Position Auctions with Budgets: Existence and Uniqueness

Itai Ashlagi[*] Mark Braverman[†] Avinatan Hassidim[‡] Ron Lavi[§]
Moshe Tennenholtz[¶]

Abstract

We design a Generalized Position Auction for players with private values and private budget constraints. Our mechanism is a careful modification of the Generalized English Auction of Edelman, Ostrovsky and Schwarz (2007). By enabling multiple price trajectories that ascent concurrently we are able to retrieve all the desired properties of the Generalized English Auction, that was not originally designed for players with budgets. In particular, the ex-post equilibrium outcome of our auction is Pareto-efficient and envy-free. Moreover, we show that any other position auction that satisfies these properties and does not make positive transfers must obtain in ex-post equilibrium the same outcome of our mechanism, for every tuple of distinct types. This uniqueness result holds even if the players' values are fixed and known to the seller, and only the budgets are private.

1 Introduction

Online advertisements via auction mechanisms are by now a major source of income for many Internet companies. Whenever an Internet user performs a search on Google, an automatic "position auction" is being conducted among several different potential advertisers, and Google places the winning ads next to the search results it outputs. Google and Yahoo! generate a revenue of several cents per each such auction, and these numbers add up to billions of dollars every year. The importance of correctly designing these auctions, and of analyzing their different economic properties, is clear.

Indeed, in this electronic setting, the interplay between theoretical models and practical implementations is rich. Many actual auction implementations were based on early theoretical insights, and the actual auction formats that have evolved over time motivated deep theoretical studies. Two examples are the papers of Varian (2007) and of Edelman et al. (2007), that analyze Google's "generalized second price" (GSP) auction, and show that it has many attractive properties. In

[*]Harvard Business School, Harvard University, 02163, USA. Email: itai.ashlagi@gmail.com.
[†]Microsoft Research New England, Cambridge, MA, USA. Email: mbraverm@cs.toronto.edu.
[‡]MIT. Email: avinatan@mit.edu.
[§]Faculty of Industrial Engineering and Management, Technion – Israel Institute of Technology. Email: ronlavi@ie.technion.ac.il. Supported by grants from the Israeli Science Foundation and the Bi-national Science Foundation.
[¶]Microsoft Israel R&D Center and Technion – Israel Institute of Technology. Email: moshet@ie.technion.ac.il.

particular, the GSP auction obtains in equilibrium an efficient (welfare-maximizing) allocation, with envy-free prices. Moreover, Edelman et al. (2007) extend these results to the incomplete-information setting via an elegant generalization of the English auction. Several other variants of position-auctions models have been studied, see e.g. Athey and Ellison (2008) and Kuminov and Tennenholtz (2009) and the references therein.

Many of the works on position auctions completely ignore the issue of budgets, and focus on the bidder's value from winning one of the slots. In sharp contrast, all actual position auctions allow bidders to specify both a value and a budget, and the latter parameter serves an important role in the strategic considerations of the bidders.[1] In fact, budgets are a weak point of the more general auction theory as well, with relatively few works that study the subject.[2] The several works that do study the effect of budgets indicate that, because the existence of budgets changes the quasi-linear nature of utilities, properly inserting budgets into the model usually results in significant modifications to the theory, both technically and conceptually. Therefore the importance of studying position auctions with budgets is two-fold: to align theoretical auction models with realistic auction systems, and to enrich the theoretical understanding of the effects of budgets on auction design.

In this paper we design a position auction for players with budgets in an incomplete information setting, where both the bidders' values and budgets are private information. As could be expected, we observe that previous analysis, and in particular the analysis of the generalized English auction, fails when budget limits exist. We obtain two results: First, we design a "generalized position auction" that retrieves all the nice properties of the generalized English auction, while taking budgets into account. In particular, the ex-post equilibrium outcome of our auction is envy-free and Pareto-efficient. Second, we show that any other mechanism that always obtains envy-freeness and Pareto-efficiency in ex-post equilibrium must choose the same slot assignment and the same payments as our mechanism, at least whenever all true types are distinct. This uniqueness result holds even if players' values are fixed and known, and the only private information of the players is their budgets. This last property is especially interesting given the argument of Edelman et al. (2007), that a complete-information assumption regarding the players' values is reasonable. Our uniqueness result shows that, in our context, such a relaxation will not make the problem easier.

We are not the first to describe a position auction with budgets. Such auctions were formulated several times already, as a special case of the more general model of unit-demand players with budget constraints. Van der Laan and Yang (2008) and Kempe, Mu'alem and Salek (2009) show that an adaptation of the Demange-Gale-Sotomayor ascending auction finds an envy-free allocation even if players have budget constraints. Aggarwal, Muthukrishnan, Pal and Pal (2009) additionally show that this mechanism is incentive-compatible. Hatfield and Milgrom (2005) study a more abstract

[1] Actual technical rules for Google's auction, for example, can be found at http://www.google.com/intl/en/ads/

[2] An up-to-date picture of the literature on auctions with budgets is given in the recent paper of Pai and Vohra (2008).

unit-demand model for players with non-quasi-linear utilities that generalizes both the Gale-Shapley stable-matching algorithm as well as the Demange-Gale-Sotomayor ascending auction, and provide an incentive-compatible and (in the case of our setting) envy-free mechanism. On top of these works, our new contributions are: (1) the uniqueness result, and (2) the new auction format that extends the generalized English auction, rather than the matching/unit-demand format. Indeed, our auction has a completely different structure, and it converges to the equilibrium outcome along a different price path. Without our uniqueness result, one could easily (incorrectly) imagine that the two proposed formats (the matching-based and the GSP-based) end-up in different outcomes. Since the generalized English auction follows Google's "next price" auction, our extension for players with budgets seems of independent interest.[3]

To highlight the specific effect of budgets on the generalized English auction, recall the basic setting: there are k slots and n players, and player i obtains a value of $\alpha_l v_i$ from receiving slot l, where the constants $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_k$ are given as an input to the mechanism (they are common knowledge), and each player is interested in at most one slot. In the generalized English auction, a single price gradually ascends, and players need to decide when to drop. Rename the players such that player 1 is the last to remain, player 2 is the second to last to remain, and so on. When the l'th player drops, she is allocated slot l for a payment that is equal to the price-point at which the $l+1$ player dropped. Thus, when l players remain, each one sees a fixed price for slot l and a gradually increasing price for the better slots, and should decide whether to drop and take slot l, or to remain and receive a better slot. The key observation in the analysis of Edelman et al. (2007) is that, if each player plans to drop at the price that makes her indifferent between slot l and slot $l-1$, the winner of slot l will not regret in retrospect the fact that she did not win a better slot. This is immediate regarding slot $l-1$, but more subtle regarding the slots that are better than $l-1$, and follows from the fact that the first to drop among the remaining l players is the one with the lowest value.

With budgets, however, this key observation fails. A player that becomes indifferent between slot l and slot $l-1$ because she has the lowest value may later be able to offer a higher price than her competitors for the slots better than $l-1$, if the competitors are limited by a low budget. For this reason, a single price trajectory fails to reach an equilibrium. The other extreme, of performing k completely separate auctions sequentially, will also not yield an ex-post equilibrium since, intuitively, the competition on slot k depends on the identity of the winners of better slots, and vice-versa. Our solution is a hybrid between these two extremes. We maintain k price trajectories, one for each slot, that ascend in a carefully-designed concurrent way. Enabling low-valued players with high budgets to "join the race" at the later stages is the main high-level conclusion that stems from our technical analysis.

In addition to the straight-forward importance of an existence and uniqueness result, which

[3] Varian (2007) also remarks on the similarity to matching models, and argues that the GSP auction is of particular interest because of actual Internet auctions.

illuminates some of the effects of budgets on auction design, in a more general context our analysis contributes another layer to the currently small literature on auctions with budgets. In particular, we wish to point out two conceptual aspects of the positive result: First, it is "detail-free" and "robust" (Wilson, 1987; Bergemann and Morris, 2005), while most previous works on auctions with budgets assume a Bayesian setting that is not robust, and sometimes not even detail-free. Second, it should be contrasted with the recent interesting impossibility of Dobzinski, Lavi and Nisan (2008). They show that there does not exist a dominant-strategy incentive-compatible and Pareto-efficient multi-item auction, even in the very restrictive setting of two identical items and two players with additive private valuations (and a private budget constraint). The existence result for position auctions demonstrates the importance of their assumption that players wish to receive *multiple* items. With unit-demand, a possibility (though unique) still exists, as demonstrated here.

The remainder of this paper is organized as follows. In section 2 we setup the formal model and explain our different technical assumptions. In section 3 we describe why the generalized English auction fails when players have budgets, and detail our modified format. Section 4 provides the analysis, with some technical details postponed to Appendix B.

2 Preliminaries

Basic model of position auctions. In a position auction there is a set $K = \{1, ..., k\}$ of items ("slots") and a set $N = \{1, ..., n\}$ of bidders, where each bidder is interested in receiving one of the slots. Each slot $l \in K$ is characterized by a known constant $\alpha_l > 0$, where $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_k$, which is an input to the mechanism. Each bidder i obtains a monetary value of $\alpha_l v_i$ from receiving slot l, where v_i is a parameter that is privately known only to player i. We assume without loss of generality that $k \leq n$, since otherwise we can just ignore the $k - n$ lowest slots.

This model has been studied in recent years (see e.g. Varian (2007) and Edelman et al. (2007)) in order to analyze the ad auctions that are conducted by search engines like e.g. Yahoo! and Google. In a nut-shell, search engines place paid online advertisements in proximity to search results that they output. Advertisers bid for the online placement of their advertisements, and the k winning bidders are positioned on the web-page according to their bids. The value v_i represents bidder i's expected profit given that the Internet user will click on her ad, and the constant α_l (the "click-through rate") represents the probability that the Internet user will indeed click on the ad, given that the ad is positioned at slot l. Slot 1 is the "best" position, i.e. has the highest click-through rate, slot 2 is the second-best, and so on and so forth.[4]

Budget constraints and valid outcomes. Previous works on Google's next-price auction have

[4] We follow the exact model of Varian (2007) and of Edelman et al. (2007), in which the click-through rate depends only on the position of the slot, and not on other factors like the quality of the different ads. Few recent works have begun to study more complex click-through-rate models, see for example the work of Kuminov and Tennenholtz (2009) and the references therein.

assumed quasi-linear utilities, i.e. bidder i's utility from receiving slot l and paying P_i is equal to $\alpha_l v_i - P_i$. In this paper we analyze the effect of adding a hard budget limit that caps the maximal payment ability of a player. More precisely, each player i has a privately-known budget b_i, and cannot pay any price $P \geq b_i$. Thus the resulting utility of a player with type (v_i, b_i) that receives slot l and pays P is

$$u((v_i, b_i), l, P) = \begin{cases} \alpha_l \cdot v_i - P & P < b_i \\ 0 & P \geq b_i \end{cases}$$

where the zero-utility for the case that $P \geq b_i$ captures the fact that if a player has to pay such P, she will default and will not complete the transaction. (our results continue to hold if this zero utility is replaced with any other negative utility). Note that the feasibility regime is any $P < b_i$. This is more convenient for us than a weak inequality due to some technical reasons that will be explained below.[5]

To summarize, we define a *valid outcome* of a position auction as a tuple $(s_i, p_i)_{i \in N}$, where every bidder i receives the slot $s_i \in K \cup \{k+1\}$ ($k+1$ is a dummy slot with $\alpha_{k+1} = 0$) and pays p_i. A valid outcome must additionally satisfy:

1. *(feasibility)* $s_i, s_j \in K, i \neq j$, implies $s_i \neq s_j$.[6]
2. *(budget limit)* $p_i < b_i$.
3. *(ex-post Individual Rationality (IR))* $p_i \leq \alpha_{s_i} v_i$.

It should be noted that valid outcomes are deterministic. Interpreting budget constraints in a randomized context is a more subtle task that we defer to future work.

Desired solution properties. Since there are many possible valid outcomes, one may wish to focus attention on those outcomes that are "efficient" and "fair", as captured by the following two classic properties:

1. *(Pareto-efficiency)* A valid outcome $o = (s_i, p_i)_{i \in N}$ is Pareto efficient if there is no other valid outcome $o' = (s'_i, p'_i)_{i \in N}$ such that $\alpha_{s'_i} v_i - p'_i \geq \alpha_{s_i} v_i - p_i$ (players weakly prefer o' to o) and $\sum_{i \in N} p'_i \geq \sum_{i \in N} p_i$ (the seller weakly prefers o' to o), with at least one strict inequality.
2. *(Envy-freeness)* A valid outcome $(s_i, p_i)_{i \in N}$ is envy-free if for every two distinct players $i, j \in N$ such that $p_j < b_i$, $\alpha_{s_i} v_i - p_i \geq \alpha_{s_j} v_i - p_j$.[7]

[5] When budgets are real numbers then either the set of all infeasible payments includes its infimum or the set of all feasible payments includes its supremum, and this choice does not seem to have any conceptual meaning.

[6] Note that many players can be assigned to the dummy slot $k+1$, meaning that they do not receive any slot.

[7] One may also consider to weaken the definition of envy-freeness to a stability condition, such that for each pair of players, at least one of them would not like to exchange slots and prices with the other. This turns out to be weaker than Pareto-efficiency: one can show that any Pareto-efficient outcome satisfies this stability condition, but there may be stable outcomes that are not Pareto-efficient. We additionally note that envy-freeness in position auctions is related to stable matchings in a two-sided graph, as discussed by Edelman et al. (2007).

The generalized English auction of Edelman et al. (2007) is envy-free, and we will show that this strong fairness property can still be achieved even when budgets are considered. Pareto-efficiency is strictly weaker than envy-freeness:

Proposition 1. *Every envy-free outcome in which all slots are allocated is Pareto-efficient.*

Proof. Assume by contradiction that $o = (s_i, p_i)_{i \in N}$ is envy-free but not Pareto-efficient, and let $o' = (s'_i, p'_i)_{i \in N}$ be a valid outcome that Pareto improves o. Without loss of generality $\sum_{i \in N} p'_i > \sum_{i \in N} p_i$, since if it is an equality then there exists a player i with a strict inequality $s'_i v_i - p'_i > s_i v_i - p_i$ and we can slightly increase p'_i to get a Pareto improving outcome in which the seller's payoff is strictly larger than her payoff in o.

For any slot l, let q_l, q'_l be the payment of the player who receives slot l is o, o' respectively. Let j be some slot such that $q'_j > q_j$, and suppose player i_j received slot j in o'. Since player i_j's utility in o' is not smaller than her utility in o it follows that i_j received a different slot in o, say $s_{i_j} = l \neq j$. We get $\alpha_l v_{i_j} - q_l \leq \alpha_j v_{i_j} - q'_j < \alpha_j v_{i_j} - q_j$. Thus player i_j envies the player who got slot j in o, contradicting the fact that o is envy-free. □

The opposite statement is not true; there exist valid outcomes that are Pareto-efficient but not envy-free. For example, the outcome that maximizes the social welfare (sum of players' values for the slots they receive) and charges no payments is Pareto-efficient, but is not envy-free.

Since the type (v_i, b_i) of player i is private information, known only to the player herself, we study the design of *mechanisms* that output (in equilibrium) a valid outcome which is Pareto-efficient and envy-free. We focus on so called "detail-free" solutions concepts, and use the equilibrium notions of ex-post Nash, and dominant strategies.[8] We refer to a direct mechanism that is incentive compatible in dominant strategies as "truthful".

Assuming distinct budgets. It turns out that, in some subtle sense, it is impossible to construct truthful and Pareto-efficient auctions, even for a single item. It is long known (Che and Gale (1998); see also Krishna (2002)) that the following single-item mechanism is truthful when true budgets are distinct: the winner is the player i with the maximal "bid" $\min(b_i, v_i)$, and she pays the second largest bid. One can verify that the dominant-strategy of a player is to declare their true value and budget, and that the outcome is Pareto-efficient and envy-free. Moreover, Dobzinski et al. (2008) show (in a more general context) that this mechanism is the unique truthful and Pareto-efficient mechanism, at least when all budgets are distinct.

To demonstrate[9] that the assumption of distinct budgets is crucial, suppose two players with budgets $b_1 = b_2 = 1$ and values $v_1 = v_2 = 3$. The $\min(v, b)$ mechanism chooses w.l.o.g. player 1 as the winner, and she pays a price of 1, for a resulting utility $v_1 - P_1 = 2$. If player 2 is able to

[8] Our direct mechanism exhibits an ex-post equilibrium, and not dominant strategies, as explained below.

[9] Similar examples were described by Van der Laan and Yang (2008) and by Aggarwal et al. (2009).

pay exactly her budget that she can gain from declaring a false budget $b' > b_2$: she becomes the winner, and pays a price of 1, for a resulting utility $v_2 - P_2 = 2 > 0$. If a player is not able to pay her exact budget parameter b_i, but only any strictly lower payment, then the outcome is infeasible and player 1 will prefer to lose and pay 0 over paying the infeasible price of 1. Either way, this mechanism is not truthful when budgets are identical.

We explicitly spell out the assumption of distinct budgets, which was implicit in some of the previous works (most probably since the event of having non-distinct budgets has zero probability). This assumption leads to the technical requirement that players can only pay any price which is strictly less than their budgets. In the zero-probability event that there exist two players with equal budgets, the auction may be canceled in order to avoid infeasible outcomes.

Alternatively, one can assume a discrete type space (in other words, assuming that all parameters are integers). With discrete types, the mechanism can avoid the infeasibility of the outcome when budgets are identical by artificially increasing the budget of each player i by an arbitrarily small $\epsilon_i < 1$ with $\epsilon_i \neq \epsilon_j$ for any two players i, j. This pre-processing step will make the mechanism truthful and Pareto-efficient even if budgets are identical (envy-freeness will still be violated with identical budgets, though). One can verify that all our results continue to hold under these modifications; we do not repeat all proofs for the discrete setting to keep the exposition concise.

3 The Generalized Position Auction

3.1 The effect of budgets on the Generalized English Auction

It is constructive to start with a short discussion on the generalized English auction of Edelman et al. (2007). This auction gradually increases a price parameter Q, and players decide whether to drop or stay. Rename the players according to the reverse order at which they dropped (player 1 never dropped, player 2 dropped last, etc.). When player $l \leq k$ drops she wins slot l and pays the price at which player $l+1$ dropped. This continuous-time description is made discrete and formal by the following definition:

Definition 1 (The Generalized English Auction (Edelman et al., 2007))**.** *Initialize $Q = 0$ (current price), $l = \min(k+1, n)$ (current slot), and $N_l = N$ (active set of bidders). Then perform:*

1. *Each player $i \in N_l$ declares a bid p_i^l (this is the price at which player i plans to drop).*

2. *The l'th highest bidder wins slot l and pays Q (recall that slot $k+1$ is a dummy slot with $\alpha_{k+1} = 0$).*

3. *If $l = 1$ then terminate. Otherwise raise Q to be the l'th bid, define N_{l-1} to be the $l-1$ highest bidders, decrease l by one, and repeat from step 1.*

Informally, when the price increases and $l \leq k$ active bidders remain, each bidder i faces two alternatives: to drop and win slot l for a price that is already fixed and known (this is the price at which the $(l+1)$'th bidder dropped), or to stay in the auction. This decision represents a trade-off between winning slot l or winning one of the better slots $1, ...l-1$ (for a higher price). In the formal definition the price does not increase continuously but the same tradeoff has to be made when the player chooses her new bid at step 1. The equilibrium strategies are derived by looking closely at this tradeoff. Assuming infinite budgets, the price P at which player i becomes indifferent between winning slot l for a price Q and winning slot $l-1$ for a price P should satisfy $\alpha_l v_i - Q = \alpha_{l-1} v_i - P$, or alternatively $P = (\alpha_{l-1} - \alpha_l) v_i + Q$. If the player bids this P in step 1 and ends up winning slot l, she is guaranteed not to regret the fact that she did not win slot $l-1$. The twist in the analysis of Edelman et al. (2007) is to show that this bidding strategy ensures that the player will not regret winning any better slot, not just slot $l-1$. In other words, this bidding strategy forms an ex-post Nash equilibrium. A simple way to observe that these strategies are indeed an ex-post equilibrium is to note that they lead to the VCG outcome, which is well-known to be incentive compatible.

With budgets, however, the picture changes and this auction no longer admits an ex-post equilibrium. The main difficulty arises from the fact that a player that prefers slot l over slot $l-1$ may still prefer slots that are better than $l-1$. To demonstrate this, consider the following example, with three players and two slots, and parameters $\alpha_1 = 1.1, \alpha_2 = 1, v_1 = 20, b_1 = 7.5, v_2 = 10, b_2 = 7.6, v_3 = 7, b_3 = 100$. With the generalized English auction, when the price reaches 7, player 3 faces a dilemma: if she will not drop, she might end up winning slot 2 for a price higher than her value for that slot (if players 1 and 2 will have infinite budgets, a piece of information she does have at the time of the decision). If she drops, she will realize in retrospect that she could have won slot 1 for a profitable price of 7.6 (while her value for slot 1 is 7.7), since players 1 and 2 turn out to be limited by their budgets, and hence cannot continue to compete with player 3 on slot 1 after the price reaches 7.6. Thus, the introduction of budgets enables the possibility that players who drop when the current slot is l might want to join again for some slot $l' < l$. Of-course, simply allowing players to re-join will cause more problems, since this implies changing the entire price hierarchy that was formed.

3.2 The Generalized Position Auction

In order to solve these difficulties, the Generalized Position Auction uses k price trajectories, one for each slot, that ascend concurrently as follows. Players first compete for the k'th slot, and each player decides when to suspend her participation in this slot's auction. The price ascent temporarily stops when exactly k players remain active. Let this price point be Q_k^1. The price ascent for slot $k-1$ starts from Q_k^1, all players (even those that suspended participation at the previous slot) may participate in the auction for slot $k-1$. Players again decide when to temporarily suspend participation, and when exactly $k-1$ players remain active the price ascent temporarily stops, and we move to slot $k-2$. This continues in a similar manner until we reach slot 1. In slot 1, the price ascent stops when exactly one player remains active. This player wins slot 1 and pays the last price that was reached (as in the English auction). At this point the auction of slot k resumes. There are now $k-1$ slots left, and so the auction continues until there remain $k-1$ active players, at this point the price ascent stops again, and the auction for slot $k-1$ resumes. This continues until the winner of slot 2 is determined. The auction of slot k is once again resumed, and this process continues in a similar manner until all slots are sold. As before, one is able to describe this process more formally via the following discrete-time mechanism.

Definition 2 (The Generalized Position Auction (indirect version))**.** *Initialize $t = 1$ (first round), $l = k$ (current slot is k), and $N_t = N$ (set of active players). Then perform:*

1. *Each player $i \in N_t$ declares a bid $p^t_{i,l}$. (this is the price at which player i will suspend participation at the auction for slot l at the current iteration t).*
2. *Let Q^t_l be the $(l+1)-(t-1)$ highest bid. (this is the price of slot l at round t – the point where the price ascent stops since the number of active players is equal to the number of remaining slots).*
3. *If $l > t$ then decrease l by 1 and repeat from step 1. Otherwise $l = t$ and,*
   - *The player i with the highest bid $p^t_{i,t}$ wins slot t and pays $P_t = Q^t_t$. (section 3.3 below describes the allowable tie-breaking rules).*
   - *If $t = k$ then terminate. Otherwise increase t by one, update N_t by removing the new winner, set $l = k$, and repeat from step 1.*

Suppose that player i had a bid $p^t_{i,l+1} > Q^t_{l+1}$ for slot $l+1$ ($t < l+1 \leq k$), and is now required to choose her bid $p^t_{i,l}$ for slot l. If she were to assume that the alternative for her is to win slot $l+1$ for a price Q^t_{l+1} then her maximal willingness to pay for slot l, as explained in subsection 3.1, is $P = (\alpha_l - \alpha_{l+1})v_i + Q^t_{l+1}$. Since she cannot exceed her budget b_i, this myopic reasoning will therefore direct her to bid $\min(b_i, (\alpha_l - \alpha_{l+1})v_i + Q^t_{l+1})$. If player i had a bid $p^t_{i,l+1} \leq Q^t_{l+1}$ for slot $l+1$ she could simply increase her willingness to pay for slot l by the added value of slot l (compared to slot $l+1$), i.e. by $(\alpha_l - \alpha_{l+1})v_i$. This leads us to define:

Definition 3 (Myopic bidding in the Generalized Position Auction)**.** *The "myopic bidding strategy" is defined by:*

$$p^t_{i,l} = \begin{cases} \min(b_i, (\alpha_l - \alpha_{l+1})v_i + \min(Q^t_{l+1}, p^t_{i,l+1})) & l < k \\ \min(b_i, \alpha_k v_i) & l = k \end{cases}$$

for any round t and any slot $l \geq t$.

Consider again the example given in section 3.1, with three players and two slots, and parameters $\alpha_1 = 1.1, \alpha_2 = 1, v_1 = 20, b_1 = 7.5, v_2 = 10, b_2 = 7.6, v_3 = 7, b_3 = 100$. In the Generalized Position Auction, when players are bidding myopically, the bids are as follows. In the first round, the slot-2-bids are $p^1_{1,2} = 7.5, p^1_{2,2} = 7.6, p^1_{3,2} = 7$. Therefore we get a cutoff price $Q^1_2 = 7$, and the slot-1-bids are $p^1_{1,1} = 7.5, p^1_{2,1} = 7.6, p^1_{3,1} = 7.7$. Hence player 3 wins slot 1 and pays 7.6. Players 1 and 2 continue to the second round, and the slot-2-bids remain as before $p^2_{1,2} = 7.5, p^2_{2,2} = 7.6$. Thus player 2 wins slot 2 and pays 7.5. One can easily verify that this is a valid outcome that is Pareto-efficient and envy-free. (recall that a player can pay only strictly less than her budget).

However, myopic bidding is not an ex-post equilibrium. For example, consider again the example from the previous paragraph. If players 1 and 2 bid myopically then player 3 can decrease her price for slot 1 by bidding $p^1_{3,2} = 0$ and $p^1_{3,1} = 7.7$. The cutoff price for slot 2 will then be zero, and because of that the bids of players 1 and 2 for slot 1 will decrease to be 2 and 1, respectively. This problem is solved by forcing consistency. We first describe a direct version of the above auction for the purpose of the analysis, and then explain how consistency verification solves the problem of the indirect auction.

Definition 4 (The Generalized Position Auction (direct version))**.**

1. *Each player i reports a type (v_i, b_i). If two players report the same budget then the auction is canceled (slots are not allocated and no price is charged).*

2. *We simulate the indirect version of the Generalized Position Auction where each player i follows myopic bidding according to her declared type (v_i, b_i).*

The main results of this paper are summarized by the following theorem:

Theorem 1. *Assuming that all true budgets are distinct,*

1. *(ex-post equilibrium) For every player i, if all other players are truthful then it is a best response for player i to be truthful as well.*

2. *(desired properties hold) If all players are truthful then the Generalized Position Auction results in a valid outcome which is Pareto-efficient and envy-free.*

3. *(uniqueness) Fix any other mechanism that always results (in ex-post equilibrium) in a valid outcome which is Pareto-efficient and envy-free, and that never makes positive transfers. Then this mechanism must output the same outcome (slot assignments and payments) as our Generalized Position Auction for any tuple of types with distinct values and distinct budgets.*

We note that the third result also implies that truthfulness is the *unique* ex-post equilibrium of the Generalized Position Auction. It is also interesting to note that when all budgets are sufficiently large the outcome of our auction is the same as the outcome of the generalized English auction, which in turn is equivalent to the outcome of VCG. Our theorem does not obey the usual rule of thumb that direct mechanisms exhibit dominant strategies, and the solution concept of ex-post equilibrium is used for indirect mechanisms. This is not the case here, although the above mechanism is direct, because of our modeling of the budget constraint. Declaring the true budget is not a dominant strategy for player i since if another player j misreports and declares b_i instead of b_j, the auction will be canceled if player i reports truthfully. As remarked in section 2, this artifact of our definitions can be avoided if we assume a discrete type space. In that case the auction will never be canceled, and truthfulness will become a dominant-strategy of the direct mechanism.

Returning to the indirect mechanism, one can make myopic bidding an ex-post equilibrium by forcing consistency[10], i.e. by verifying that the bidding behavior of each bidder is consistent with myopic bidding according to some possible type. Since myopic bidding is straight-forward, this could be easily done "on the fly", as the auction progress. An inconsistent bidder is disqualified. Such a consistency check maintains the advantages of indirect auctions, mainly that the winner of the highest slot does not reveal her type, and other bidders implicitly reveal their types only when competition forces them to do so. A standard result shows:

Corollary 3.2.1. *Myopic bidding according to one's true type is an ex-post equilibrium of the generalized position auction (indirect version with consistency check).*

Proof. Fix a player i and suppose all other players are bidding myopically according to their true types t_{-i}. Let u_i be i's resulting utility from bidding myopically according to her true type. Assume by contradiction that there exists a different strategy that results in utility $\tilde{u}_i > u_i$. That strategy must be consistent with some type $\tilde{t}_i$ since otherwise the player is disqualified, with utility zero, and since the mechanism is individually rational we have $u_i \geq 0$. But when i is consistent the result of the indirect auction is identical to the result of the direct auction with declaration $(\tilde{t}_i, t_{-i})$, and since the direct auction is truthful we have $\tilde{u}_i \leq u_i$, a contradiction. □

It is interesting to note that, while indeed most of the auctions being conducted in real settings are indirect, the electronic position auctions of Google and Yahoo! are actually direct mechanisms, where the advertisers are required to simply bid a value and a budget.

3.3 Tie-breaking

The issue of tie-breaking requires some attention. In general, when there are several highest bids in step 3 of the generalized position auction, either all highest bidders have the same value, or at most one of them has a higher value, but her bid is cut at her budget. For example, suppose one item and two players that declare $(v_1, b_1) = (7, 10)$ and $(v_2, b_2) = (8, 7)$. Then at price 7 there will be a tie. Since player 2 cannot pay her budget, we must choose player 1 as the winner.

[10] It is not clear whether the indirect mechanism of Aggarwal et al. (2009) requires a similar consistency check, as it is described only in its direct version. The mechanism of Hatfield and Milgrom (2005) is also direct.

More formally, we prove the following intuitive property: if player i has larger value than player j, but her bid at some slot is smaller than j's bid, then it must be the case that i's bid was cut at her budget. This property will be extensively used throughout the analysis.

Claim 3.3.1. *Fix any round t, slot l, and $i, j \in N_t$. If $v_i \geq v_j$ and $p^t_{i,l} \leq p^t_{j,l}$, with at least one strict inequality, then $p^t_{i,l} = b_i$.*

Proof. We prove the claim by induction. For slot k the proof is immediate from the definition. Therefore we assume correctness for slot $l+1$ and prove for slot l. Assume by contradiction that $p^t_{i,l} \neq b_i$. If $p^t_{i,l+1} \geq Q^t_{l+1}$ then $p^t_{i,l} = (\alpha_l - \alpha_{l+1})v_i + Q^t_{l+1}) \geq (\alpha_l - \alpha_{l+1})v_j + Q^t_{l+1}) \geq p^t_{j,l}$, which is a contradiction since by assumption either $v_i > v_j$ or $p^t_{i,l} < p^t_{j,l}$. Otherwise $p^t_{i,l+1} < Q^t_{l+1}$. If $p^t_{i,l+1} = b_i$ we get a contradiction since $b_i \geq p^t_{i,l} \geq p^t_{i,l+1} = b_i$. Therefore by the induction assumption we must have $p^t_{j,l+1} \leq p^t_{i,l+1}$, and this inequality is strict if $v_i = v_j$. Thus $p^t_{i,l} = p^t_{i,l+1} + (\alpha_l - \alpha_{l+1})v_i > p^t_{j,l+1} + (\alpha_l - \alpha_{l+1})v_j \geq p^t_{j,l}$, a contradiction. □

Corollary 3.3.1. *Fix any round t, slot l, and $i, j \in N_t$. If $p^t_{i,l} = p^t_{j,l}$ then either $p^t_{i,l} = b_i$, or $p^t_{j,l} = b_j$, or $v_i = v_j$.*

Thus, if there exist two or more highest bidders in step 3 of the generalized position auction, we choose the winner to be some highest bidder i such that $b_i \neq Q^t_t$. Note that there exists at most one highest bidder with $b_i = Q^t_t$ since budgets are distinct. The tie-breaking among all players with equal value may be arbitrary, but consistent throughout the auction. We denote the tie-breaking order over the players by $\succ$, i.e. for two players i, j, $i \succ j$ implies that in case of a tie i will be chosen. This tie-breaking rule ensures that a player will always pay strictly less than her budget, and thus the outcome is ex-post individually rational.

4 Analysis

We use few additional terms and notations throughout the analysis: B^t_j denotes the set of $j - t + 1$ highest bidders at slot j and iteration t. Ties for inclusion in B^t_j are settled the same way as described above, and in particular for any $i \in B^t_j$ we have $Q^t_j < b_i$. A player $i \in B^t_l$ is "strong" at slot l and iteration t, otherwise the player is "weak". We call P_t the "price of slot t". We say that slot l is *better* than slot $\tilde{l}$ if $l < \tilde{l}$ (and slot $\tilde{l}$ is *worse* than slot l). We sometimes use

$$q^t_{i,l} = \min(p^t_{i,l}, Q^t_l).$$

This gives $p^t_{i,l} = \min(b_i, q^t_{i,l+1} + (\alpha_l - \alpha_{l+1})v_i)$ for every player, slot, and round, which will simplify notation. Note that $p^t_{i,l} \geq p^t_{j,l}$ implies $q^t_{i,l} \geq q^t_{j,l}$.

One important observation that follows in a straight-forward way from the definition of the mechanism is that the outcome of round t depends only on the set of remaining players N_t, because the bids $p^t_{i,k}$ are fixed and identical in all rounds t. Thus a new round is simply a recursive call to the same auction, with a new set of players and a new set of slots.

Several monotonicity properties of the bids, for any round t, any player $i \in N_t$, and any slot l, will turn out useful:

1. $Q^t_l \geq Q^t_{l+1}$: we have $p^t_{i,l} \geq q^t_{i,l+1}$, and for every $i \in B^t_{l+1}$ we have $q^t_{i,l+1} = Q^t_{l+1}$. Thus for at least $|B^t_{l+1}|$ players i we have $p^t_{i,l} \geq Q^t_{l+1}$. Since Q^t_l is the $|B^t_{l+1}| - 1$ highest bid for slot l in round t the claim follows.

2. $q^t_{i,l} \geq q^t_{i,l+1}$: we have $p^t_{i,l} \geq q^t_{i,l+1}$, thus if $p^t_{i,l} = q^t_{i,l}$ we are done, and otherwise $q^t_{i,l} = Q^t_l \geq Q^t_{l+1} \geq q^t_{i,l+1}$.

3. $p^{t+1}_{i,l} \geq p^t_{i,l}$ and therefore also $q^{t+1}_{i,l} \geq q^t_{i,l}$. This follows by induction on slot $l = k, ..., 1$. For slot k the claim is by definition, this now implies the claim for slot $k-1$, and so on. This fact also implies that $Q^{t+1}_l \geq Q^t_l$.

4. If $i \notin B^1_l$ and $p^t_{i,l} = q^t_{i,l} = b_i$, then player i will not win any slot $s \leq l$. (this follows from the previous two properties).

4.1 Envy-Freeness

The first property we prove is envy-freeness. For notational simplicity, throughout the subsection we rename the players such that player i wins slot i, for $i = 1, ..., k$, and every player $i > k$ does not win any slot.

We prove envy-freeness in steps, building intuition by using the case of two slots to demonstrate key ideas.

Two slots: player 1 does not envy player 2. We start by showing that, with two slots, player 1 (who wins the highest slot, slot 1) does not envy player 2 (who wins the lower slot). We have $\min(b_1, (\alpha_1 - \alpha_2)v_1 + q^1_{1,2}) = p^1_{1,1} \geq P_1$, and $P_1 < b_1$ by the tie-breaking rule. This implies $(\alpha_1 - \alpha_2)v_1 + q^1_{1,2}) \geq P_1$. Rearranging, we get $\alpha_1 v_1 - P_1 \geq \alpha_2 v_1 - q^1_{1,2} \geq \alpha_2 v_1 - P_2$, where the second inequality follows since $q^1_{1,2} \leq Q^1_2 \leq Q^2_2 = P_2$.

The general case: player s does not envy player $l > s$. With more than two slots, we need a very similar argument to show that player some player s does not envy a player $l > s$ (that received a slot worse than s). The only complication is the fact that the two slots s, l might not be adjacent as before, and a simple inductive argument is being used to overcome the difficulty.

Claim 4.1.1. *Fix any player i and any two slots l, s with $s < l \leq k+1$. Then $\min(b_i, q^t_{i,l} + (\alpha_s - \alpha_l)v_i) \geq p^t_{i,s} \geq q^t_{i,s}$ (where we define $q^t_{i,k+1} = \alpha_{k+1} = 0$). Furthermore, if $i \notin B^t_j$ for any $s \leq j < l$ then the two inequalities become equalities.*

Proof. We prove by induction on $s = k, ..., 1$. For $s = k$ the claim is by definition. Now fix $s < k$ and assume correctness for $s + 1$ and any $l' > s + 1$. We need to show correctness for s and any $l > s$. We have by definition $q^t_{i,s} \leq p^t_{i,s} = \min(b_i, (\alpha_s - \alpha_{s+1})v_i + q^t_{i,s+1})$. If $l = s + 1$ we are done. Otherwise $l > s+1$ and we have by induction $q^t_{i,s+1}) \leq \min(b_i, q^t_{i,l} + (\alpha_{s+1} - \alpha_l)v_i)$. Combining the two equations, the first part of the claim follows. If $i \notin B^t_j$ for any $s \leq j < l$ then the first inequality is equality by definition, and the second inequality is equality by the induction assumption. Thus the second part of the claim follows as well. □

Now, exactly as in the two-slots case, from the above claim we get $\min(b_s, (\alpha_s - \alpha_l)v_s + q^s_{s,l}) \geq q^s_{s,s} \geq P_s$, and $P_s < b_s$ by the tie-breaking rule. This implies $(\alpha_s - \alpha_l)v_s + q^s_{s,l}) \geq P_s$. Rearranging, we get $\alpha_s v_s - P_s \geq \alpha_l v_s - q^s_{s,l} \geq \alpha_l v_s - P_l$, where the second inequality follows since $q^s_{s,l} \leq Q^s_l \leq Q^l_l = P_l$.

Therefore we have shown that a player does not envy any other player that receives a worse slot. We now continue to show the other direction, that a player does not envy any other player that receives a better slot, which is a bit more complicated. As above, we start with the case of two slots.

Two slots: player 2 does not envy player 1. If we had $q^1_{2,2} = q^2_{2,2}$ then we could use an argument similar to above to show that player 2 does not envy player 1: $P_1 = Q^1_1 \geq p^1_{2,1} = \min(b_2, (\alpha_1 - \alpha_2)v_2 + q^1_{2,2})$. Thus, if $P_1 < b_2$ (and assuming $q^1_{2,2} = q^2_{2,2} = Q^2_2 = P_2$) then $P_1 \geq (\alpha_1 - \alpha_2)v_2 + q^1_{2,2} = (\alpha_1 - \alpha_2)v_2 + P_2$, which, by rearranging, gives us $\alpha_2 v_2 - P_2 \geq \alpha_1 v_2 - P_1$ as we need.

However it may well be that $q^1_{2,2} < q^2_{2,2}$, as is the case in the running example of section 3, where player 2 wins slot 2, and $q^2_{2,2} = Q^2_2 = 7.5 > 7 = Q^1_2 \geq q^1_{2,1}$. Also notice that in this example $P_1 = b_2$ and therefore player 2 does not envy the winner of the first slot (who is player 3 in the example). It turns out that this is in fact what happens in general: either $q^1_{2,2} = q^2_{2,2}$, or $P_1 \geq b_2$. More specifically, If $q^1_{1,2} \geq q^1_{2,2}$ (where player 1 is assumed to be the winner of slot 1) then the former case is true, as claim 4.1.2 below shows, and if $q^1_{1,2} < q^1_{2,2}$ then the latter case is true, as claim 4.1.3 shows. We phrase and prove the claims in general terms, as they will turn out useful in the sequel as well.

Claim 4.1.2. *Let player i be the winner of slot t, and fix some slot $l > t$ and some player $j \in N_{t+1}$ such that $p^t_{j,l} \leq p^t_{i,l}$ (and therefore also $q^t_{j,l} \leq q^t_{i,l}$). Then $p^t_{j,l} = p^{t+1}_{j,l}$ and $q^t_{j,l} = q^{t+1}_{j,l}$.*

Proof. We first prove $p^t_{j,l} = p^{t+1}_{j,l}$ by induction on the slots. For $l = k$ the proof is immediate since $p^t_{j,k} = p^{t+1}_{j,k}$ for any player j. We assume correctness for slot $l + 1$ and prove for l. If $i \in B^t_{l+1}$ then by induction $p^t_{j,l+1} = p^{t+1}_{j,l+1}$ for any player $j \notin B^t_{l+1}$, which implies $B^{t+1}_{l+1} = B^t_{l+1} \setminus \{i\}$, hence also $Q^{t+1}_{l+1} = Q^t_{l+1}$. This implies $p^t_{j,l} = p^{t+1}_{j,l}$ for all players in N_{t+1}. Otherwise assume that $i \notin B^t_{l+1}$. This implies $p^t_{j,l} = \min(b_j, (\alpha_l - \alpha_{l+1})v_j + p^t_{j,l+1})$. If $p^t_{j,l+1} \leq p^t_{i,l+1}$ then by induction $p^t_{j,l+1} = p^{t+1}_{j,l+1}$, hence $p^t_{j,l} = \min(b_i, (\alpha_l - \alpha_{l+1})v_j + p^t_{j,l+1}) = \min(b_i, (\alpha_l - \alpha_{l+1})v_j + p^{t+1}_{j,l+1}) \geq p^{t+1}_{j,l} \geq p^t_{j,l}$, and the claim follows. Otherwise suppose $p^t_{j,l+1} > p^t_{i,l+1}$. This implies $v_i < v_j$: if $v_i \geq v_j$, claim 3.3.1 implies that $p^t_{i,l+1} = b_i$, which implies that player i cannot win slot t, a contradiction. Thus $v_i < v_j$, and claim 3.3.1 implies $p^t_{j,l} = b_j$ (by reversing the roles of i, j in claim 3.3.1, since we now have $v_i < v_j$ and $p^t_{j,l} \leq p^t_{i,l}$). Thus $p^t_{j,l} = b_j \geq p^{t+1}_{j,l} \geq p^t_{j,l}$, implying the first part of the claim.

We now prove that $q^t_{j,l} = q^{t+1}_{j,l}$. If $p^t_{i,l} \geq Q^t_l$ then the above paragraph implies $Q^{t+1}_l = Q^t_l$ and hence $q^t_{j,l} = q^{t+1}_{j,l}$. If $p^t_{i,l} < Q^t_l$ then $p^t_{j,l} = q^t_{j,l}$ and $p^t_{i,l} = q^t_{i,l}$. The above paragraph implies $q^t_{j,l} = p^t_{j,l} = p^{t+1}_{j,l} \geq q^{t+1}_{j,l} \geq q^t_{j,l}$, and the claim follows. □

In words, the claim states that, if player i wins slot t in round t, then every player j that bids lower than i in some slot $l > t$ in round t (i.e. $p^t_{j,l} \leq p^t_{i,l}$) will have the same bid $p^t_{j,l} = p^{t+1}_{j,l}$ for slot l in the next iteration $t + 1$. As an immediate implication we get that, if the winner i of slot t is strong in slot $l > t$ and round t then $p^t_{j,l} = p^{t+1}_{j,l}$ for any player $j \in N_{t+1}$, and therefore also $Q^t_l = Q^{t+1}_l$. Alternatively put, if $p^t_{i,l} \geq Q^t_l$ then $Q^t_l = Q^{t+1}_l$.

In particular, for the two-slots case, If $q^1_{1,2} \geq q^1_{2,2}$ then $q^1_{2,2} = q^2_{2,2}$ and player 2 does not envy player 1. To complete the argument we need to show that if $q^1_{1,2} < q^1_{2,2}$ then $P_1 \geq b_2$. Towards this, we first show:

Claim 4.1.3. *Fix any player $s > 1$ such that $q^1_{s,1} < b_s$. Then for any slot l, $q^1_{1,l} \geq q^1_{s,l}$, and if $s \in B^1_l$ then $1 \in B^1_l$.*

Proof. We first show that if $p^1_{1,l} < p^1_{s,l}$ then $1 \in B^1_l$. Otherwise it must follow that $p^1_{1,l} = q^1_{1,l} \neq b_1$, and claim 3.3.1 implies $v_1 < v_s$. Since $p^1_{1,1} \geq p^1_{s,1}$ it follows that $p^1_{s,1} = b_s$, a contradiction. This implies $q^1_{1,l} \geq q^1_{s,l}$. This also implies that if $s \in B^1_l$ but $1 \notin B^1_l$ then $p^1_{1,l} = p^1_{s,l} = Q^1_l$. Since $p^1_{1,l} \neq b_1$ (as player 1 wins slot 1) and $p^1_{s,l} \neq b_s$ (as $q^1_{s,l} = q^1_{s,1} < b_s$) then $v_1 = v_s$, and by corollary 3.3.1 also $p^1_{1,1} = p^1_{s,1}$. But this contradicts the consistency of the tie-breaking rule, since in slot l the tie-breaking preferred player 1 over player s and in slot 1 the tie-breaking preferred player s over player 1. □

This completes the case of two slots: if $q^1_{1,2} < q^1_{2,2}$ then $q^1_{2,1} = b_2$ (since by claim 4.1.3, $q^1_{2,1} < b_2$ implies $q^1_{1,2} \geq q^1_{2,2}$). Since $q^1_{1,1} \geq q^1_{2,1}$ then $P_1 = Q^1_1 \geq q^1_{2,1} = b_2$, and the two-slots case follows.

The general case: player s does not envy player $l < s$. The proof for the two-slots case relied on the claim that if $q^s_{s,1} < b_2$ then $q^s_{s,s} = q^1_{s,s}$, taking $s = 2$. The general argument relies on the same claim, and to prove it we first need two claims:

Claim 4.1.4. *Fix any player $s > 1$ such that $q^1_{s,1} < b_s$, and any slot $l < s$. If $s \in B^1_l$ then $l' \in B^1_l$ for any $l' \leq l$.*

Proof. We prove by induction on the number of slots k. For $k = 1$ the claim is empty. Assume correctness for any $k' < k$ slots and let us prove for k. We have $1 \in B^1_l$ by claim 4.1.3. Therefore $B^2_l = B^1_l \setminus \{1\}$. Claim 4.1.3 also implies $q^1_{1,2} \geq q^1_{s,2}$ which by using claim 4.1.2 implies $q^2_{s,2} = q^1_{s,2} \leq q^1_{s,1} < b_s$. Since $s \in B^2_l$ the induction assumption implies $l' \in B^2_l$ for any $1 < l' \leq l$, and the claim follows. □

Claim 4.1.5. *Fix any player $s > 1$ such that $q^1_{s,1} < b_s$. Then $B^1_s = \{1, ..., s\}$, and $s \notin B^1_l$ for any $l < s$. This gives two corollaries:*

1. *If $s > k$ then player s is always weak.*

2. *$P_s = Q^1_s = q^1_{s,s}$.*

Proof. If $s \in B^1_l$ for some $l < s$ then by claim 4.1.4 we have that $\{1, ..., l, s\} \subseteq B^1_l$ which contradicts the fact that $|B^1_l| = l$. If $s \notin B^1_s$ then by combining claims 4.1.3 and 4.1.2 we have $p^1_{s,s} = p^2_{s,s} = \cdots = p^s_{s,s}$, which implies that $s \notin B^s_s$, a contradiction. Thus using claim 4.1.4 again we have $B^1_s = \{1, ..., s\}$. The first corollary is immediate from the claim, and the second corollary follows by claim 4.1.2. □

By using this last claim we could prove envy-freeness in the same way that was used for the two-slots case. However for the sequel we wish to extract one additional interesting property of the resulting prices.

Claim 4.1.6. *For any slot $l = 1, ..., k$, $P_l = \max_{s>l} \min(b_s, (\alpha_l - \alpha_s)v_s + P_s)$, where we define $\alpha_s = 0$ and $P_s = 0$ for any player $s > k$.*

Proof. It is enough to prove the claim only for $l = 1$, since the price P_l for $l > 1$ is determined by a recursive auction for l slots and a set of players N_l, and in that auction slot l is the first slot. We will show that, for any player $s > 1$, $q^1_{s,1} = \min(b_s, (\alpha_1 - \alpha_s)v_s + P_s)$. Since $P_1 = \max_{s>1} q^1_{s,1}$, the claim will then immediately follow. If $q^1_{s,1} = b_s$ then by claim 4.1.1 we have $b_s = q^1_{s,1} \leq p^1_{s,1} \leq \min(b_s, q^1_{s,s} + (\alpha_1 - \alpha_s)v_s) \leq \min(b_s, (\alpha_l - \alpha_s)v_s + P_s) \leq b_s$, implying the claim. Otherwise $q^1_{s,1} < b_s$ and by claim 4.1.5 we have that $s \notin B^1_l$ for any $l < s$. By claims 4.1.1 and 4.1.5 we get $q^1_{s,1} = \min(b_s, q^1_{s,s} + (\alpha_1 - \alpha_s)v_s) = \min(b_s, P_s + (\alpha_1 - \alpha_s)v_s)$. □

Lemma 1. *The outcome of the Generalized Position Auction (direct version) with truthful bidding is envy-free. Furthermore, if $s < l$ and $p^s_{s,s} > P_s$ then one direction of envy-freeness holds with a strict inequality: $\alpha_s v_s - P_s > \alpha_l v_s - P_l$.*

Proof. Consider any two players s, l. We will show that s does not envy l. If $s < l$ then the only non-trivial possibility is $l \leq k$. In this case $(\alpha_s - \alpha_l)v_s + Q^s_l \geq (\alpha_s - \alpha_l)v_s + q^s_{s,l} \geq p^s_{s,s} \geq P_s$, where the second inequality follows by claim 4.1.1. This implies $\alpha_s v_s - P_s \geq \alpha_l v_s - Q^s_l \geq \alpha_l v_s - P_l$, and if $p^s_{s,s} > P_s$ then the first inequality is strict. If $s > l$ then, by claim 4.1.6, $P_l \geq \min(b_s, (\alpha_l - \alpha_s)v_s + P_s)$. Thus, if $P_l < b_s$ then $P_l \geq (\alpha_l - \alpha_s)v_s + P_s$ which again implies $\alpha_s v_s - P_s \geq \alpha_l v_s - P_l$. □

This finishes the proof of envy-freeness. For the proof of uniqueness in section 4.3 below it will be useful to state one more easy implication of the above:

Claim 4.1.7. *Fix any slot s, and suppose that there exists a player $j > s$ such that $P_s = (\alpha_s - \alpha_j)v_j + P_j < b_j$, and $v_j \neq v_s$. Then for any slot $l > s$ we have $\alpha_s v_s - P_s > \alpha_l v_s - P_l$.*

Proof. We show that $p^s_{s,s} > P_s$ which implies the claim by Lemma 1. By the proof of claim 4.1.6 we have $p^s_{j,s} = q^s_{j,s} = \min((\alpha_s - \alpha_j)v_j + P_j, b_j) = P_s$. Therefore if $p^s_{s,s} = P_s$ then $p^s_{s,s} = p^s_{j,s}$. Since $p^s_{s,s} \neq b_s$ and $p^s_{j,s} \neq b_j$ we get by corollary 3.3.1 that $v_s = v_j$, a contradiction. Thus $p^s_{s,s} > P_s$, and the claim follows. □

4.2 Incentive Compatibility

We prove incentive-compatibility by first identifying some basic properties that the auction exhibits when one player changes her bid while all other players' bids are fixed.

Claim 4.2.1. *Fix a player, i, and arbitrary declarations of the other players. Consider two declarations of player i, (v, b) and $(\tilde{v}, \tilde{b})$ and suppose player i wins slot l and pays P_l when declaring (v, b), and wins slot $\tilde{l}$ and pays $\tilde{P}_{\tilde{l}}$ when declaring $(\tilde{v}, \tilde{b})$. (l and/or $\tilde{l}$ can take the value $k+1$ to denote that player i loses). Then,*

1. *If $\tilde{v} \geq v$ and either $b = \tilde{b}$ or $b > \min(P_{\tilde{l}}, \tilde{P}_{\tilde{l}})$ then:*
   (a) *$\tilde{l} \leq l$.*
   (b) *For any slot $s \geq l$, $\tilde{P}_s = P_s$.*
   (c) *$\tilde{P}_{\tilde{l}} \geq P_{\tilde{l}}$.*
2. *If $v = \tilde{v}$ and $\tilde{b} > b > \tilde{P}_{\tilde{l}}$ then $\tilde{l} = l$.*
3. *If $v = \tilde{v}$ and $\tilde{b} > b$ then $\tilde{l} \leq l$.*

While these properties are rather intuitive, the proof is technical, and is deferred to appendix A. Despite the fact that all properties are intuitive, they may be misleading, and the qualifiers and requirements detailed in the properties are really necessary (this also explains why the proof gets technical). For example, property 1b might appear true even without the requirement that $b = \tilde{b}$ or $b > \min(P_{\tilde{l}}, \tilde{P}_{\tilde{l}})$. Therefore it is interesting to see a counter example to this property when these requirements does not hold: Consider a setting of two slots with $\alpha_1 = 1000$ and $\alpha_2 = 1$, and three players with types $\theta_1 = (1, 1000)$, $\theta_2 = (10, 10)$ and $\theta_3 = (11, 11)$ (recall that the first number is the value and the second number is the budget). Suppose player 3 changes her type to $\tilde{\theta}_3 = (11, 1001)$. Quite surprisingly, the price of slot 2 then strictly decreases.

We next bootstrap these properties to show full incentive compatibility. Throughout, we fix the true type of player i to be (v_i, b_i), and denote by $u_i(v, b)$ player i's utility when declaring some type (v, b) (the declaration of all other players is fixed throughout). We need to show that $u_i(v_i, b_i) \geq u_i(v, b)$, for any other type (v, b). Since we already established that player i does not envy a losing player, we have $u_i(v_i, b_i) \geq 0$. Thus we consider only types (v, b) such that $u_i(v, b) > 0$ (otherwise $u_i(v_i, b_i) \geq 0 \geq u_i(v, b)$). We show separately for each coordinate that reporting the true value in that coordinate weakly increases the player's utility, and then aggregate.

Claim 4.2.2. *For any $b > b_i$ and any v, $u_i(v, b) \le u_i(v, b_i)$.*

Proof. Suppose player i wins slot s and pays P_s when declaring (v, b). Since $u_i(v, b) > 0$ we have $b_i > P_s$ and by property 2, when declaring (v, b_i) player i still wins slot s and still pays P_s. Therefore $u_i(v, b) \le u_i(v, b_i)$ and the claim follows. □

Claim 4.2.3. *For any $b \le b_i$ and any v, $u_i(v, b) \le u_i(v_i, b)$.*

Proof. Suppose player i wins slot s and pays P_s when declaring (v_i, b) and wins slot $\tilde{s}$ and pays $\tilde{P}_{\tilde{s}}$ when declaring (v, b) (s and/or $\tilde{s}$ can take the value $k + 1$ to denote that i loses). Since $b \le b_i$, i's payment is at most her budget, and so she has a non-negative utility from both declarations. By envy-freeness, $\alpha_s v_i - P_s \ge \alpha_{\tilde{s}} v_i - P_{\tilde{s}}$, where $P_{\tilde{s}}$ denotes the price of slot $\tilde{s}$ when player i declares (v_i, b). If $v > v_i$ then $\tilde{s} \le s$ by property 1a of claim 4.2.1 and then $\tilde{P}_{\tilde{s}} \ge P_{\tilde{s}}$ by property 1c. If $v < v_i$ then $\tilde{s} \ge s$ by property 1a and then $\tilde{P}_{\tilde{s}} = P_{\tilde{s}}$ by property 1b. In any case, we have $\alpha_{\tilde{s}} v_i - P_{\tilde{s}} \ge \alpha_{\tilde{s}} v_i - \tilde{P}_{\tilde{s}}$. We get $u_i(v_i, b) = \alpha_s v_i - P_s \ge \alpha_{\tilde{s}} v_i - \tilde{P}_{\tilde{s}} = u_i(v, b)$, as claimed. □

Claim 4.2.4. *For any $b \le b_i$, $u_i(v_i, b) \le u_i(v_i, b_i)$.*

Proof. Let $f(v, b)$ denote the slot assigned to player i when declaring (v, b), and $P(v, b)$ be i's payment when declaring (v, b). Define $g(v, b) = \alpha_{f(v,b)} \cdot v - P(v, b)$, i.e. this is i's utility if she declares (v, b) and if her true value is indeed v. We will argue that $g(v, b) = \int_0^v \alpha_{f(x,b)} dx$. For $v' > v$ we have by property 1a that $f(v', b) \le f(v, b)$. In addition, if $f(v', b) = f(v, b)$ then $P(v', b) = P(v, b)$ by property 1b. Let $v_1^*, ..., v_L^*$ be the discontinuity points of $f(\cdot, b)$ (i.e. when b is fixed and v increases from 0 to ∞). In other words, for any index $1 \le l \le L - 1$ and any $v_l^* < x_1 < x_2 < v_{l+1}^*$ we have $f(x_1, b) = f(x_2, b)$ and $P(x_1, b) = P(x_2, b)$. Therefore $\frac{\partial g(v,b)}{\partial v}|_{v=x_1} = \frac{\partial g(v,b)}{\partial v}|_{v=x_2} = \alpha_{f(x_1,b)}$. Since there is a finite number $L \le k$ of such discontinuity points we get $g(v, b) = \int_0^v \alpha_{f(x,b)} dx$. By property 3 we have $f(x, b) \ge f(x, b')$ for any $b \le b'$, implying using the above that $g(v, b) \le g(v, b')$. Since $b \le b_i$ we get $u_i(v_i, b) = g(v_i, b) \le g(v_i, b_i) = u_i(v_i, b_i)$, and the claim follows. □

Lemma 2. *Truthfulness is an ex-post equilibrium of the Generalized Position Auction.*

Proof. We need to show that any false declaration (v, b) yields weakly smaller utility than the true declaration (v_i, b_i). If $b > b_i$ we have $u_i(v, b) \le u_i(v, b_i) \le u_i(v_i, b_i)$, where the first inequality follows from claim 4.2.2 and the second inequality follows from claim 4.2.3. If $b \le b_i$ we have $u_i(v, b) \le u_i(v_i, b) \le u_i(v_i, b_i)$, where the first inequality follows from claim 4.2.3 and the second inequality follows from claim 4.2.4. □

4.3 Uniqueness

We finish the analysis by showing that the Generalized Position Auction is the unique mechanism that satisfies all the desirable properties discussed at the beginning. We need one additional natural requirement:

Definition 5 (No Positive Transfers (NPT))**.** *A mechanism has the "No Positive Transfers" (NPT) property if no player receives a positive payment from the mechanism.*

This property is necessary for the uniqueness result. Consider for example a setting of one item and two players, with $b_1 = 1, b_2 = 2$, and $v_1 = 5, v_2 = 3$. The Generalized Position Auction sells the item to player 2 for a price of 1. A different mechanism that violates NPT is: first pay each player a subsidy of 4 dollars (this increases the bidders' budgets). Then run our mechanism using the updated budgets. It is not hard to verify that this is truthful, individually rational, and envy-free. However the result will now be different: player 1 will receive the item and will pay 3 dollars. It is interesting to note that the usual quasi-linear setting does not exhibit such a phenomena, and it is well-known that one can normalize the payment of a losing player to be 0 without affecting the outcomes of the mechanism being considered. As this simple example shows, when budgets limits are a real constraint this is not quite the case.

Together with ex-post IR, NPT implies that the payment of a losing player is exactly zero. This is in fact the only use of the NPT property, and one can replace the NPT requirement with a "zero payment for losers" requirement. This seems like a natural and common property.

A second issue that requires some attention is ruling out ties. Clearly, if the Generalized Position Auction encounters a tie during its execution, it can be decided in several ways, affecting the outcome. Thus, the uniqueness result can only hold when there are no ties, i.e. when all types are distinct w.r.t. both the value and the budget.

Let M denote the Generalized Position Auction, and fix any other truthful mechanism M' that satisfies NPT, envy freeness, Pareto optimality, and ex-post individual rationality.

Lemma 3. *For any tuple of types $(\vec{v}, \vec{b})$ such that $v_i \neq v_j$ and $b_i \neq b_j$, M and M' output the same slot assignment and the same payments. Moreover, this holds even if the values of the players are fixed and are publicly known, and only the budgets are private information.*[11]

Proof. Fix any tuple of types $(\vec{v}, \vec{b})$ in T^*. Define $w(s), w'(s)$ as the winners of slot s in mechanisms M, M', respectively, and let P_l, P'_l be the payment of the winner of slot l in mechanisms M, M', respectively. We start with two claim and then prove the lemma by induction.

Claim 4.3.1. *$P'_s \geq P_s$ for any slot $1 \leq s \leq k$.*

Proof. Let A contain all slots $1 \leq s \leq k$ such that $P_s > P'_s$, and suppose by contradiction that A is not empty. For any $s \in A$, let l be the slot that $i = w(s)$ wins in M' (i.e. $w(s) = w'(l) = i$). We claim that $l \in A$: if $P'_l \geq P_l$ then we get $\alpha_s v_i - P_s \geq \alpha_l v_i - P_l \geq \alpha_l v_i - P'_l \geq \alpha_s v_i - P'_s > \alpha_s v_i - P_s$, where the first inequality follows from envy-freeness of M since $P_l \leq P'_l < b_i$, and the third inequality follows from envy-freeness of M' since $P'_s < P_s < b_i$, and we get a contradiction. Thus, a player

[11] Alternatively, it can be stated that the budgets are common knowledge and the values are private information. For simplicity we restrict attention to just one version.

wins a slot in A in M if and only if she wins wins a slot in A in M'. We will show that there exists at least one player that does not receive a slot in A in M but must win a slot in A in M', and will thus get a contradiction.

Let $s^* = \max(s \in A)$. By claim 4.1.6 let $i = w(l)$ for $l > s^*$ be a player such that $P_{s^*} = \min(b_i, P_l + (\alpha_s - \alpha_l)v_i)$ (we may choose $l = k+1$ to denote the fact that i loses in M). We have $P'_{s^*} < P_{s^*} \le b_i$, and $\alpha_l v_i - P_l < \alpha_{s^*} v_i - P'_{s^*}$. Note that i wins a slot $l \notin A$ in M (since $l > s^*$). We will show that i must win a slot in A in M', which will be a contradiction. For any slot $j \notin A$ (including $j = k+1$ to consider the possibility that i loses in M'), either $P'_j \ge b_i$, or $P_j \le P'_j < b_i$, in which case $\alpha_j v_i - P'_j \le \alpha_j v_i - P_j \le \alpha_l v_i - P_l < \alpha_{s^*} v_i - P'_{s^*}$, where the second inequality follows by the envy-freeness of M since $P_j < b_i$. Since $P'_{s^*} < b_i$ and M' is envy-free it follows that i cannot win slot j in M'. Thus player i must win some slot in A, a contradiction. □

Claim 4.3.2. *Define the set B to contain all slots $1 \le l \le k$ such that $P_l = P'_l$. Then the set of players that win a slot in B is identical in both M and M', i.e. $\{\, w(s) \mid s \in B \,\} = \{\, w'(s) \mid s \in B \,\}$.*

Proof. Assume by contradiction that there exists a player i that wins a slot $s \in B$ in M, and a slot $l \notin B$ in M' (as before we can have $s = k+1$). Note that by claim 4.3.1 and since $l \notin B$ we have $P'_l > P_l$. We get $\alpha_s v_i - P'_s = \alpha_s v_i - P_s \ge \alpha_l v_i - P_l > \alpha_l v_i - P'_l$, where the first inequality follows by envy-freeness of M, since $P_l < P'_l < b_i$. Since $P'_s = P_s < b_i$ this contradicts the envy-freeness of M'. □

Claim 4.3.3. *Let B be as defined in claim 4.3.2. Then for any $s \in B$ we have $w(s) = w'(s)$.*

Proof. Fix a slot $s \in B$. We assume that for any $l \in B$ with $l < s$ we have $w(l) = w'(l)$ and prove $w(s) = w'(s)$, which implies the claim by induction. Let $i = w(s)$. Suppose by contradiction that $w'(s) = j \ne i$. Suppose player j wins slot s_j in M. By claim 4.3.2 we have $s_j \in B$ and by assumption we have $s_j > s$. By claim 4.1.6 we have $P_s \ge \min(b_j, (\alpha_s - \alpha_{s_j})v_j + P_{s_j})$. Since $P'_s = P_s$ and $P'_{s_j} = P_{s_j}$, envy-freeness of M' implies $P_s = (\alpha_s - \alpha_{s_j})v_j + P_{s_j} < b_j$. Claim 4.1.7 then implies that for any slot $l > s$ we have $\alpha_s v_i - P_s > \alpha_l v_i - P_l$. Now suppose player i wins slot s_i in M. By claim 4.3.2 we have $s_i \in B$ and by assumption we have $s_i > s$. Since $P'_s = P_s$ and $P'_{s_i} = P_{s_i}$ we get $\alpha_s v_i - P'_s > \alpha_{s_i} v_i - P_{s_i}$. Since $P'_s = P_s < b_i$ we get a contradiction to the envy-freeness of M'. Thus $w(s) = w'(s)$ and the claim follows. □

We now prove by induction on $l = k, ..., 0$ that, for all type declarations: (1) the set of players that win slots $1, ..., l$ is the same in both mechanisms (they do not necessarily win the same slots), and (2) for any slot $k \ge s > l$, the same player wins slot s in both mechanisms, and $P'_s = P_s$. The lemma will then follow by taking $l = 0$.

To prove the base case of $l = k$ we need to argue that the same set of players lose in both mechanisms: for any slot $s \le k$, if $s \in B$ (as defined in claim 4.3.2 above) then a losing player i in

M cannot win s in M' by claim 4.3.3. If $s \notin B$ then by claim 4.3.1 we have $P'_s > P_s \geq \min(b_i, \alpha_s v_i)$ and since M' is ex-post IR it follows that $w'(s) \neq i$. Hence i must lose in M' as well.

We now assume correctness for some index $l \leq k$ and prove the inductive claim for $l-1$. All we need to show is that $w(l) = w'(l)$, and $P_l = P'_l$. Let $i = w(l)$ be the winner of slot l in mechanism M, and suppose that $i = w'(l')$. Note that $l' \leq l$ by the induction assumption. We first prove that $P_{l'} = P'_{l'}$. By claim 4.3.1 we have $P_{l'} \leq P'_{l'}$, and assume by contradiction that the inequality is strict. Since $b_i > P'_{l'}$ we have by envy-freeness that $\alpha_l v_i - P_l \geq \alpha_{l'} v_i - P_{l'}$. Since $P_{l'} < P'_{l'}$ we can pick a small enough $\epsilon > 0$ such that $\alpha_l v_i - (P_l + \epsilon) > \alpha_{l'} v_i - P'_{l'}$. Now if player i declares a different type $(v_i, P_l + \epsilon)$ (i.e. the same value and a budget just above her price in M) then by property 2 of claim 4.2.1 we have that player i wins slot l in M in the new type declaration as well. By the induction assumption player i wins some slot $l'' \leq l$ in M' in the new declaration, and her new payment P'' is at most her new budget $P_l + \epsilon$. We get $\alpha_{l''} v_i - P'' \geq \alpha_l v_i - (P_l + \epsilon) > \alpha_{l'} v_i - P'_{l'}$. Thus player i strictly increased her utility be misreporting her type, contradicting the truthfulness of M'. Thus $P_{l'} = P'_{l'}$. Therefore $l' \in B$, and by claim 4.3.3 we have $w'(l') = w(l')$. Since $w(l) = w'(l')$ by assumption we get $l' = l$, and the claim follows. □

5 Conclusions

We have designed a generalized position auction, for players with private values and private budget constraints. Our auction is built on top of the generalized English auction, and its ex-post equilibrium outcome is individually rational, Pareto-optimal and envy-free. Moreover, any auction that satisfies these properties, and in addition does not make positive transfers to the players, must yield in ex-post equilibrium the same outcome as our auction, for every tuple of distinct types. This uniqueness result holds even if values are public knowledge and only budgets are private.

While the generalized English auction uses only one price trajectory, our auction must use k different price trajectories, that concurrently ascend. This implies that our auction needs to exchange more messages than the generalized English auction, specifically, we need an order of $n \cdot k^2$ messages while the generalized English auction requires an order of $n \cdot k$ messages. This is an artifact of the introduction of budgets, and the only other mechanism for position auctions with budgets, that is based on the Demange-Gale-Sotomayor ascending auction (and was described in the Introduction), shows a similar increase in the amount of required messages. In fact, Aggarwal et al. (2009) prove that the amount of messages exchanged in their auction is in the order of $n \cdot k^3$, and our auction performs slightly better than that, requiring only an order of $n \cdot k^2$ messages. Thus, from a practical point of view, our auction has a slight advantage. In future research it may be interesting to determine what is the *minimal* possible number of messages needed in order to reach the unique incentive compatible and envy-free outcome.

Our setup here is a one-shot setup, in which the auction runs only once. A more advanced (and realistic) setup would assume a repeated stochastic scenario, in which the same position auction is being conducted several times, where the number of occurrences and their frequency is uncertain. This change of setup complicates the analysis even without budgets, and with the existence of budgets it adds an important dimension that is now missing from our analysis. In particular, in such a setup new strategic issues are being added since a bidder that artificially increases competition in current auction exhausts competitors' budgets and thus affects their future ability to compete. This issue was considered for other auction formats, for example by Benoit and Krishna (2001), but in the context of position auctions this issue is hardly understood. While our setup does not directly add to its understanding, as there is just one single auction being conducted, our analysis is a necessary first step that starts to shed some light on the complicated effects of budgets in common position-auction formats.

A Proof of Claim 4.2.1

We rename player i to be i_l, to avoid notational confusion later on. Recall that we consider two declarations of player l, (v, b) and $(\tilde{v}, \tilde{b})$, where $\tilde{v} \geq v$ and $\tilde{b} \geq b$. Suppose player i_l wins slot l and pays P_l when declaring (v, b), and wins slot $\tilde{l}$ and pays $\tilde{P}_{\tilde{l}}$ when declaring $(\tilde{v}, \tilde{b})$. (l and/or $\tilde{l}$ can take the value $k+1$ to denote that player i_l loses). We use $\tilde{x}$ to describe the variable x in the execution for $(\tilde{v}, \tilde{b})$, for example $\tilde{B}_l^1, \tilde{q}_{i,l}^1$, and so on. The following lemma will be repeatedly used as a tool to prove the five properties. Its proof is given in appendix B.

Lemma 4.

1. *If either $b = \tilde{b}$ or $b > \min(P_{\tilde{l}}, \tilde{P}_{\tilde{l}})$ then there exists a slot $1 \leq j^* \leq l$ such that the set of winners of slots $1, ..., j^*$ in both declarations is the same set.*

2. *If $v = \tilde{v}$ and $b > \min(P_{\tilde{l}}, \tilde{P}_{\tilde{l}})$ then there exists a slot $1 \leq j^* \leq \tilde{l}$ such that the set of winners of slots $1, ..., j^*$ in both declarations is the same set.*

Proof of properties 1a and 1b. We prove the two properties by induction on the number of slots k. In addition we inductively prove that the winner of every slot $s > l$ is the same player in both declarations, and the losing players are the same. If $k = 1$ then the claim is immediate from the definition of the mechanism. We assume correctness for $k' < k$ slots and prove for k slots. By lemma 4 there exists a slot $j^* \leq l$ such that the winners of slots $1, ..., j^*$ are the same in both declarations. If $j^* < l$ then at iteration $j^* + 1$ in both declarations we are left with the same set of players, and a mechanism for $k - j^* < k$ slots, and the induction assumption implies the claim. If $j^* = l$ then clearly the first property holds since player i_l wins a slot $1, ..., j^*$ in both declarations. In addition the set of players at iteration $j^* + 1$ is the same for both declarations, hence each slot $j > j^*$ has the same winner in both declarations, which implies by claim 4.1.6 that $\tilde{P}_s = P_s$ for any slot $s \geq l$, as claimed.

Proof of property 1c. If $\tilde{l} = l$ then the claim is immediate from the above. Otherwise some other player l_1 wins slot l in the declaration $(\tilde{v}, \tilde{b})$, and suppose l_1 won slot s_1 in declaration (v, b). Note that $s_1 < l$ since by the previous proof the winners of slots $l+1, ..., k$ plus all losers are the same in both declarations. Let l_2 be the player that wins slot s_1 in declaration $(\tilde{v}, \tilde{b})$, and suppose l_2 won slot s_2 in declaration (v, b). We again have $s_2 < l$. When this terminates we must reach a player l_r that won slot $s_r = \tilde{l}$ in declaration (v, b). Denote $s_0 = l$. We argue by induction on $i = 0, ..., r$ that $\tilde{P}_{s_i} \geq P_{s_i}$. The base case of $i = 0$ follows from the previous proof. We now assume by induction that $\tilde{P}_{s_i} \geq P_{s_i}$ and prove that $\tilde{P}_{s_{i+1}} \geq P_{s_{i+1}}$. Note that $b_{l_{i+1}} > \tilde{P}_{s_i} \geq P_{s_i}$. If $\tilde{P}_{s_{i+1}} \geq b_{l_{i+1}} > P_{s_{i+1}}$ then we immediately get the inductive claim. Otherwise assume $\tilde{P}_{s_{i+1}} < b_{l_{i+1}}$. Player l_{i+1} wins slot s_{i+1} in (v, b), hence $\alpha_{s_{i+1}} v_{l_{i+1}} - P_{s_{i+1}} \geq \alpha_{s_i} v_{l_{i+1}} - P_{s_i}$. On the other hand player l_{i+1} wins slot s_i in $(\tilde{v}, \tilde{b})$, hence $\alpha_{s_i} v_{l_{i+1}} - \tilde{P}_{s_i} \geq \alpha_{s_{i+1}} v_{l_{i+1}} - \tilde{P}_{s_{i+1}}$. Since $\tilde{P}_{s_i} \geq P_{s_i}$ it follows that $\tilde{P}_{s_{i+1}} \geq P_{s_{i+1}}$, as claimed.

Proof of property 2. We first note that by property 1a we have $\tilde{l} \leq l$. We prove by induction on the number of slots k. If $k = 1$ then the claim is immediate from the definition of the mechanism. We assume correctness for $k' < k$ slots and prove for k slots. By lemma 4, using its second part for slot $\tilde{l}$, there exists a slot $j^* \leq \tilde{l}$ such that the winners of slots $1, ..., j^*$ are the same in both declarations. If $j^* < \tilde{l}$ then at iteration $j^* + 1$ in both declarations we are left with the same set of players, and a mechanism for $k - j^* < k$ slots, and the induction assumption implies the claim. If $j^* = \tilde{l} \leq l$ then since player i_l wins one of the slots $1, ..., j^*$ in both declarations it must follow that $\tilde{l} = l$.

Proof of property 3. Suppose by contradiction that $\tilde{l} > l$. Then we have $b > P_l \geq P_{\tilde{l}}$, where the second inequality follows from envy-freeness (claim 4.1.6). But then according to property 1a we get $\tilde{l} \leq l$, a contradiction.

B Proof of Lemma 4

Proof. We start with a basic property that states that, in the first iteration of the auction, the weakest player i in slot s among all players j that win some slot $s_j < s$ must be strong at slot s_i (s_i is the slot that i receives). This implies, for example, that if all weak players in slot s remain weak in all better slots $s' < s$ (in the first iteration) then the set of winners of slots $1...s$ is exactly B^1_s.

Claim B.0.4. *Fix some slot s. Let $W_s = \{ j \notin B^1_s$ and j wins some slot $s' \leq s \}$, suppose that W_s is not empty, and fix some $i \in argmin_{j \in W} p^1_{j,s}$. Let $s_i \leq s$ be the slot that i wins. Then $i \in B^1_{s_i}$.*

Proof. Assume by contradiction that $i \notin B^1_{s_i}$. Then there must exist a player j that wins some slot $s_j < s_i$ and $p^1_{j,s_i} < p^1_{i,s_i}$, otherwise by claim 4.1.2 we have $p^1_{i,s_i} = p^{s_i}_{i,s_i}$ which by bid monotonicity implies $i \notin B^{s_i}_{s_i}$, contradicting the fact that i wins s_i. Since j wins $s_j < s_i$ and $j \notin B^1_{s_i}$ we have

$p^1_{j,s_i} \neq b_j$. By claim 3.3.1 we get $v_i > v_j$. However since j wins $s_j < s$, the minimality of i's bid at s implies $p^1_{j,s} \geq p^1_{i,s}$. Since $v_i > v_j$ we get $p^1_{i,s} = b_i$. Since $i \notin B^1_s$ this contradicts the fact that i wins $s_i \leq s$. □

Corollary B.0.1. *Fix some slot s. Suppose that for any player $i \notin B^1_s$ we have that either $i \notin B^1_j$ for all slots $j < s$ or that i does not win any slot $j < s$. Then the set of players that win slots $1, ..., s$ is B^1_s.*

Let $s^* \in \{l, \tilde{l}\}$ be the slot that satisfies the conditions of claim 4.

Claim B.0.5. *$p^1_{i,s^*} = \tilde{p}^1_{i,s^*}$ for any player $i \neq i_l$ such that $i \notin B^1_{s^*}$ and $p^1_{i,s^*} \leq p^1_{i_l,s^*}$. In addition, if $s^* = \tilde{l}$ then $p^1_{i_l,s^*} = \tilde{p}^1_{i_l,s^*}$.*

Proof. Assume first that $s^* = \tilde{l}$. Since $\tilde{b} > b > P_{\tilde{l}} \geq Q^1_{\tilde{l}}$ and $v = \tilde{v}$ we have $\tilde{p}^1_{i,s} = p^1_{i,s}$ for any player i and any slot $s \geq \tilde{l}$.

Now assume that $s^* = l$. We prove by induction on the slot $s = k...l$. By definition $p^1_{i,k} = \tilde{p}^1_{i,k}$ for any player $i \neq i_l$. Assume correctness for slot $s + 1$ and let us prove for s. If $i_l \in B^1_{s+1}$ then $p^1_{i,s} = \min(b_i, (\alpha_s - \alpha_{s+1})v_i + \min(Q^1_{s+1}, p^1_{i,s+1})) = \tilde{p}^1_{i,s}$ for every player $i \neq i_l$, since by the induction assumption $Q^1_{s+1} = \tilde{Q}^1_{s+1}$ and $p^1_{i,s+1} = \tilde{p}^1_{i,s+1}$. Otherwise assume $i_l \notin B^1_{s+1}$. For every player i with $p^1_{i,s+1} \leq p^1_{i_l,s+1}$ we again get by definition $p^1_{i,s} = \tilde{p}^1_{i,s}$.

Otherwise $p^1_{i,s+1} > p^1_{i_l,s+1}$. Since $i_l \notin B^1_{s+1}$ and i_l wins slot $l < s + 1$ we have $p^1_{i_l,s+1} \neq b_{i_l}$, which implies by claim 3.3.1 that $v_i > v_{i_l}$. Therefore for any i with $p^1_{i,s} \leq p^1_{i_l,s}$ we have $p^1_{i,s} = b_i \geq \tilde{p}^1_{i,s} \geq p^1_{i,s}$. Hence $\tilde{p}^1_{i,s} = p^1_{i,s} = b_i$, implying $\tilde{p}^1_{i,s} = p^1_{i,s} = b_i$. If $i_l \in B^1_s$ then all players $i \notin B^1_s$ have $p^1_{i,s} \leq p^1_{i_l,s}$, and the claim follows. □

Note that, by this lemma, if $i_l \in B^1_{s*}$ then we get $p^1_{i,s^*} = \tilde{p}^1_{i,s^*}$ for any player $i \notin B^1_{s^*}$, and hence $\tilde{B}^1_{s^*} = B^1_{s^*}$.

We also note that $\tilde{q}^1_{i,s} \geq q^1_{i,s}$ and $\tilde{p}^1_{i,s} \geq p^1_{i,s}$ for any player i and any slot s (this follows by a simple induction on the slot $s = k, ..., 1$). We say that a player $i \notin B^1_{s^*}$ "jumped" if $i \neq i_l$, $p^1_{i_l,s^*} \geq p^1_{i,s^*}$ and there exists a slot $j \leq s^* - 1$ such that $i \in B^1_j$.

Claim B.0.6. *If a player $i \neq i_l$ with $i \notin B^1_{s^*}$ and $p^1_{i_l,s^*} \geq p^1_{i,s^*}$ did not jump then $p^1_{i,j} = \tilde{p}^1_{i,j}$ and $i \notin \tilde{B}^1_j$ for any slot $j \leq s^*$.*

Proof. Since $i \notin B^1_{s^*}$ and $p^1_{i_l,s^*} \geq p^1_{i,s^*}$ but i did not jump we have $i \notin B^1_j$ for any slot $j \leq s^*$. We show the claim by induction on $j = s^*, s^* - 1, ..., 1$. The base case $j = s^*$ follows since slot s^* is an anchor. Assume that $\tilde{p}^1_{i,j+1} = p^1_{i,j+1}$ and $i \notin \tilde{B}^1_{j+1}$ for some $j < s^*$. Then $\tilde{p}_{i,j} = \min(b_i, (\alpha_j - \alpha_{j+1})v_i + \tilde{p}_{i,j+1}) = \min(b_i, (\alpha_j - \alpha_{j+1})v_i + p_{i,j+1}) = p_{i,j}$, completing the first part of the inductive step. Since $\tilde{p}_{i',j} \geq p_{i',j}$ for any player i', $i \notin B^1_j$ implies $i \notin \tilde{B}^1_j$. □

Claim B.0.7. *If $i_l \in B^1_{s^*}$ and there does not exist a player that jumped then the set of players that win slots $1, ..., s^*$ is identical in both declarations (v, b) and $(\tilde{v}, \tilde{b})$.*

Proof. Every player $i \notin B^1_{s^*}$ satisfies $p^1_{i_l,s^*} \geq p^1_{i,s^*}$, and, since no such player jumped, corollary B.0.1 implies that the players in $B^1_{s^*}$ win slots $1, ..., s^*$ in declaration (v, b). We will show that the players in $\tilde{B}^1_{s^*}$ win slots $1, ..., s^*$ in declaration $(\tilde{v}, \tilde{b})$, which will imply the claim since $B^1_{s^*} = \tilde{B}^1_{s^*}$. Assume by contradiction that some player $i \notin \tilde{B}^1_{s^*}$ wins slot $s_i \leq s^*$ (w.l.o.g. i has a minimal bid in slot s^* among all such players). By claim B.0.4 it follows that $i \in \tilde{B}^1_{s_i}$. On the other hand since $B^1_{s^*} = \tilde{B}^1_{s^*}$ we have $i \notin B^1_{s^*}$, and $i \neq i_l$. Thus claim B.0.6 implies $i \notin \tilde{B}^1_{s_i}$, a contradiction. □

Using this, if $i_l \in B^1_{s^*}$ and there does not exist a player that jumped then we can conclude the proof of lemma 4 by choosing $j^* = s^*$. The next claim shows that in any other case there must be a player that jumped.

Claim B.0.8. *If $i_l \notin B^1_{s^*}$ then there exists a player i that jumped such that $p^1_{i,s^*} < p^1_{l,s^*}$.*

Proof. If $s^* = l$ then by claim 4.1.2 there is a player i' that wins slot $s_{i'} < s^*$ and $p^1_{i',s^*} < p^1_{l,s^*}$, and by claim B.0.4 there exists a player i with $p^1_{i,s^*} \leq p^1_{i',s^*}$ that wins slot $s_i < s^*$ and $i \in B^1_{s_i}$ (i may be i'). Therefore i jumped. If $s^* = \tilde{l}$ then by assumption $p^1_{i_l,s^*} = \tilde{p}^1_{i_l,s^*}$ and therefore $i_l \notin \tilde{B}^1_{s^*}$. As above this implies that there exists a player $i \neq i_l$ that wins slot $s_i < s^*$ such that $\tilde{p}^1_{i,s^*} < \tilde{p}^1_{i_l,s^*}$ and $i \in \tilde{B}^1_{s_i}$. Player i also satisfies $i \notin B^1_{s^*}$ and $p^1_{i_l,s^*} > p^1_{i,s^*}$. We argue that i jumped: otherwise claim B.0.6 implies $i \notin \tilde{B}^1_j$ for any slot $j \leq s^*$, a contradiction. □

Therefore we assume that there exists a player that jumps. For two players i, j and a slot s, we denote $p^1_{i,s} \succ p^1_{j,s}$ if $p^1_{i,s} > p^1_{j,s}$, or $p^1_{i,s} = p^1_{j,s}$ and $i \succ j$. Let i^* be a player with minimal bid p^1_{i,s^*} w.r.t. $\succ$ among all players that jumped. Let $j^* \leq s^* - 1$ be some slot such that $i^* \in B^1_{j^*}$.

Claim B.0.9. *For any player $i \neq i_l$ such that $i \notin B^1_{j^*}$, and for any slot $j \leq j^*$, we have: (1) $i \notin B^1_j$, (2) $i \notin \tilde{B}^1_j$, and (3) $p_{i,j} = \tilde{p}_{i,j}$. In addition, if $i_l \notin B^1_{j^*}$ then $\tilde{l} > j^*$ and $l > j^*$.*

Proof. Consider a player $i \notin B^1_{j^*}$. Assume first that $p^1_{i^*,s^*} \succ p^1_{i,s^*}$ (note that this implies that $i \neq i_l$ since $p^1_{i^*,s^*} < p^1_{i_l,s^*}$). By the minimality assumption on i^* we have that $i \notin B^1_j$ for any slot $j < s^*$. By claim B.0.6 we also have $p_{i,j} = \tilde{p}_{i,j}$ and $i \notin \tilde{B}^1_j$ for any slot $j < s^*$. If $p_{i,s^*} = p_{i^*,s^*}$ and $i \succ i^*$ (note that this still implies $i \neq i_l$) then since $i \notin B^1_{j^*}$ and $i^* \in B^1_{j^*}$ we must have $p^1_{i,j^*} = b_i$, which implies the three properties.

Otherwise $p_{i,s^*} > p_{i^*,s^*}$. We must have $v_i > v_{i^*}$, otherwise we get by claim 3.3.1 that $p_{i^*,s^*} = b_{i^*}$ which is a contradiction since $i^* \notin B^1_{s^*}$ and $i^* \in B^1_{j^*}$. Since $p_{i,j^*} \leq p_{i^*,j^*}$ we get $p^1_{i,j^*} = b_i$, and since $i \notin B^1_{j^*}$ then $i \notin B^1_j$ for any $j < j^*$. In addition, if $i \neq i_l$ or $i = i_l$ and $b = \tilde{b}$ then $b_i = \tilde{p}^1_{i,j^*}$, implying $i \notin \tilde{B}^1_j$ and $p_{i,j} = \tilde{p}_{i,j}$ for any $j \leq j^*$.

This establishes the three properties for $i \neq i_l$, and that, if $b = \tilde{b}$ and $i_l \notin B^1_{j^*}$ then player i_l does not win any slot $j \leq j^*$. If $i_l \notin B^1_{j^*}$ and $b < \tilde{b}$ then we get $p^1_{l,j^*} = b$ from the above paragraph. Thus player i_l cannot win any slot $j \leq j^*$ in declaration (v, b), hence $l > j^*$. It remains to show $\tilde{l} > j^*$. Since $b < \tilde{b}$ we have by assumption $b > \min(P_{\tilde{l}}, \tilde{P}_{\tilde{l}})$. If $b > P_{\tilde{l}}$ then since $P_j \geq p^1_{i_l,j} = b$ for any $j \le j^*$ we get $\tilde{l} > j^*$. Similarly, if $b > \tilde{P}_{\tilde{l}}$ then since $\tilde{P}_j \ge \tilde{p}^1_{i_l,j} \ge p^1_{i_l,j} = b$ for any $j \le j^*$ we again get $\tilde{l} > j^*$. □

By claim B.0.9, the conditions of corollary B.0.1 hold for slot j^* and declaration (v, b) (note that by claim B.0.9, if $i_l \notin B^1_{j^*}$ then player i_l wins slot $l > j^*$ in declaration (v, b)). Therefore the players in $B^1_{j^*}$ win slots $1, ..., j^*$ in this declaration. To finish the proof of lemma 4 we argue that these players are the winners of slots $1, ..., j^*$ in declaration $(\tilde{v}, \tilde{b})$ as well.

Assume by contradiction that there exists a player $x \notin B^1_{j^*}$ that wins a slot $s_x \le j^*$ in declaration $(\tilde{v}, \tilde{b})$. By claim B.0.9 we have $x \ne i_l$ since $\tilde{l} > j^*$. Assume without loss of generality that x has a minimal bid $\tilde{p}^1_{x,j^*}$ among all players $x \notin B^1_{j^*}$ that win some slot $s \le j^*$ in declaration $(\tilde{v}, \tilde{b})$. By claim B.0.9, $p^1_{x,j^*} = \tilde{p}^1_{x,j^*}$. Since $p^1_{i,j^*} \le \tilde{p}^1_{i,j^*}$ for any player i it follows that $x \notin \tilde{B}^1_{j^*}$ as well. By claim B.0.9 we have $x \notin \tilde{B}^1_{s_x}$, and therefore by claim B.0.4 there must exist $y \notin \tilde{B}^1_{j^*}$ such that $\tilde{p}^1_{y,j^*} < \tilde{p}^1_{x,j^*}$ and y wins some slot $s_y \le j^*$. By the minimality assumption on the choice of x we must have $y \in B^1_{j^*}$. Therefore $p^1_{y,j^*} \ge p^1_{x,j^*}$. But we also have $p^1_{y,j^*} \le \tilde{p}^1_{y,j^*} < \tilde{p}^1_{x,j^*} = p^1_{x,j^*}$, a contradiction. □

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