Routing Games with an Unknown Set of Active Players

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ABSTRACT

In many settings there exists a set of potential participants, but the set of participants who are actually active in the system, and in particular their number, is unknown. This topic has been first analyzed by Ashlagi, Monderer, and Tennenholtz [AMT] in the context of simple routing games, where the network consists of a set of parallel links, and the agents can not split their jobs among different paths. AMT used the model of pre-Bayesian games, and the concept of safety-level equilibrium for the analysis of these games. In this paper we extend the work by AMT. We deal with splitable routing games, where each player can split his job among paths in a given network. In this context we generalize the analysis to all two-node networks, in which paths may intersect in unrestricted manner. We characterize the relationships between the number of potential participants and the number of active participants under which ignorance is beneficial to each of the active participants.

1. INTRODUCTION

The study of congestion games [22, 18, 16] is central to game theory, computer science, and electronic commerce. Indeed, the study of congestion games has become a central ingredient in work connecting the above disciplines (see e.g. [20, 19, 14, 4, 13, 12, 6]). Most of the related studies assume complete information, and, to the best of our knowledge, all of them, except for [3], are assuming complete information about the set (and in particular the number) of participants in the system.[1] However, in many settings, although the set of registered/potential participants may be known, the actual set of active participants is unknown. Hence, incorporating uncertainty about the set of actual participants into congestion settings is a desirable task.

Routing games are defined by congestion networks, and as such they are a special of congestion games, which are defined by congestion forms.[2] In this paper we focus on symmetric congestion games, and therefore we focus on two-node congestion networks only.[3] In a two-node congestion network there is a two-terminal directed graph, and every edge in the graph is associated with a per-unit cost function. The per-unit cost of moving one unit of good through this edge is a function of the number of units that are moved through this edge. Distinct edges may have distinct cost functions. In this paper we assume that the edge cost functions are continuously differentiable, convex, and increasing. There are two types of games that can be associated with a congestion network and a finite set of users. In a routing game with divisible units, each unit of good is continuously divisible, and hence every user should decide about the proportion of her good to be moved through each route. In the other model, the goods are not divisible. In this paper we focus on routing games with divisible units. In a routing game with complete information every player knows the network structure, the cost functions, and the number of users. In a routing game with incomplete information discussed in this paper, every active player knows all of the above except for the number of active players; She does know the number of potential players.[4]

Games with incomplete information with a commonly known prior probability over the set of states of nature are called Bayesian Games. When such a prior probability does not exist in a given game, and we want to stress its non-existence we call this game pre-Bayesian.[5] In previous work Ashlagi, Monderer and Tennenholtz [AMT] [3] suggested to model

[1]Some recent works deal with incomplete information about other paramters in the Bayesian setting [7, 8].

[2] A congestion form is a particular structure consisting of a set of resources together with a class of subsets of this set. Every congestion form, and a finite set of users uniquely define a congestion game. However, distinct congestion forms can generate identical congestion games. A particular type of form is the one generated by a network. Games defined by congestion networks are called routing games.

[3]It is obvious, and also can be derived from [15] that every symmetric congestion game is a symmetric routing game, which can be derived from a two-node congestion network.

[4]In our model there is a finite number of agents. The initial research of congestion games [22], as well as much of the recent research [21, 19] discuss congestion games with continuum of agents, which are called non-atomic congestion games.

[5]In the economics' literature the term " game with incomplete information" has been used as a synonym to "Bayesian Game". Only recently researchers have been worked on games without priors, and such games received several titles in distinct papers. In this paper we follow the terminology of [9], and we refer to such games as pre-Bayesian.

behavior in pre-Bayesian games by the concept of safety-level equilibrium. A safety-level equilibrium is a strategy profile in which each agent minimizes his worst case cost over all possible states of the environment, assuming the other agents stick to their prescribed strategies. In the above context the possible states of the environment correspond to the possible sets of active players. AMT concentrated on simple (parallel) routing games with nondivisible units. They showed that in the linear case the lack of information about the number of active participants is beneficial for the players. Their result was obtained under the assumption that the participants use a symmetric equilibrium in both, the complete and incomplete information games.

As we said the focus of this paper is on symmetric routing games with divisible units. Therefore, we deal with general two-node networks. Unlike [AMT] We do not assume symmetric behavior but rather prove that such behavior holds in equilibrium. We characterize the relationships between the number of potential participants and the number of active participants under which ignorance is beneficial to each of the active participants.

In order to study the value of ignorance one needs to define a corresponding index for the value of ignorance. Let $c(k)$ be the cost of each player in equilibrium in the complete information case when there are k players, and $c(k,n)$ be the cost of each player in a safety-level equilibrium in the related game with incomplete information when there are k active players and n potential players. We define the value of ignorance to be $\nu(k,n) = c(k) - c(k,n)$. If this value is non-negative, ignorance is beneficial for the players. In order for the above index to make sense the cost in equilibrium at each of the above settings should be uniquely defined. As we will show, this is indeed the case in the model discussed in this paper.

We show that for simple linear routing games with divisible units, for sufficiently large k, $\nu(k,n) \geq 0$ for $k < n < k(k+1) - 1$. That is, each of the k active players "enjoys" ignorance when the number of potential players, n, is in the above interval. Under the minor assumption that there are two linear edge-cost functions which have different additive coefficients, each agent strictly gains due to that uncertainty, i.e. $\nu(k,n) > 0$. In addition we show that $\nu(k,n) \leq 0$ for $n > k(k+1) - 1$, and $\nu(k,n) = 0$ for $n = k(k+1) - 1$.

Next we deal with general linear two-node networks. We show that $\nu(k,n) \geq 0$ for $k < n \leq 2k-1$. Moreover we show that $\nu(k,n-1) \leq \nu(k,n)$ for every $k < n \leq 2k-1$ and that the reverse inequalities hold for every $n > 2k-1$. In particular for a fixed k, $\nu(k,n)$ is minimized at $n = 2k-1$.

Our results have interesting implications in the context of protocol design in congestion settings with incomplete information. Consider an organizer who knows the number of participants at each given point, and wishes to maximize social surplus. That is, the organizer's goal is to minimize the agents' costs. In ranges in which the value of ignorance is positive (i.e. when the number of potential participants is not too large with respect to the number of active participants) the organizer should not reveal the number of actual participants. Analogously, in ranges in which the value of ignorance is negative the organizer should reveal the number of actual participants. Note that if the costs are paid to a revenue-maximizing organizer, the above policies should be reversed.

2. BASIC DEFINITIONS AND MODELS

2.1 Two-Node Congestion Networks

Let $G = (V, E, v_s, v_t)$ be a 2-terminal directed graph without self edges, where V is a finite set of nodes, E is a finite set of edges, and $v_s, v_t \in V$ are two distinct nodes called the source node and target node, respectively. For every $v \in V$ we denote by $Out(v)$ and $In(v)$ the set of out-going and incoming edges of v, respectively. A *route* is a directed path with distinct nodes that connects v_s to v_t. For every edge $e \in E$ and a route R we write $e \in R$ whenever e is part of the route R. Let RO be the set of routes. We assume that $RO \neq \emptyset$.

Every edge $e \in E$ is associated with a cost function $d^e : \Re \rightarrow \Re$ which satisfies the following properties:

- Every edge cost function d^e is continuously differentiable, convex, increasing, and $d^e(x) > 0$ for every $x > 0$.

$d^e(x)$ is interpreted as the cost per unit that is moved through e when the load on e is x.[6]

Let $\mathbf{d} = (d^e)_{e \in E}$ be the vector of edge cost functions. The tuple $\mathcal{N} = (G, \mathbf{d})$ is called a *congestion network*. A congestion network is called *simple* if $V = \{v_s, v_t\}$. It is called linear if there exist positive constants a_e, and non-negative constants b_e, $e \in E$ such that $d^e(x) = a_e x + b_e$ for every $e \in E$ and for every $x \in \Re$.

2.2 Route Flows and Edge Flows

Consider an agent who has to move a continuously divisible unit of good from the source to the target. A splitting policy for such an agent is therefore a function $g : RO \rightarrow [0,1]$ with $\sum_{R \in RO} g(R) = 1$. That is, for every route R, $g(R)$ is interpreted as the proportion of the unit sent through the route R. Such a splitting policy is also called a *route flow*. For every route flow g and for every $e \in E$ we denote $f_g^e = \sum_{R \in RO | e \in R} g(R)$. That is, f_g^e is the number of units routed through e. It is well-known that for every route flow g the following two conditions hold for the vector $f = (f^e)_{e \in E} = (f_g^e)_{e \in E}$:

$$\sum_{e \in Out(v)} f^e = \sum_{e \in In(v)} f^e + r^v \quad \text{for every} \quad v \in V. \qquad (1)$$

$$f^e \geq 0, \quad \text{for every} \quad e \in E, \qquad (2)$$

where

$$r^v = \begin{cases} 1 & v = v_s \\ -1 & v = v_t \\ 0 & otherwise. \end{cases}$$

Every vector $f = (f^e)_{e \in E}$ that satisfies the above two conditions is called an *edge flow*, and f_g is called the edge flow induced by the route flow g. The set of route flows is denoted by $\Delta(RO)$, and the set of edge flows is denoted by F. Hence every route flow $g \in \Delta(RO)$ induces an edge flow $f_g \in F$, but it is obvious, and well-known that not every edge flow is induced by some route flow. A necessary condition for an edge flow to be induced by a route flow is

[6] The values of $d^e(x)$ for $x < 0$ are not relevant to any of our discussions, but it is technically useful to let d^e be defined over the whole real line.

given below. A cycle in G is a simple closed directed path. Let f be an edge flow, and let C be a cycle. We say that C is positive with respect to f if $f^e > 0$ for every $e \in C$.

LEMMA 1. *Let $\mathcal{N}$ be a two-node congestion network. Every flow f with no positive cycles is induced by some route flow.*

The proof follows from a more general theorem named - the flow decomposition theorem (see [2]).

Note that an edge flow may be induced by several distinct route flows, as can be seen in the following example.

EXAMPLE 1. *Consider the following graph (Figure 1). Let g_a and g_b be two route flows defined as follows: $g_a(v_s - a - b - v_t) = 0.1$, $g_a(v_s - a - c - d - b - v_t) = 0.2$, $g_a(v_s - c - d - v_t) = 0.7$. $g_b(v_s - a - b - v_t) = 0.1$, $g_b(v_s - a - c - d - v_t) = 0.2$, $g_b(v_s - c - d - b - v_t) = 0.2$, $g_b(v_s - c - d - b - v_t) = 0.5$. Observe that both g_a and g_b induce the edge flow shown in the figure.*

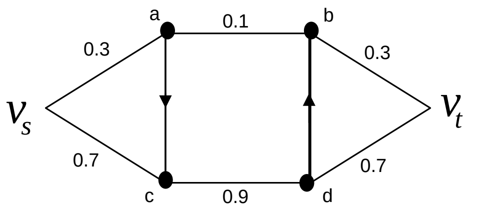

Figure 1: Two-node Congestion Network.

2.3 Symmetric Routing Games

Let $\mathcal{N} = (G, \mathbf{d})$ be a two-node congestion network, and let I be a nonempty finite set of players. Whenever it is convenient and harmless we assume that $I = \{1, \cdots, n\}$, $n \geq 1$. We are about to define strategies and cost functions in the corresponding game denoted by $\Gamma_{\mathcal{N}}(I)$. This game, when derived from a general congestion network is called a *routing game*.[7] In our case the routing game is symmetric. In this game, every player i chooses a route flow g_i, and thus a *route flow profile* $\mathbf{g} = (g_1, g_2, \cdots, g_n) \in \Delta(RO)^I$ is generated. Each such profile of route flows generates a profile of edge flows, $\mathbf{f_g} = (f_{g_1}, f_{g_2}, \cdots, f_{g_n}) \in F^I$. The cost function of every player, $c_i(\mathbf{g})$ would depend on the profile of edge flows $\mathbf{f_g}$ via the formula

$$c_i(\mathbf{g}) = C_i(\mathbf{f_g}),$$

where $C_i : F^I \to \Re$ is a function defined over profiles of edge flows as follows:

$$C_i(\mathbf{f}) = \sum_{e \in E} f_i^e d^e(\mathbf{f}^e), \qquad (3)$$

where

$$\mathbf{f}^e = \sum_{i=1}^{n} f_i^e \quad \text{for every } e \in E$$

Routing games will be called simple (linear) if they are derived from simple (linear) congestion networks.

A route flow profile $\mathbf{g}$ is in *equilibrium* in $\Gamma_{\mathcal{N}}(I)$ if

$$c_i(\mathbf{g}) \leq c_i(h_i, \mathbf{g}_{-i})$$

for every player i and for every route flow h_i, where $\mathbf{g}_{-i}$ denotes the profile of route flows of all players but i.

In the following theorem we show that every symmetric routing game possesses an equilibrium. We further show that although the symmetric routing game may have multiple equilibria, the concept of *equilibrium cost* is well-defined. That is, there exists a level of cost $c(n)$ such that in every equilibrium profile $\mathbf{g}$ in $\Gamma_{\mathcal{N}}(I)$, every player pays $c(n)$. That is, $c_i(\mathbf{g}) = c(n)$ for every player i.

THEOREM 1. *Let $\mathcal{N} = (G, \mathbf{d})$ be a two-node congestion network, let n be a positive integer, and let I be a set of n players.*

1. *The routing game $\Gamma_{\mathcal{N}}(I)$ possesses an equilibrium.*
2. *There exists a symmetric profile of edge flows, $\mathbf{f}[n] = (f[n], \cdots, f[n])$ such that every equilibrium route flow profile in $\Gamma_{\mathcal{N}}(I)$ induces $\mathbf{f}$. That is, for every equilibrium $\mathbf{g}$, $\mathbf{f_g} = \mathbf{f}[n]$.*
3. *Consequently, there exists a level of cost, $c(n)$ such that in every equilibrium of the routing game $\Gamma_{\mathcal{N}}(I)$ every player pays $c(n)$; $c(n)$ is called the equilibrium cost in $\Gamma_{\mathcal{N}}(I)$.*

PROOF. It is useful to extend the routing game to a game in which the players can choose edge flows directly. In this game, which we call the *edge flow routing game* and denote it by $\tilde{\Gamma}_{\mathcal{N}}(I)$, every player is able to choose an edge flow rather then just a route flow. Hence, the strategy set of every player is F, and the cost function of player i is given in (3).

It was proved in Theorem 5 in [17] that there exists a unique equilibrium in $\tilde{\Gamma}_{\mathcal{N}}(I)$. Obviously every permutation of this equilibrium profile is also an equilibrium. Therefore the unique equilibrium must be symmetric. In order to complete the proof of the theorem we have to relate route equilibrium profiles in the routing game $\Gamma_{\mathcal{N}}(I)$ to the unique edge equilibrium profile in $\tilde{\Gamma}_{\mathcal{N}}(I)$.

Let $\mathbf{f}_{-i}$ be a profile of edge flows of all players but i. An edge flow f_i is called a best response for i versus $\mathbf{f}_{-i}$ if $\min_{h_i \in F} C_i(h_i, \mathbf{f}_{-i})$ is attained at $h_i = f_i$. Because the edge cost functions are positive in $(0, \infty)$, such a best response f_i cannot have a positive cycle. Therefore, by Lemma 1, f_i is induced by some route flow g_i, that is $f_{g_i} = f_i$. Hence the following claim holds:

Claim 1

1. For every equilibrium profile, $\mathbf{f} \in F^I$ in the edge flow routing game $\tilde{\Gamma}_{\mathcal{N}}(I)$ there exists a route flow profile $\mathbf{g} \in \Delta(RO)$ that induces $\mathbf{f}$, that is $\mathbf{f_g} = \mathbf{f}$. Moreover, every such route flow profile $\mathbf{g}$ is in equilibrium in the routing game $\Gamma_{\mathcal{N}}(I)$.
2. Let $\mathbf{g} \in \Delta(RO)^I$ be an equilibrium route flow profile in $\Gamma_{\mathcal{N}}(I)$. Then $\mathbf{f_g}$ is an equilibrium profile in $\tilde{\Gamma}_{\mathcal{N}}(I)$.

[7] Some authors call it a network game.

Combining the existence and uniqueness of equilibrium in $\tilde{\Gamma}_{\mathcal{N}}(I)$ with Claim 1 completes the proof.

□

2.4 Incomplete Information About the Number of Active Players

Routing games with unknown active players, are pre-Bayesian games as discussed in [3]. Let $\mathcal{N}$ be a two-node congestion network, and let I be a finite set of *potential players*. A *state* is a nonempty subset, K of players. That is, the set of states is $\Omega = 2^I \setminus \{\emptyset\}$. The set of active players at the state K is K itself. An active player knows that he is active, but he does not know the true state. Hence, an active player knows nothing about the other players, and in particular he does not know the number of active players. In a *routing game with unknown active players* denoted by $H_{\mathcal{N}}(I)$, at every state K the players in K are playing the game $\Gamma_{\mathcal{N}}(K)$, but they do not know it. The lack of knowledge about the set of active players does not have an effect on the set of strategies available to each potential player. A strategy for every potential player i is a route flow g_i, which he will use once he is active. Note however, that an active player cannot compute his cost even if he knows the complete route flow profile $\mathbf{g} = (g_i)_{i=1}^n$. All he knows is that he will get $c_i(\mathbf{g_K})$ if the set of active players is K, where $\mathbf{g_K} = (g_i)_{i \in K}$. When players are considering worst-case scenarios regarding the missing information about the set of active players, and they are in equilibrium, they form a *safety-level equilibrium* as defined in [3].[8] According to this definition, a profile of route flows $\mathbf{g}$ is a *safety level equilibrium* in $H_{\mathcal{N}}(I)$ if for every player i the maximal value of $\min_{K \subseteq I, i \in K} c_i(h_i, \mathbf{g_{K \setminus \{i\}}})$ over all $h_i \in \Delta(RO)$ is obtained at $h_i = g_i$. Since all cost functions are non-decreasing the worst case scenario is obtained in state $K = I$. We obtain the following result.

LEMMA 2. *Let $\mathcal{N} = (G, \mathbf{d})$ be a two-node congestion network , and let I be a finite set of players. Let $\mathbf{g} \in \Delta(RO)^I$ be a route flow profile. $\mathbf{g}$ is a safety-level equilibrium in the routing game with incomplete information $H_{\mathcal{N}}(I)$ if and only if $\mathbf{g}$ is an equilibrium in $\Gamma_{\mathcal{N}}(I)$.*

PROOF. The proof is similar to the proof of Lemma 4 in [3] and is therefore omitted. □

Theorem 1 implies that when there are n potential players and k active players each of the active players is using at every safety-level equilibrium a route flow that induces the edge flow $f[n]$, which is the edge flow induced in equilibrium in the complete information game with n players. Let $c(k,n)$ be the actual cost of each of the active k players when each of them is using $f[n]$. That is, for an arbitrary player i, $c(k,n) = C_i(\mathbf{f[n]_K})$.

3. THE VALUE OF IGNORANCE

We proceed to analyze the value of ignorance in routing games as a function of the relationship between the number of active participants, k, and the number of potential participants, n.

Consider a two-node congestion network $\mathcal{N} = (G, \mathbf{d})$, and the associated routing game with unknown active players, $H_{\mathcal{N}}(I)$, where $|I| = n$. Suppose that the real state of the world is K where $|K| = k$ and $k < n$. If this state is commonly known then each player $i \in K$ pays $c(k)$. If the real state is unknown then every active player pays $c(k,n)$. Therefore it is natural to call the difference the *value of ignorance*. We denote the value of ignorance by $\nu(k,n)$. That is,

$$\nu(k,n) = c(k) - c(k,n)$$

The value of ignorance indicates how much players "enjoy" the ignorance about the actual set of players. Observe that ignorance is beneficial (in a weak sense) for the players if and only if $\nu(k,n) \geq 0$. In the following example we demonstrate the value of ignorance in a simple congestion network.

EXAMPLE 2. *Consider the congestion network $\mathcal{N}$ in Figure 2. Let $I = \{1,2,3\}$, i.e. there are 3 potential players. Let the real state be $K = \{1,2\}$. Hence, there are two active players. First we find the equilibrium in the routing game with complete information with 2 players. Assume the first player sends $y \geq 0$ on the upper edge and $1 - y \geq 0$ on the lower edge. Then the second player's objective is to minimize $x(x+y) + (1-x)(1-x+1-y+1)$, where x is the amount she will send on the upper edge. The solution to this is $x = \frac{2-y}{2}$. Since the induced edge flow profile in equilibrium is symmetric (see Theorem 1) it must be that $x = y$. Therefore $x = \frac{2}{3}$. The cost for each of the players in this case is $c(2) =$*

$$\frac{2}{3} * \frac{4}{3} + \frac{1}{3} * (\frac{2}{3} + 1) = \frac{13}{9}.$$

We next find the equilibrium in the routing game with complete information with 3 players. Assuming the total amount two players send in the upper edge is $y \geq 0$, then the third player's objective is to minimize $x(x+y)+(1-x)(1-x+2-y+1)$, where x is the amount she will send on the upper edge. The solution to this is $x = \frac{5-2y}{4}$. By the symmetry of the induced edge flow profile in equilibrium we obtain $x = \frac{5-4x}{4}$ and therefore $x = \frac{5}{8}$. If the state K is not known to the players then by playing the safety-level equilibrium each of the players in K will send $\frac{5}{8}$ in the upper edge an therefore their costs will be $c(2,3) =$

$$\frac{5}{8} * \frac{10}{8} + \frac{3}{8}(\frac{6}{8} + 1) = \frac{23}{16}.$$

Hence the value of ignorance is $\nu(2,3) = \frac{13}{9} - \frac{23}{16} > 0$.

[8] The leading solution concept for pre-Bayesian games is ex post equilibrium. However, ex post equilibrium rarely exists. In contrast, It was shown in [3] that safety-level equilibrium exists in general in pre-Bayesian games. Independently, [1] Another type of equilibrium that exists in general in pre-Bayesian games is a minimax-regret equilibrium [10]. However, we do not discuss this equilibrium in this paper.

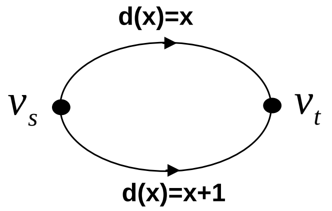

Figure 2.

In Example 2 we showed that the value of ignorance may be positive. In our next results we provide a rich class of games in which this phenomena occurs. We proceed to estimate the value of ignorance in linear routing games over simple and general networks.

3.1 Linear Routing Games: Simple Congestion Networks

Consider a linear congestion network in which $d^e(x) = a^e x + b^e$.

THEOREM 2. *Let $\mathcal{N} = (G, \mathbf{d})$ be a simple and linear congestion network . There exist an integer $N(\mathcal{N})$ such that for every $k \geq N(\mathcal{N})$.*

1. *$\nu(k,n) \geq 0$ for every $k(k+1)-1 > n > k$. Moreover, the inequality is strict if and only if there exists $e_1, e_2 \in E$ such that $b_{e_1} \neq b_{e_2}$.*
2. *$\nu(k,n) \leq 0$ for every $n > k(k+1)-1$. Moreover, the inequality is strict if and only if there exists $e_1, e_2 \in E$ such that $b_{e_1} \neq b_{e_2}$.*
3. *For $n = k(k+1)-1$, $\nu(k,n) = 0$.*

In order to prove Theorem 2 we need the following lemma:

LEMMA 3 ([11]). *Let $\mathcal{N} = (G, \mathbf{d})$ be a simple linear congestion network . For every n let $\Gamma_{\mathcal{N}}(n)$ be a routing game with n players. Let $A = \sum_{e\in E} \frac{1}{a^e}$ and let $B = \sum_{e\in E} \frac{b^e}{a^e}$.*

1. *If at equilibrium each player sends a positive amount on each edge, that is $f^e[n] > 0$ for every $e \in E$, then*
$$f^e[n] = \frac{1}{a^e A}[1 + \frac{B - b^e A}{(n+1)}] \quad \text{for every} \quad e \in E. \quad (4)$$
2. *$f^e[n] > 0$ for every $e \in E$ if and only if*
$$\frac{1}{A}[1 + \frac{B}{n+1}] > \max_{e\in E} \frac{b^e}{n+1}. \quad (5)$$

Proof of Theorem 2: By part 2 of Lemma 3 there exists an integer N depending on $\mathcal{N}$ such that for every $n \geq N$ inequality (5) holds. Let $N(\mathcal{N}) = N$, and let $n > k \geq N(\mathcal{N})$. Denote $C = \sum_{e\in E} \frac{b^{e2}}{a^e}$. We are about to prove that

$$\nu(k,n) =$$
$$\frac{(AC - B^2)((n-k)(k^2+k-n-1))}{A(n+1)^2(k+1)^2}, \quad (6)$$

where A and B are defined in the statement of Lemma 3. Since $b^{e2} + {b^l}^2 \geq 2b^e b^l$ for every $e, l \in E$, $AC - B^2 \geq 0$. Moreover, $AC - B^2 > 0$ if and only if there exist a couple of edges, $\hat{e}, \hat{l}$ such that $b_{\hat{e}} \neq b_{\hat{l}}$. In addition $k^2 + k - n - 1$ is positive for $n < k(k+1)-1$, negative for $n > k(k+1)-1$ and zero otherwise. Therefore the proof of the theorem follows from (6). We have to prove (6). Indeed, by Lemma 3 and because (5) holds, for every $e \in E$, $f^e[n] = \frac{1}{a^e A}[1 + \frac{B - b^e A}{(n+1)}]$, and $f^e[k] = \frac{1}{a^e A}[1 + \frac{B - b^e A}{(k+1)}]$. Therefore,

$$\nu(k,n) =$$
$$\sum_{e\in E} k a^e[(f^e[k])^2 - (f^e[n])^2] + \sum_{e\in E} b^e(f^e[k] - f^e[n]). \quad (7)$$

As

$$\sum_{e\in E} b^e(f^e[k] - f^e[n]) =$$
$$\sum_{e\in E} \frac{(B - b^e A)(n-k)b^e}{(n+1)(k+1)a^e A} =$$
$$\frac{(B^2 - AC)(n-k)}{(n+1)(k+1)A}, \quad (8)$$

and

$$\sum_{e\in E} k a^e[(f^e[k])^2 - (f^e[n])^2] =$$
$$k\sum_{e\in E} a^e[(\frac{1}{a^e A} + \frac{B - b^e A}{(k+1)a^e A})^2 -$$
$$k\sum_{e\in E} a^e[(\frac{1}{a^e A} + \frac{B - b^e A}{(n+1)a^e A})^2] =$$
$$k\sum_{e\in E}[\frac{2(B - b^e A)(n-k)}{(n+1)(k+1)a^e A^2}] +$$
$$k\sum_{e\in E}[\frac{(B - b^e A)^2}{(k+1)^2 a^e A^2} - \frac{(B - b^e A)^2}{(n+1)^2 a^e A^2}] =$$
$$k\sum_{e\in E}[\frac{(B - b^e A)^2}{(k+1)^2 a^e A^2} - \frac{(B - b^e A)^2}{(n+1)^2 a^e A^2}] =$$
$$k[\frac{CA - B^2}{(k+1)^2 A} - \frac{CA - B^2}{(n+1)^2 A}].$$

We obtain that $\nu(k,n) =$

$$\frac{(AC - B^2)(k(n+1)^2 - k(k+1)^2}{A(n+1)^2(k+1)^2} -$$
$$\frac{(AC - B^2)(n-k)(n+1)(k+1))}{A(n+1)^2(k+1)^2}.$$

Since

$$(k(n+1)^2 - k(k+1)^2 - (n-k)(n+1)(k+1)) =$$
$$k[(n-k)(n+k) + 2(n-k)] - (n-k)(n+1)(k+1)] =$$
$$(n-k)(k^2 + k - n - 1),$$

(6) follows.□

3.2 Linear Routing Games: Two-Node Congestion Networks

Let $\mathcal{N} = (G, \mathbf{d})$ be a linear two-node congestion network. As in the previous section, our goal is to estimate the value of ignorance. Our next result shows that ignorance about the real state can be beneficial for the players in a rich and natural set of situations, assuming an arbitrary general network.

THEOREM 3. *Let $\mathcal{N} = (G, \mathbf{d})$ be a linear two-node congestion network. Let $k, n \geq 1$ be integers. If $2k-1 \geq n > k$ then $\nu(k,n) \geq 0$, and $\nu(k, n-1) \leq \nu(k,n)$. Moreover, if there exists an edge e for which $f^e[n]$ admits at least two different values at the interval $n \in [k, 2k-1]$, the above inequalities are strict.*

We need some preparations for the proof of Theorem 3. Recall that an edge flow is a vector $f \in R^E$ that satisfies conditions (1) and (2). However, the right-hand-side of (3) is well-defined for every vector indexed by the edges. This enables us to extend the cost functions C_i to $(\Re^E)^I$.

Let $\mathbf{f} \in (\Re^E)^I$. The marginal cost of each user $i \in I$ on the edge e with respect to f_i^e is

$$\frac{\partial C_i(\mathbf{f})}{\partial f_i^e} = d^e(\mathbf{f}^e) + f_i^e \frac{\partial d^e(\mathbf{f}^e)}{\partial \mathbf{f}^e}.$$

We further need the following notation. For every fictitious edge flow profile $\mathbf{f}$ and every couple of reals $\alpha, \beta \in \Re$ let $K_i(\mathbf{f}, \alpha, \beta) = \frac{\partial C_i(\mathbf{f})}{\partial f_i^e} + \alpha - \beta$.

Let $\mathcal{N}$ be a congestion network, and let I be a finite set of players. Let $\mathbf{f_{-i}}$ be a profile of edge flows of all players but i. We say that f_i is a best response to $\mathbf{f_{-i}}$ if f_i is an optimal solution for the following minimization problem:

$$PR_i : \quad \min_{z_i \in F} C_i(z_i, \mathbf{f_{-i}}).$$

Note that PR_i can be specifically written as the following minimization problem with the decision variables z_i^e, $i \in I, e \in E$

$$PR_i \begin{cases} \min \sum_{e\in E} z_i^e d^e(z_i^e + \sum_{j\neq i} f_j^e) \\ \text{s.t.} \quad z_i \in \Re^E \quad \text{and} \\ \sum_{e\in Out(v)} z_i^e = \sum_{e \in In(v)} z_i^e + r_v^i \qquad v \in V \\ z_i^e \geq 0, \quad \text{for every} \quad e \in E. \end{cases}$$

As the objective function in PR_i is convex, and all constraints are defined by linear inequalities and equalities, PR_i is a convex minimization problem with linear constraints.

Therefore by Theorem 5 in Section 4, necessary and sufficient conditions for optimality are provided by the Karush-Kuhn-Tucker (KKT) conditions. Thus, $f_i \in F$ is an optimal solution for PR_i if and only if there exist Lagrange multipliers λ_i^v, $v \in V$ such that for every edge $e \in E$:

$$K_i(\mathbf{f}, \lambda_i^{\hat{t}(e)}, \lambda_i^{\hat{h}(e)}) \geq 0, \quad \text{and} \quad K_i(\mathbf{f}, \lambda_i^{\hat{t}(e)}, \lambda_i^{\hat{h}(e)}) f_i^e = 0, \tag{9}$$

where $\hat{t}(e)$ and $\hat{h}(e)$ are the tail and head nodes of the edge e respectively.

Therefore by Claim 1 (appearing at the proof of Theorem 1) an edge flow profile $\mathbf{f} \in F^I$ is induced by a route flow equilibrium profile if and only if for every player $j \in I$ there exist lagrange multipliers λ_j^v, $v \in V$ such that for every edge $e \in E$:

$$K_j(\mathbf{f}, \lambda_j^{\hat{t}(e)}, \lambda_j^{\hat{h}(e)}) \geq 0, \quad \text{and} \quad K_j(\mathbf{f}, \lambda_j^{\hat{t}(e)}, \lambda_j^{\hat{h}(e)}) f_j^e = 0. \tag{10}$$

As $d^e(x) = a^e x + b^e$, and $I = \{1, ..., n\}$,

$$K_j(\mathbf{f}, \lambda_j^{\hat{t}(e)}, \lambda_j^{\hat{h}(e)}) = a^e f_j^e + a^e \sum_{i=1}^n f_i^e + b^e + \lambda_j^{\hat{t}(e)} - \lambda_j^{\hat{h}(e)} \tag{11}$$

for every $j \in I$.

By Theorem 1, $(f[n], ..., f[n])$ is the unique edge flow profile induced by every equilibrium. Therefore, there exist Lagrange multipliers λ^v, $v \in V$ such that for all $e \in E$:

$$(n+1)a^e f^e[n] + b^e + \lambda^{\hat{t}(e)} - \lambda^{\hat{h}(e)} \geq 0, \quad \text{and}$$

$$[(n+1)a^e f^e[n] + b^e + \lambda^{\hat{t}(e)} - \lambda^{\hat{h}(e)}] f^e[n] = 0. \tag{12}$$

For every real $t \geq 0$ consider the following convex optimization problem $(SYM)_t$:

$$(SYM)_t : \quad min_{f\in F} \sum_{e\in E} a^e t (f^e)^2 + b^e f^e.$$

Remark: the problem $(SYM)_n$ can be interpreted as finding the minimal cost for every player when all players are restricted to use the same edge flow.

By Theorem 5 in Section (4), $f \in F$ is an optimal solution for $(SYM)_t$ if and only if there exist Lagrange multipliers λ^u, $u \in V$ such that for every edge $e \in E$:

$$2ta^e f^e + b^e + \lambda^{\hat{t}(e)} - \lambda^{\hat{h}(e)} \geq 0 \quad \text{and}$$

$$[2ta^e f^e + b^e + \lambda^{\hat{t}(e)} - \lambda^{\hat{h}(e)}] f^e = 0. \tag{13}$$

We are now able to prove the following lemma.

LEMMA 4. *Let $\mathcal{N} = (G, \mathbf{d})$ be a linear congestion network and let $n \geq 1$. $f \in F$ is a solution to $(SYM)_n$ if and only if $f = f[2n-1]$. In addition*

$$\sum_{e\in E} f^e[n](a^e \frac{n+1}{2} f^e[n] + b^e) \leq \sum_{e\in E} f^e[k](a^e \frac{n+1}{2} f^e[k] + b^e)$$

for every integer $k \geq 1$.

PROOF. The proof follows after observing that by setting $t = \frac{n+1}{2}$ in (13) we get (12). □

Proof of Theorem 3: We show that $c(k, n+1) \leq c(k,n)$ for every n such that $2k-2 \geq n \geq k$. For every integer $\hat{n} > 0$ we extend the function $c(\cdot, \hat{n})$ to non-integer positive numbers α as follows: $c(\alpha, n) = \sum_{e\in E} f^e[\hat{n}](a^e \alpha f^e[\hat{n}] + b^e)$. Note that

$$c(k,n) = A_k(f[n]) + B(f[n])$$

, where

$$A_k(f[n]) = k \sum_{e\in E} a^e (f^e[n])^2$$

and

$$B(f[n]) = \sum_{e\in E} b^e f^e[n].$$

Let $2k-2 \geq n \geq k$. By lemma 4, $c(\frac{n+1}{2}, n) \leq c(\frac{n+1}{2}, n+1)$, and also $c(\frac{n+2}{2}, n) \geq c(\frac{n+2}{2}, n+1)$. Therefore

$$c(\frac{n+2}{2},n) - c(\frac{n+2}{2},n+1) =$$

$$A_{\frac{n+2}{2}}(f[n]) + B(f[n]) - A_{\frac{n+2}{2}}(f[n+1]) - B(f[n+1]) =$$

$$s[A_{\frac{n+1}{2}}(f[n]) - A_{\frac{n+1}{2}}(f[n+1])] + B(f[n]) - B(f[n+1]) \geq 0,$$

where $s = \frac{n+2}{2}/\frac{n+1}{2}$. Let $D(t) =$

$$t[A_{\frac{n+1}{2}}(f[n]) - A_{\frac{n+1}{2}}(f[n+1])] + B(f[n]) - B(f[n+1]).$$

We showed that $D(1) \leq 0$ and $D(s) \geq 0$. Therefore $D(t) \geq 0$ for every $t \geq s$ by the monotonicity of $D(t)$ in t. However $k \geq \frac{n+2}{2}$. Therefore by setting $t = k/\frac{n+1}{2}$ we obtain the desired result since $t \geq s$. If $f^e[n] \neq f^e[n+1]$ for some $e \in E$ then $c(k,n+1) < c(k,n)$ by the convexity of the program $(SYM)_n$. □

Next we show that $\nu(k,n)$ is non-increasing in n for $n \geq 2k-1$.

THEOREM 4. *Let $\mathcal{N} = (G, \mathbf{d})$ be a linear congestion network. Let $k \geq 1$ be an integer. $\nu(k,n+1) \leq \nu(k,n)$ for every n such that $n \geq 2k-1$.*

PROOF. We need the following claim.

Claim 2 Let $\rho_k \geq 0$ $k = 1,2,...$ be an increasing sequence of real numbers. Let $F : \mathbb{R}_+^m \to \mathbb{R}_+$ and let $G : \mathbb{R}_+^m \to \mathbb{R}_+$. Define $H : \mathbb{R}_+^m \times N \to \mathbb{R}_+$ by $H(x,k) = F(x) + \rho_k G(x)$. Let x_k be a solution of the minimization problem of $H(x,k)$ s.t. $x \in D \subset \mathbb{R}_+^m$ where D is a bounded convex set. $F(x_k) \leq F(x_{k+1})$ for every $k \geq 1$.

Proof (claim): Notice that for every $k \geq 1$ $H(x_k,k) \leq H(x_{k+1},k+1)$ since $\rho_k \leq \rho_{k+1}$. We next show that $G(x_k) \geq G(x_{k+1})$. Observe that

$$F(x_k) + \rho_k G(x_k) \leq F(x_{k+1}) + \rho_k G(x_{k+1})$$

and that

$$F(x_{k+1}) + \rho_{k+1} G(x_{k+1}) \leq F(x_k) + \rho_{k+1} G(x_k).$$

Therefore

$$(\rho_{k+1} - \rho_k) G(x_k) \geq (\rho_{k+1} - \rho_k) G(x_{k+1})$$

which yields $G(x_k) \geq G(x_{k+1})$. Since

$$F(x_{k+1}) + \rho_k G(x_{k+1}) \geq F(x_k) + \rho_k G(x_k)$$

it must be that $F(x_k) \leq F(x_{k+1})$. □

We proceed with the main proof. Let i be some arbitrary player. For every edge flow $f \in F$ we define $F(f) = \sum_{e \in E} f^e(a^e k f^e + b^e)$ and $G(f) = \sum_{e \in E} a^e (f^e)^2$. Let $H(f,m) = F(f) + \frac{m}{2} G(f)$. by lemma 4, for every $m = 0,1,2,...$ the optimization problem $\min_{f \in F} H(f,m)$ is minimized at $f = f[2k+m-1]$. Therefore by the claim $F(f[n+1]) \geq F(f[n])$ for every $n \geq 2k-1$. Observe that $c(k,n)) = F(f[n])$. Hence $c(k,n+1)) \geq c(k,n))$ for every $n \geq 2k-1$. □

4. THE KARUSH-KUHN-TUCKER (KKT) CONDITIONS

In this section we describe the relevant theory of the KKT conditions that was required in our proofs. The material is taken from [5].

Consider the following problem:

$$(IC) \quad \min\{f(x) : g_j(x) \leq 0, j = 1,...,m,$$
$$h_k(x) = 0, k = 1,...,p, \quad x_i \geq 0, i = 1,..,n, x \in \mathcal{R}^n\}.$$

We say that (IC) is a convex program if $f, g_1, ..., g_m$ are real valued convex and differentiable functions on $\Re^n$, and $h_1, ..., h_p$ are linear.

For every $x \in \Re^n$ let

$$L(x) = f(x) + \sum_{j=1}^{m} \mu_j g_j(x) + \sum_{k=1}^{p} \lambda_k h_k(x).$$

The following are the well known *Karush-Kuhn-Tucker* (KKT) conditions at a feasible point x^*:

$$KKT \begin{cases} \text{There exists lagrange multipliers } \mu_j \; j = 1,...,m \\ \text{and } \lambda_k \; k = 1,...,p \quad \text{such that} \\ \mu_j g_j(x^*) = 0, \quad \mu_j \geq 0 \quad j = 1,...,m, \\ \quad \frac{\partial L_i(x^*)}{\partial x_i} \leq 0, \qquad x_i^* \frac{\partial L_i(x^*)}{\partial x_i} = 0 \quad i = 1,...,n \end{cases}$$

THEOREM 5. *(The Karush-Kuhn-Tucker Theorem)*
Let (IC) be a convex program and let x^* be a feasible solution to (IC). If there exists $\bar{x} \in \mathcal{R}_+^n$ such that at $\bar{x}$ the non-linear g_j are strictly negative, and linear g_j are non-positive then the KKT conditions are both necessary and sufficient for x^* to be optimal for (IC).

5. SUMMARY

In many congestion settings there exists a set of potential users, but the set of users which are actually active in the system is unknown. Routing games naturally fall into these settings.

The value of ignorance is an index, which measures how much the users "enjoy" the ignorance about the actual set of players. AMT [3] analyzed the value of ignorance in routing games in which players cannot split their goods and the underlying network is simple - all paths are parallel. In this paper we significantly extend their work. We analyze the value of ignorance in all symmetric routing games, in which players can split their goods. We deal both with simple and two-node networks. We show that in both types of networks ignorance is helpful if the number of potential participants is not too big with respect to the number of active participants. Our results in simple networks differ from the results obtained in AMT. We show how the value of ignorance changes as a function of the number of potential players. As in AMT we use the concept of pre-Bayesian games, and the concept of safety-level equilibrium for the analysis of the value of ignorance.

The number of active players in a routing game is just one example for data that may not be available to the players. Other natural examples are network structure, edge cost functions, and job sizes. The study of the Pre-Bayesian games associated with lack of information about such data is an interesting direction for future research.

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