(s_i, s'_{-i}) = i$.

Note that our definition of weakly dominant strategies allows for several weakly dominant strategies that an agent is indifferent between, as we do not require a strict preference over every other strategy.

Definition 2.4. Let G be a selection game. The *Weakly Dominant Strategies* pure solution concept $\mathcal{C}_{WD}$ is the set of all strategy profiles s where every player plays a weakly dominant strategy. That is, for all $i \in N : s_i \in D_G(i)$.

The *Mixed Dominant Strategies* solution concept $\mathcal{C}_{MD}$ is the set of all mixed strategy profiles s where every player plays a weakly dominant mixed strategy. That is, for all $i \in N$: $s_i \in \Delta(D_G(i))$.

The *Uniform Dominant Strategies* mixed solution concept $\mathcal{C}_{UD}$ consists of the mixed strategy profile s^U where for all $i \in N$: s_i^U is a uniform mixture over $D_G(i)$.

Example 2.5. Consider the selection game $G = (N, S, v)$, where $N = \{1, 2, 3, 4\}$, $S = (\{a_1, b_1\}, \{a_2, b_2\}, \{a_3, b_3\}, \{\varepsilon\})$, and v is described below:

	a_3		b_3	
	a_2	b_2	a_2	b_2
a_1	1	4	2	2
b_1	1	3	4	3

In this game all strategies for all players are weakly dominant. Therefore, all 8 strategy profiles are solutions in weakly dominant strategies. Furthermore, any mixture $((\alpha, 1-\alpha), (\beta, 1-\beta), (\gamma, 1-\gamma), (1))$ is a solution in mixed dominant strategies. However, there is only one solution in uniform dominant strategies, which is when the three active agents play their "a" strategy with a probability of exactly $\frac{1}{2}$.

3 Quasi-Symmetry

A basic requirement from mechanisms for deterministic lotteries is fairness in the sense that all agents have the same influence, as captured by the notion of quasi-symmetry. In a quasi-symmetric game all agents have the same options (i.e. strategy set), and the outcome is determined only by the multiset of strategies played, with no regard to which agent played what strategy:

Definition 3.1. A selection game $G = (N, S, v)$ is called quasi-symmetric if all strategy sets are equal ($S_i = S_j \forall i, j \in N$) and for every permutation $\pi : N \mapsto N$ and for every strategy profile $s \in \mathbb{S}$: $v(\pi(s)) = v(s)$.

Note that this definition is different from *symmetry* as defined for normal-form games, which requires the payoffs be also permuted. Such symmetry, however, is impossible with selection games because it implies that when all agents play the same strategy there will be a tie.

Our first major result shows that any selection game can be extended to a quasi-symmetric one while preserving the dominant strategy sets for the agents. That is, in any selection game we can add dominated strategies for the agents to produce a quasi-symmetric selection game.

Theorem 3.2. *Let $G = (N, S, v)$ be a selection game where all agents' strategy sets are disjoint. Then, there exists a quasi-symmetric selection game $G' = (N, S'^n, v')$ such that for all $s \in \mathbb{S}$: $v'(s) = v(s)$, and for every agent $i \in N$: $D_G(i) = D_{G'}(i)$.*

Proof. Assume $N = \{1, 2, \ldots, n\}$ and assume $S_i = \{s_i^1, \ldots, s_i^{m_i}\}$ for all $i \in N$. Let $S' = \cup S_i$. That is, all agents can play any of the agents' strategies. The v' function is defined as follows: In order to preserve quasi-symmetry, we will define v' on multisets of strategies in S'. Let $p \in \mathbb{S}'$ be some multiset of size n over S'.

- If $\{i | \exists j : s_i^j \in p\} = N$, that is, exactly one strategy of each agent is played, then let $p'_i = s_i^j$ for all $i \in N$ and $s_i^j \in p$. In this case, we define $v'(p) \triangleq v(p')$. That is, the original game is played with the strategies assigned for each agent — note that these strategies may have been played by different agents than the original ones.
- Otherwise, we define $v'(s) \triangleq \text{argmax}_i(C_p(i) + \frac{i-1}{n})$, where
$$C_p(i) = \sum_{j=1}^{m_i} \#_p(s_i^j)$$
and $\#_p(s_i^j)$ is the number of instances of s_i^j in p. That is, when more than one strategy of an agent is played, the winner is determined by plurality on the number of times the agent's strategy is played, with ties broken based on agent number.

This game trivially satisfies the requirement that for all $s \in \mathbb{S}$: $v'(s) = v(s)$. It is now left to show that the dominant strategy sets of the agents remain the same.

Let $i \in N$ be some agent. Let $d_i \in D_G(i)$ be a dominant strategy for i in G. We shall now show d_i is dominant in G'. We must now show that for every strategy profile $p \in \mathbb{S}$ in G': $v'(p) = i \Rightarrow v'(d_i, p_{-i}) = i$. Let $p \in \mathbb{S}$ be some strategy profile and consider the following cases:

- If $\{i | \exists j : s_i^j \in p\} = N$ and $p_i = s_i^j$ for some j, that is, exactly one strategy of each agent is played and i plays its own strategy:
$$\begin{aligned} &v'(p) = i \Rightarrow v(p') = i \Rightarrow \\ &\Rightarrow v(d_i, p'_{-i}) = i \Rightarrow v'(d_i, p_{-i}) = i \end{aligned}$$
- If $\{i | \exists j : s_i^j \in p\} = N$ and $p_i = s_k^j$ for some j and $k \neq i$, then in (d_i, p_{-i}) agent i's strategies will be played twice while all other agents' strategies would be played at most once, and thus $v'(d_i, p_{-i}) = i$.
- If $\{i | \exists j : s_i^j \in p\} \neq N$ and $p_i = s_i^j$ for some j, then $v'(p) = v'(d_i, p_{-i})$ because $C_p(r) = C_{d_i, p_{-i}}(r)$ for all $r \in N$ and the original game is not played.
- If $\{i | \exists j : s_i^j \in p\} \neq N$ and $p_i = s_k^j$ for some j and $k \neq i$, then $C_{d_i, p_{-i}}(i) > C_p(i)$, while $C_{d_i, p_{-i}}(r) \leq C_p(r)$ for all $r \in N \setminus \{i\}$, and thus $v'(p) = i \Rightarrow v'(d_i, p_{-i}) = i$.

Now assume $d_i \in D_{G'}(i) \setminus D_G(i)$ is a dominant strategy. If $d_i = s_i^j$ for some j, then the same example proving s_i^j is not dominant in G is also valid in G'. Therefore, $d_i = s_k^j$ for some j and $k \neq i$. Consider two cases:

- Assume there exists some strategy profile $p \in \mathbb{S}$ in G where $v(p) = i$ and $s_k^j \in p$. Consider the strategy profile (d_i, p_{-i}). In this strategy profile s_k^j is played twice while ever other agent's strategy is played at most once and thus $v'(d_i, p_{-i}) = k \neq i$. Note that $v'(p) = i$, which is a contradiction to the fact that d_i is a dominant strategy in G'.
- Otherwise, there exists some strategy profile $p \in \mathbb{S}$ in G where $v(p) \neq i$ and $s_k^j \in p$. Consider the strategy profile $p' = p[i \leftrightarrow k]$, which is the same as p when agent i plays k's strategy and vice versa. Note that $p'_i = d_i$ and that $v'(p') = v(p) \neq i$. Now consider the strategy profile (s_i^j, p'_{-i}), where $s_i^j \in S_i$. In this strategy profile $C(i) = 2$ while for all other agents $r \in N$: $C(r) \leq 1$. Therefore, $v'(s_i^j, p'_{-i}) = i$ in contradiction to the fact that d_i is a dominant strategy in G'.

□

It is important to note that although this quasi-symmetric extension technically gives the agents more options to choose from, the constructed incentive structure implies that it is in each agent's best interest to play one of its original strategies, as all other strategies are dominated.

Example 3.3. The quasi-symmetric extension of the game in Example 2.5 is a $7 \times 7 \times 7 \times 7$ game. Here is a partial table of v' in this game where players 3 and 4 play a_3 and ε respectively (cells in **bold** indicate when the original game is played):

$s_1 \backslash s_2$	a_1	b_1	a_2	b_2	a_3	b_3	ε
a_1	1	1	**1**	**4**	3	3	4
b_1	1	1	**1**	**3**	3	3	4
a_2	**1**	**4**	2	2	3	3	4
b_2	**1**	**3**	2	2	3	3	4
a_3	3	3	3	3	3	3	4
b_3	3	3	3	3	3	3	4
ε	4	4	4	4	4	4	4

We can see in this example, that it is indeed best for each agent to play one of its original weakly dominant strategies.

4 Non-Imposition

One application of selection games is in the context of elections among members of some group, also known as Ranking Games or Systems[6, 1]. In this context, a very simple requirement is non-imposition, which means every agent has a situation in which it is elected[2]. However, in the context of selection games, we want to ensure this situation will actually occur when the game is played. Therefore, we require that all outcomes must be possible in a *solution* of the game:

Definition 4.1. Let $\mathcal{C}$ be a (pure) solution concept for selection games. A selection game $G = (N, S, v)$ is called *non imposing under solution concept* $\mathcal{C}$ if for all $i \in N$ there exists some solution $s \in \mathcal{C}(G)$ such that $v(s) = i$.

As the following proposition shows, if the number of agents is at most three, it is impossible to satisfy even this modest fairness requirement under weakly dominant strategies. Needless to say, this prevents the implementation of general distributions.

Proposition 4.2. *There exists no selection game that is non-imposing under weakly dominant strategies with* $|N| \leq 3$.

Proof. Assume $|N| = 2$. Let (s_1, s_2) be a solution under weakly dominant strategies where $v(s_1, s_2) = 1$ and let (s'_1, s'_2) be a solution under weakly dominant strategies where $v(s'_1, s'_2) = 2$. By weak dominance of s'_1: $v(s'_1, s_2) = 1$, but by weak dominance of $s_2 : v(s'_1, s_2) = 2$, which is a contradiction.

Assume $|N| = 3$. Let (s_1^1, s_2^1, s_3^1), (s_1^2, s_2^2, s_3^2), (s_1^3, s_2^3, s_3^3) be solutions under weakly dominant strategies where $v(s_1^i, s_2^i, s_3^i) = i$. By dominance of strategies $\{s_1^2, s_2^3, s_3^1\}$, we have:

$$\begin{aligned} v(s_1^2, s_2^1, s_3^1) &= 1 \\ v(s_1^2, s_2^3, s_3^2) &= 2 \\ v(s_1^3, s_2^3, s_3^1) &= 3 \end{aligned}$$

Now consider the strategy profile (s_1^2, s_2^3, s_3^1). If $v(s_1^2, s_2^3, s_3^1) = 1$, then by dominance of s_1^3: $v(s_1^3, s_2^3, s_3^1) = 1 \neq 3$, in contradiction to the above. Similarly, if $v(s_1^2, s_2^3, s_3^1) = 2$ then $v(s_1^2, s_2^1, s_3^1) = 2 \neq 1$ and if $v(s_1^2, s_2^3, s_3^1) = 3$ then $v(s_1^2, s_2^3, s_3^2) = 3 \neq 2$, which leads to a contradiction. □

We will revisit the significance of this result to ranking systems in our discussion in Section 8.

We shall now complete the classification by demonstrating the existence of non-imposing selection games with four or more agents.

Proposition 4.3. *There exist selection games that are non-imposing under weakly dominant strategies for all* $|N| \geq 4$.

Proof. Let $G_N = (N, S, v)$, where $S_1 = \{a_1, b_1, 5, 6, \ldots, n\}$, $S_2 = \{a_2, b_2\}$, $S_3 = \{a_3, b_3\}$ and $S_i = \{\varepsilon\}$ for all $i \geq 4$. The v function is defined as follows:

	a_3		b_3	
	a_2	b_2	a_2	b_2
a_1	1	4	2	2
b_1	1	3	4	3
5	1	5	5	5
6	1	6	6	6
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
n	1	n	n	n

Note that the three active agents are indifferent between all their strategies under every strategy profile, and thus all outcomes are solutions in weakly dominant strategies. As every agent has at least one outcome in which she appears, G_N satisfies non-imposition under weakly dominant strategies. □

5 Implementation of General Distributions

We shall now present our second main result: the existence of selection games that implement any distribution under mixed dominant strategies. First, we must define the concept of implementation of a distribution:

Definition 5.1. A selection game $G = (N, S, v)$ *implements a distribution* $\mathcal{D} \subseteq \Delta(N)$ *under correlated solution concept* $\mathcal{C}$ if there exists some solution $s \in \mathcal{C}(G)$ such that for all $i \in N$: $Pr\left[v(s) = i\right]$ is distributed according to $\mathcal{D}$.

A selection game $G = (N, S, v)$ is called *uniform under correlated solution concept* $\mathcal{C}$ it implements the uniform distribution under $\mathcal{C}$.

The existence of selection games that are uniform under uniform dominant strategies has been previously shown by [3] in the context of leader election. We shall now extend this result to any rational distribution:

Definition 5.2. A probability distribution $\mathcal{D}$ over a finite set N is called *rational* if for all $x \in N$: $\mathcal{D}(x) \in \mathbb{Q}$.

Theorem 5.3. *Let N be a finite player set where $|N| \geq 4$, and let $\mathcal{D}$ be a rational distribution over N. There exists a selection game that implements $\mathcal{D}$ under uniform dominant strategies.*

Proof. Assume $N = \{1, 2, \ldots, n\}$ and $\mathcal{D} = (d_1, \ldots, d_n)$ where $n \geq 4$. Let k be a multiplier such that $d_i \cdot k \in \mathbb{N}$ for all $i \in N$. Assume wlog that $d_1 \leq d_2 \leq \cdots \leq d_n$. Let $k' = (d_1 + d_2)k^2$.

Denote $B = \{k' \cdot (d_1 + d_2) + 1, \ldots k'\}^2$, and let $A \subseteq B$ be a set such that $|A| = d_3 \cdot k'^2$. Such a set exists because

$$\begin{aligned} d_3 \cdot k'^2 &\leq \frac{d_3 + d_4}{2} k'^2 \leq \frac{1 - d_1 - d_2}{2} k'^2 \leq \\ &\leq (1 - d_1 - d_2)^2 k'^2 = |B|. \end{aligned}$$

Let $G = (N, S, v)$ be the selection game where $S_i = \{1, \ldots, k'\}$ for $i \in \{1, 2, 3\}$ and $S_i = \{\varepsilon\}$ for $i \in \{4, \ldots, n\}$. The v function is defined to satisfy:

$$\begin{aligned} v(s_1, s_2, s_3) = 1 \quad &\Leftrightarrow \quad s_3 \leq \frac{k' \cdot d_1}{d_1 + d_2} \wedge \\ &\wedge \quad s_2 \leq k' \cdot (d_1 + d_2) \\ v(s_1, s_2, s_3) = 2 \quad &\Leftrightarrow \quad s_3 > \frac{k' \cdot d_1}{d_1 + d_2} \wedge \\ &\wedge \quad s_1 \leq k' \cdot (d_1 + d_2) \\ v(s_1, s_2, s_3) = 3 \quad &\Leftrightarrow \quad (s_1, s_2) \in A. \end{aligned}$$

The remaining combinations of (s_1, s_2, s_3) are mapped to the rest of the agents such that agent i will satisfy $v(s_1, s_2, s_3) = i$ for exactly $d_i \cdot k'^3$ combinations.

Note that the three active agents are indifferent between all their strategies under every strategy profile, and thus all outcomes are solutions in weakly dominant strategies. Further note that for $i \in \{1, 2, 3\}$, agent i appears exactly $d_i k'^3$ times which is d_i of the matrix, and that the remaining agents are placed in the remainder of the matrix. Thus the given distribution of the outcomes $\mathcal{D}$ will be implemented when all agents uniformly mix on their weakly dominant strategies. □

For the sake of mathematical completeness, we will now show that is also possible to implement irrational distributions, but only under mixed dominant strategies.

Fact 5.4. *Let $\mathcal{D}$ be an irrational distribution over a finite agent set N. There exists no selection game that implements $\mathcal{D}$ under uniform dominant strategies.*

Proof. Trivial. A uniform distribution on a finite strategy set leads to a rational probability for selection of each agent. □

Theorem 5.5. *Let N be a finite player set where $|N| \geq 4$. There exists a selection game G_N such that for any distribution $\mathcal{D}$ over N: G implements $\mathcal{D}$ under mixed dominant strategies.*

Proof. Assume $N = \{1, 2, \ldots, n\}$ and $\mathcal{D} = (d_1, d_2, \ldots, d_n)$ where $n \geq 4$. Assume wlog that $d_1 \leq d_2 \leq \cdots \leq d_n$.

Let $G_N = (N, S, v)$ be the selection game where $S_1 = \{a_1^4, \ldots a_1^n, b_1^4, \ldots, b_1^n\}$, $S_2 = \{a_2, b_2\}$, $S_3 = \{a_3, b_3\}$ and $S_i = \{\varepsilon\}$ for $i \in \{4, \ldots, n\}$. The v function is defined as follows:

<table>
<tr><th></th><th colspan="2">a_3</th><th colspan="2">b_3</th></tr>
<tr><th></th><th>a_2</th><th>b_2</th><th>a_2</th><th>b_2</th></tr>
<tr><td>a_1^4</td><td>4</td><td rowspan="6">1</td><td rowspan="3" colspan="2">2</td></tr>
<tr><td>$\vdots$</td><td>$\vdots$</td></tr>
<tr><td>a_1^n</td><td>n</td></tr>
<tr><td>b_1^4</td><td rowspan="3">3</td><td rowspan="3">3</td><td>4</td></tr>
<tr><td>$\vdots$</td><td>$\vdots$</td></tr>
<tr><td>b_1^n</td><td>n</td></tr>
</table>

Note that the three active agents are indifferent between all their strategies under every strategy profile, and thus all outcomes are solutions in weakly dominant strategies. Therefore, any mixture by the players is a mixed dominant strategy.

Now consider the following system of equations where x, y, z are the probabilities the three players play their "a" strategies respectively:

$$\begin{aligned} (1-y) \cdot z &= d_1 \\ (1-z) \cdot x &= d_2 \\ (1-x) \cdot y &= d_3 \\ 0 \leq x, y, z &\leq 1 \end{aligned}$$

Solving, we get the following solutions:

$$\begin{aligned} x &= \frac{1 - d_1 - d_3 + d_2 \pm \sqrt{\Delta}}{2(1-d_1)} \\ y &= \frac{1 - d_2 - d_1 + d_3 \pm \sqrt{\Delta}}{2(1-d_2)} \\ z &= \frac{1 - d_3 - d_2 + d_1 \pm \sqrt{\Delta}}{2(1-d_3)} \\ \Delta &= (1 - d_1 - d_2 - d_3)^2 - 4d_1 d_2 d_3 \end{aligned}$$

These solutions are in fact well-defined and satisfy the constraints (the cases for y and z are symmetric):

$$\begin{aligned} \Delta &= (1 - d_1 - d_2 - d_3)^2 - 4d_1 d_2 d_3 \geq \\ &\geq (\frac{1}{4})^2 - 4(\frac{1}{4})^3 = 0 \\ \sqrt{\Delta} &\leq (1 - d_1 - d_2 - d_3) \\ x &\geq \frac{1 - d_1 - d_3 + d_2 - (1 - d_1 - d_2 - d_3)}{2(1-d_1)} = \\ &= \frac{2d_2}{2(1-d_1)} \geq 0 \\ x &\leq \frac{1 - d_1 - d_3 + d_2 + (1 - d_1 - d_2 - d_3)}{2(1-d_1)} = \\ &= \frac{2(1 - d_1 - d_3)}{2(1-d_1)} \leq 1 \end{aligned}$$

Choose one of the solutions above, and let s^* be the strategy profile where player 1 plays strategy a_1^j with probability $x \cdot d_j / \sum_{i=4}^{n} d_i$, and strategy b_1^j with probability $(1-x) \cdot d_j / \sum_{i=4}^{n} d_i$; and players 2 and 3 play strategies a_2 and a_3 with probabilities y and z respectively. As the above equations are satisfied, we get that when s^* is played, every agent i wins with the correct probability d_i, and thus G_N implements $\mathcal{D}$. □

Applying Theorem 3.2, these results can be extended to quasi-symmetric games.

Corollary 5.6. *Let N be a finite player set where $|N| \geq 4$. Let $\mathcal{D}$ be a distribution over N. There exists a quasi-symmetric selection game G that implements $\mathcal{D}$ under weakly dominant strategies. Moreover, if $\mathcal{D}$ is rational then G implements $\mathcal{D}$ under uniform dominant strategies.*

6 No Dummy

We now revisit the notion of fairness. Although quasi-symmetry does indeed capture fairness in the sense that all agents have exactly the same potential influence, due to the incentive structure it may be the case that some agents have only one weakly dominant strategy, which they will always play. Therefore, it is of interest to consider the design of selection mechanisms where all agents have at least two different weakly dominant options to choose from.

Definition 6.1. Let $G = (N, S, v)$ be a selection game. A player $i \in N$ is called a *dummy player* if for all weakly dominant strategy profiles $s_{-i} \in S_{-i}$ and for all weakly dominant strategies $s_i, s'_i \in D_G(i)$:

$$v(s_i, s_{-i}) = v(s'_i, s_{-i}).$$

G is said to satisfy *no dummy* if no player $i \in N$ is a dummy player.

We now show that this no dummy property can in fact be satisfied by non-imposing selection games.

Proposition 6.2. *There exist selection games that are non-imposing under weakly dominant strategies and satisfy no dummy for every* $|N| \geq 4$.

Proof. We will prove the existence of such selection games by induction. For the base of the induction we will show games with 4, 5, and 6 players. In each of these games all players have two strategies $\{a_i, b_i\}$, and the v functions are described below (the cells with more than one value have different values depending on the number of players):

	a_6				b_6			
	a_4		b_4		a_4		b_4	
	a_2	b_2	a_2	b_2	a_2	b_2	a_2	b_2
a_5 a_3 a_1	1	3	2	2	2	2	1	3
b_1	1	4	3	4	3	4	1	4
b_3 a_1	4/5	3	4/5	1	5	1	5	3
b_1	2	2	3	1	3	1	2	2
b_5 a_3 a_1	2	2	1	3	1	3	2	2
b_1	3	4/6	1	4/6	1	6	3	6
b_3 a_1	5	1	5	3	5	3	5	1
b_1	3	1	2	2	2	2	3	1

It is easy to see that all strategies in these games are weakly dominant and that the games are in fact non-imposing under weakly dominant strategies. Furthermore, no player in these games is a dummy player.

Let $N = \{1, \ldots, n\}$, where $n \geq 7$. Let G_{n-3} be a selection game that is satisfies no dummy and is non-imposing under weakly dominant strategies for the player set $\{4, \ldots, n\}$. In this case, the following game G_n is played:

	a_3		b_3	
	a_2	b_2	a_2	b_2
a_1	1	$(*)$	2	2
b_1	1	3	$(*)$	3

If a strategy profile marked by $(*)$ is played, the game G_{n-3} determines the outcome. All strategies in this game are weakly dominant and this game is non-imposing by induction on G_{n-3}. Furthermore, no player in the game is a dummy player, as required. ☐

The classification of the set of distributions that are implementable in mixed or uniform dominant strategies by selection games with the constraint of no dummy is an open problem. We conjecture that any distribution is implementable in mixed dominant strategies in such games. We shall now present a partial result in that direction.

Proposition 6.3. *Let* $N = \{1, \ldots n\}$ *be a player set such that* $n = 4 + 3k$ *for some* $k \in \mathbb{N}$. *There exists a selection game* G_n *that satisfies no dummy and implements any distribution* $\mathcal{D} \in \Delta(N)$ *under mixed dominant strategies.*

Proof. Let $\mathcal{D} = (d_1, d_2, \ldots, d_n)$. Assume wlog that $d_1 \leq d_2 \leq \cdots \leq d_n$. The game G_n is the game recursively defined in the proof of Proposition 6.2. We have already shown this game satisfies no dummy and that all strategies are weakly dominant. It remains to show a solution in mixed dominant strategies in which $\mathcal{D}$ is implemented.

The solution in mixed dominant strategies that implements $\mathcal{D}$ is as follows: Agent n plays a_n with probability 1. For all $k \in \{0, \ldots \frac{n-4}{3}\}$, the other agents' strategies are defined as follows (following the proof of Theorem 5.5):

$$\begin{aligned}
s_{3k+1} &= \frac{r_k + 2d_{3k+2}/S_k \pm \sqrt{\Delta}}{2(1 - d_{3k+1}/S_k)} \\
s_{3k+2} &= 1 - \frac{r_k + 2d_{3k+3}/S_k \pm \sqrt{\Delta}}{2(1 - d_{3k+2}/S_k)} \\
s_{3k+3} &= \frac{r_k + 2d_{3k+1}/S_k \pm \sqrt{\Delta}}{2(1 - d_{3k+3}/S_k)} \\
S_k &= \sum_{i=3k+1}^{n} d_i \\
r_k &= \frac{1}{S_k} \sum_{i=3k+4}^{n} d_i
\end{aligned}$$

$$\Delta = r_k^2 - 4d_{3k+1}d_{3k+2}d_{3k+3}/S_k^3.$$

The distribution $\mathcal{D}$ is implemented because in every subgame the three participants are selected with the correct relative probability, and the remaining games are played in the remainder of the cases. □

In order to demonstrate our conjecture, we will now show an example of a selection game that satisfies no dummy and is uniform under uniform dominant strategies.

Example 6.4. The following 8-player selection game satisfies no dummy and is uniform under uniform dominant strategies:

		$a_{7\oplus8}$		$b_{7\oplus8}$	
		$a_{3\oplus4}$	$b_{3\oplus4}$	$a_{3\oplus4}$	$b_{3\oplus4}$
$a_{7\oplus8}$	$a_{1\oplus2}$	1	7	3	3
	$b_{1\oplus2}$	1	5	8	5
$b_{7\oplus8}$	$a_{1\oplus2}$	2	7	4	4
	$b_{1\oplus2}$	2	6	8	6

Each pair of agents' strategy is XOR'd together to affect each of the four dimensions of this matrix, thus every agent has an effect on the outcome. As all strategies for all agents are dominant, this game satisfies no dummy. As every cell is played with equal probability and every agent appears exactly twice, this selection game is in fact uniform under uniform dominant strategies.

7 Strong Nash Equilibrium

One can consider other solution concepts in parallel to the ones based on weakly dominant strategies. In particular, another strong concept is the strong Nash equilibrium[4]. Here we consider a stronger version of this equilibrium concept, where the deviating coalitions are allowed to transfer utility.

Definition 7.1. Let G be a selection game. The *Strong Nash Equilibrium* mixed solution concept $\mathcal{C}_{SNE}$ is the set of all mixed strategy profiles s where for all $C \subseteq N$ and for all mixed strategy profiles $s' \in \Delta(S_1) \times \Delta(S_2) \times \cdots \times \Delta(S_n)$ where $s'_i = s_i \quad \forall i \in N \setminus C$:

$$Pr[v(s') \in C] \le Pr[v(s) \in C]$$

As it turns out, while the study of dominant strategy solutions for the implementation of arbitrary distributions has been to the best of our knowledge first introduced in this paper, similar results regarding strong Nash equilibrium can be easily derived from classical mechanisms in cryptography[5].

Proposition 7.2. *There exist quasi-symmetric selection games that implement every rational distribution $\mathcal{D}$ under strong Nash equilibrium for every N, where no agent has a dominated strategy.*

Proof. Assume $N = \{1, \ldots n\}$ let $\mathcal{D} = (d_1, \ldots, d_n)$ be a rational distribution over N. Let k be a multiplier such that $d_i \cdot k \in \mathbb{N}$ for all $i \in N$. Let $G = (N, S, v)$ be a selection game where every agent has strategy set $S_i = \{0, \ldots k-1\}$, and for all $i \in N$: $v(s) = i$ iff

$$k \cdot \sum_{j=1}^{i-1} d_j \le \left[\sum_{j=1}^{n} s_j\right] \bmod k < k \cdot \sum_{j=1}^{i} d_j.$$

This game is trivially quasi-symmetric. Consider the strategy profile where all agents uniformly mix over their pure strategies. In this strategy profile the sum $[\sum_{j=1}^n s_j] \bmod k$ is distributed uniformly on $\{0, \ldots k-1\}$. By the definition of v, the distribution $\mathcal{D}$ is correctly implemented. Furthermore, this is a strong Nash equilibrium because a coalition $C \subsetneq S$ cannot change the distribution of $[\sum_{j=1}^n s_j] \bmod k$, while the coalition $C = S$ always includes the winning agent. Therefore, no coalition can profit from deviation. □

8 Discussion and Implications

A slight variation of the selection games setting discussed in this paper is when the incentive structure is reversed, and each agent prefers *not* to be selected. Any selection game where all agents have only weakly dominant strategies, and thus are indifferent between all their strategies, retains this feature in the reverse incentive structure. As our second main result was built using this kind of games, it equally applies in the reverse utility setting. Indeed, apart from the results regarding quasi-symmetry, all of the results in this paper apply in the reverse utility setting.

Selection games are as a special case of ranking games[6], where the agents care only about whether or not they are ranked first (or last). Mechanism design in this setting has been studied in work on ranking systems[1, 2]. For example, when outgoing links are considered as votes, any page ranking system, such as PageRank[11] or the HITS algorithm[8] can be described as a ranking system, implying a selection game in which agents care only about being ranked first. Our results can therefore be interpreted also from the perspective of the study of existence of non-imposing incentive compatible ranking systems. In particular, although it has been shown that under the linear utility function there exists an incentive-compatible non-imposing ranking system for three agents[2], we have shown in Proposition 4.2 that no such selection mechanism exists.

It is also interesting to put this work also in perspective of work on incentive-compatible social choice. The celebrated Gibbard-Sattewhaite theorem[7, 12] shows an impossibility result for non-imposing incentive compatible mechanisms if there are at least three candidates in a social choice setting. In contrast, our results show that in a selection game setting, although impossibility is obtained when there are *at most* three agents, a constructive possibility result is obtained for four or more agents.

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