Mediators in Position Auctions*

Itai Ashlagi Dov Monderer Moshe Tennenholtz
Technion–Israel Institute of Technology
Faculty of Industrial Engineering and Management
Haifa 32000, Israel
ashlagii@tx.technion.ac.il
{dov,moshet}@ie.technion.ac.il

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Abstract

A mediator is a reliable entity which plays on behalf of the players who give her the right of play. The mediator is guaranteed to behave in a pre-specified way based on messages received from the agents. However, a mediator cannot enforce behavior; that is, agents can play in the game directly without the mediator's help. A mediator generates a new game for the players, the mediated game. The outcome in the original game of an equilibrium in the mediated game is called a mediated equilibrium. Monderer and Tennenholtz introduced a theory of mediators for games with complete information. We extend the theory of mediators to games with incomplete information, and use the new theory to study position auctions, a central topic in practical and theoretical electronic commerce. We provide a minimal set of conditions on position auctions, which is sufficient to guarantee that the VCG outcome function is a mediated equilibrium in these auctions.

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1 Introduction

In a game with incomplete information with private values every player holds some private information, which is called the player's type, and has a set of possible actions. Every profile of the players'actions generates an outcome in a given set of possible outcomes. The utility of a player depends on the vector of types and on the outcome generated by the profile of actions. A strategy of a player is a function that maps each of its possible types to an action. The game is called a Bayesian game, when a commonly known probability measure on the profiles of types is commonly known to the participants. Otherwise, it is called a pre-Bayesian game. In this paper we deal only with pre-Bayesian games. The leading solution concept for pre-Bayesian games is the ex post equilibrium: A profile of strategies, one for each player, such that no player has a unilateral profitable deviation, independent of the types of the other players. Consider the following simple example of a pre-Bayesian game, which possesses an ex post equilibrium. The game is denoted by $\mathbb{H}$, and the set of outcomes in this game is identified with the set of profiles of actions.

	a	b
a	$5,2$	$3,0$
b	$0,0$	$4,2$

A

	a	b
a	$2,2$	$0,0$
b	$3,3$	$5,2$

B

In $\mathbb{H}$ there are two players. Both players can choose among two actions: a and b. The column player, player 2, has a private type, A or B. The row player, player 1, has only one possible type. A strategy of player 1 is g_1,where $g_1 = a$ or $g_1 = b$. A strategy of player 2 is a function $g_2 : \{A, B\} \rightarrow \{a, b\}$.That is, player 2 has 4 possible strategies. In this game the strategy profile (g_1, g_2) is an ex post equilibrium, where $g_1 = b$ and $g_2(A) = b, g_2(B) = a$.

Unfortunately, pre-Bayesian games do not, in general, possess ex post equilibria, even if we allow mixed strategies. In order to address this problem and to enable the players to reach a desired outcome of a given game as an ex post equilibrium we suggest in this paper the use of *mediators*.

A mediator is a reliable entity that can interact with the players and perform on their behalf actions in a given game. However, a mediator cannot enforce behavior. Indeed, an agent is free to participate in the game without the help of the mediator. The mediator's behavior on behalf of the agents that give it the right of play is pre-specified, and is conditioned on information the agents provide to the mediator. This notion is natural; in many systems there is some form of reliable party or administrator that can be used as a mediator. Brokers and routers are simple examples of such mediators. Notice that we assume that the game is given, and all the mediator can do is to perform actions on behalf of the agents that explicitly allow it to do so.[1] A weaker form of a mediator discussed in the game theory literature is captured by the notion of correlated equilibrium [4]. This notion was generalized to communication equilibrium in [9, 22]. An additional type of mediator is discussed in [19]. However, in all these settings the mediator can not perform actions on behalf of the agents that allow it to do so. Another type of mediators is captured by the notion of conditional contracts. In a conditional contract, a joint strategy for the group of all players is suggested, and is executed only if all players agree to accept this suggestion. However, the idea that a mediation device will play also on behalf of a strict subset of the players, who give it the right of play, has not been considered in that literature.

In this paper, a mediator will specify the actions to be performed on behalf of the set of players who give it the right of play, as a function of the identity of the players in that set, and the information provided by these players. Such mediators that can obtain the "right of play" but cannot enforce the use of their services have been defined and discussed for general n-person games with complete information in [21].[2] In this paper we introduce the use of such mediators in games with incomplete information.

In order to illustrate the power of mediators for games with incomplete information consider the following pre-Bayesian game $\mathbb{G}$ that does not possess an ex post equilibrium. In $\mathbb{G}$, the column player has two possible types: A and B.

[1] This natural setting is different from the one discussed in the classical theories of implementation and mechanism design, where a designer designs a new game from scratch in order to yield some desired behavior.

[2] For games with complete information the main interest is in leading agents to behaviors which are stable against deviations by coalitions. A special case of mediators was already discussed in [14], where the authors discussed mediators for a two-person game, which is known to the players but not to the mediators, and they looked for Nash equilibrium in the new game generated by the mediator. The topic of mediators for games with complete information has been further generalized and analyzed in [24].

	a	b
a	$5,2$	$3,0$
b	$0,0$	$2,2$

A

	a	b
a	$2,2$	$0,0$
b	$3,0$	$5,2$

B

A mediator for $\mathbb{G}$ should specify the actions it will choose on behalf of the players that give it the right of play. If player 2 wants to give the mediator the right of play it should also report a type. Consider the following mediator:

If both players give the mediator the right of play, the mediator will play on their behalf (a, a) if player 2 reports A, and (b, b) if player 2 reports B. If only player 1 gives the mediator the right of play, the mediator will choose a on his behalf. If only player 2 gives the mediator the right of play, the mediator will choose action a (resp. b) on his behalf, if B (resp. A) has been reported.

The mediator generates a new pre-Bayesian game,which is called the *mediated game*. In the mediated game player 1 has three actions: Give the mediator the right of play, denoted by m, or play directly a or b. Player 2 has four actions: $m - A$, $m - B$,a,b, where $m - A$ ($m - B$) means reporting A (B) to the mediator and giving it the right of play. The mediated game is described in the following figure:

	$m - A$	$m - B$	a	b
m	$5,2$	$2,2$	$5,2$	$3,0$
a	$3,0$	$5,2$	$5,2$	$3,0$
b	$2,2$	$0,0$	$0,0$	$2,2$

A

	$m - A$	$m - B$	a	b
m	$2,2$	$5,2$	$2,2$	$0,0$
a	$0,0$	$2,2$	$2,2$	$0,0$
b	$5,2$	$3,0$	$3,0$	$5,2$

B

It is easy to verify that giving the mediator the right of play, and reporting truthfully, is an ex post equilibrium at the mediated game. That is, (f_1, f_2) is an ex post equilibrium, where $f_1 = m$, and $f_2(A) = m - A$, $f_2(B) = m - B$. In this case we say that the mediator *implements the outcome function* $\varphi^m : \{A, B\} \rightarrow \mathbf{O}$ defined by $\varphi^m(A) = (a, a)$ and $\varphi^m(B) = (b, b)$, where $\mathbf{O}$ is the set of outcomes of $\mathbb{G}$.

The aim of this paper is twofold. We introduce mediators for games with incomplete information, and apply them in the context of position auctions.[3]

Our choice of position auctions as the domain of application is not a coincidence; indeed, position auctions have become a central issue in advertisement and sponsored search, and the selection of appropriate position auctions for that task is a subject of considerable amount of study [18, 8, 15, 6, 7, 26]. There exist many types of 3rd parties (i.e. neither advertisers, nor the companies running the auctions) which do the bidding on behalf of advertisers in position auctions.[4] The typical role of such a 3rd party is to decide on the exact keywords for which a bid will be made and on these bids' values. As thousands of position auctions are being held by the leading search engines and many advertisers seek the support of such 3rd parties, the market of 3rd parties for bidding in position auctions is flowering. The type of 3rd parties discussed in this paper allows for coordination of advertisers' bids in position auctions. Such coordination is known in economics, for example in the context of the so-called bidding rings.[5] The mediators discussed in this paper are 3rd parties with specific characteristics, allowing for such bid coordination. The goal of a mediator is to maximize its own profit. This goal is not explicitly modeled in this paper. However, it is implicitly assumed that maximizing the mediator's profit is highly correlated with the goal of guaranteeing high utility to rational agents. We assume rational agents follow equilibrium behavior, and require the equilibrium to be an ex-post equilibrium; that is, an equilibrium in which each agent best strategy does not depend on the distribution of other agents' valuations. Such equilibrium behavior also provides stability, as it does not require the agents to make a trial and error procedure, learning the other agents' valuations, which might deteriorate the agents' payoffs. Given a position auction, the mediator should induce an ex-post equilibrium, in which the agents get high utility. By the revelation principle, it is sufficient to consider only mediators such that in an ex-post equilibrium agents use the mediator's service and report their true valuations to it. Notice that given a position auction, a mediator with such desired properties induces another (new) position auction. As it is shown in [2], the VCG

[3]Mediators for Bayesian one-item auctions (in particular first price and second price auctions) have been already discussed in [10, 17, 5]. However, the mediators in these papers are endowed with the additional strong characteristic – the ability to re-distribute payoffs.

[4]E.g., http://www.salsainternet.com.au and http://www.topclickmedia.co.uk/sevices.htm.

[5]See the references mentioned in Footnote 3.

position auction is the *only* anonymous position auction that possesses an ex post equilibrium. Therefore, we deal in this paper with implementation of the outcome of the VCG position auction by truthful mediation. The VCG outcome is considered to lead to relatively high utilities for the agents. For example, in [26], Varian considers the next-price position auction[6], which is currently used in practice, and a natural subset of its equilibria under complete information. He shows that the utility of each player is maximized in the equilibrium which yields the VCG utilities. [7]

One such mediator has already been discussed, for other purposes, in the literature: An English auction type of algorithm was constructed in [7] that takes as input the valuations of the players and outputs bids for the next-price position auction. It was proved there that reporting the true type to this algorithm by each player forms an ex post equilibrium, which generates the VCG outcome. In our language this algorithm can "almost" be considered as a mediator for the next-price position auction that implements the VCG outcome function; What is missing, is a component that "punishes" players who send their bids directly to the auctioneer, and a proof that using the mediator services and reporting the true type by each player is an ex post equilibrium in the mediated game defined by the algorithm and by the additional component. Notice that, in principle, a mediator may generate a desired outcome function by punishing the players who do not use its services using very high bids by the players that use its services. However, we believe that such mediators are not realistic, and therefore we concentrate on the search for *individually rational* mediators that implement the VCG outcome function and satisfy an additional rationality condition: the payoff of an agent who gives the mediator the right of play and reports his type truthfully cannot be negative regardless of the actions taken by the agents who did not choose the mediator's services, or agents who report false types to the mediator.

We first prove the existence of such desired mediators for next-price position auctions. Next, we provide a minimal characterization for the existence of individually rational mediators that implement the VCG outcome function in position auctions; We provide three conditions on position auctions that imply the existence of such an individually rational mediator for any given

[6]In [7] next-price position auctions are called *generalized second price (GSP) auctions*.

[7]Notice that since maximizing agent utilities is opposite to revenue maximization by the auctioneer, we do not expect the auctioneer to provide such 3rd party services.

position auction satisfying these conditions. We also show that the set of those three conditions is minimal. Using this result, we prove the existence of such individually rational mediators for a rich class of position auctions, including all k-price position auctions, $k > 1$. For k=1, the self-price position auction, we show an impossibility result. However, in auctions with the first arrival rule, in which ties are impossible, we prove the existence of individually rational mediators that implement the VCG outcome function.

In Section 2 we present the general theory of mediators for pre-Bayesian games. In Sections 3-7 we apply mediators to position auctions. It Section 8 we generalize the results on position auctions to results on position auctions with quality factors, in which the auction organizer can express preferences over players. In such auctions we deal with the implementation of the outcome function of an appropriate weighted VCG position auction.

2 Mediators in pre-Bayesian Games

In this section we present a general theory of mediators in pre-Bayesian games. In a pre-Bayesian game, $\mathbb{G}$, there is a fixed set of players $N = \{1, 2, \cdots, n\}$, each of the players is endowed with a set of actions $b_i \in B_i$. There is a set of outcomes $\mathbf{O}$, and a function $\psi : \mathbf{B} \to \mathbf{O}$ that maps action profiles to outcomes, where $\mathbf{B} = B_1 \times \cdots \times B_n$. There is a set of states $\omega \in \Omega$. The payoff of player i, $w_i(\omega, a)$ depends on the realized state $\omega \in \Omega$, and on the outcome $a \in \mathbf{O}$. However, the realized state is not known to the players. Every player i receives a state-correlated type, $v_i = \tilde{v}_i(\omega) \in V_i$ on which he conditions his action. Let $\mathbf{V} = V_1 \times \cdots \times V_n$. In this paper we deal with private value auctions. Therefore, as is common for such auctions, we assume that $\Omega = \mathbf{V}$. Hence, for every $\mathbf{v} = (v_1, \ldots, v_n) \in \mathbf{V}$, $\tilde{v}_i(v) = v_i$.

Let $\mathbb{G} = (N, \mathbf{V}, \mathbf{O}, (w_i)_{i \in N}, \mathbf{B}, \psi)$ be a pre-Bayesian game. We define the *utility function* of i, $u_i : \mathbf{V} \times \mathbf{B} \to \mathbb{R}$ as follows:

$$u_i(\mathbf{v}, \mathbf{b}) = w_i(\mathbf{v}, \psi(\mathbf{b})).$$

A *strategy* of player i in $\mathbb{G}$ is a function f_i that assigns an action, $b_i = f_i(v_i) \in B_i$ to every possible type $v_i \in V_i$. A profile of strategies $\mathbf{f} = (f_1, \cdots, f_n)$ is an *ex post equilibrium* in $\mathbb{G}$ if for every player i, and for every $\mathbf{v} \in \mathbf{V}$

$$u_i(\mathbf{v}, \mathbf{f}(\mathbf{v})) \geq u_i(\mathbf{v}, b_i, \mathbf{f}_{-i}(\mathbf{v}_{-i})), \quad \forall b_i \in B_i,$$

where, $\mathbf{f}(\mathbf{v}) = (f_1(v_1), \ldots, f_n(v_n))$, and $\mathbf{f_{-i}}(\mathbf{v}_{-i})$ is the vector $\mathbf{f}(\mathbf{v})$ without the i's component.[8] f_i is a *dominant strategy* for i if the above inequalities hold for every profile f_{-i} of the other players' strategies.

In mechanism design theory, there is a given *environment*, $\mathbb{E} = (N, T, \mathbf{O}, (w_i)_{i \in N})$ that contains all components of a pre-Bayesian game except for the rules of the game; that is, the environment does not contain the action sets and the mapping from actions to outcomes. In addition, the mechanism designer has an *outcome function* $\varphi : \mathbf{V} \to \mathbf{O}$, which she wishes to implement. Hence, the goal of the designer is to find *rules*, $(\mathbf{B}, \psi)$ and an ex post equilibrium $\mathbf{f}$ in the pre-Bayesian game $\mathbb{G} = (\mathbb{E}, \mathbf{B}, \psi)$ that implements φ in the sense that

$$\psi(\mathbf{f}(\mathbf{v})) = \varphi(\mathbf{v}), \quad \mathbf{v} \in \mathbf{V}.$$

In contrast, in this paper we consider situations in which the pre-Bayesian game is given, and cannot be changed. It is well-known that ex post equilibrium need not exist even in very simple pre-Bayesian games. This makes the choice of a strategy for a participant very difficult. To deal with the non-existence problem, and/or to increase players' utility when a particular ex post equilibrium is not desirable, and/or to guarantee economic efficiency, we suggest the use of mediators. A mediator is a reliable party that acts in a pre-specified way in behalf of the players who give her the right to do so. All other players can independently play or use other mediators.

Mediators for games with complete information have been defined and analyzed in [21]. In games with complete information the only input the mediator collects is the "right of play".[9] Hence, her action depends on the set of players that give her the right of play. Therefore, a mediator for a game with complete information is defined by a vector $\mathbf{b} = (\mathbf{b}_S)_{S \subseteq N}$, where

[8] It is well-known that an ex post equilibrium in a pre-Bayesian game is a Bayesian equilibrium for every choice of prior probability and vice versa. Indeed, the classical literature in economics/game theory actually discussed ex post equilibrium in a particular Bayesian game, and defined it as a Bayesian equilibrium which is robust to changes in the prior probability. Only recently the concept of pre-Bayesian games have been explicitly defined and analyzed. See e.g.,[12, 11], where pre-Bayesian games are called *games in informational form* and *games without probabilistic information*,[13], where they are called *games with incomplete information with strict type uncertainty*, [1], where they are called *distribution-free games with incomplete information*, and [3] where they are called pre-Bayesian games.

[9] What we call here a *mediator* for games with complete information is called a *minimal mediator* in [21]. It is proved in [21] that a minimal mediator can implement every outcome which can be implemented by a mediator.

$\mathbf{b}_S = (b_i)_{i\in S} \in \mathbf{B}_S$ is the vector of actions used by the mediator if S is the set of players that give her the right of play.[10]

In a pre-Bayesian game $\mathbb{G}$, the mediator can use the information provided by the agents. Hence, a player is required not only to give the mediator the right of play but also to report his type.

Definition 1 (Mediators) *Let* $\mathbb{G} = (\mathbb{E}, \mathbf{B}, \psi)$ *be a pre-Bayesian game, where* $\mathbb{E} = (N, \mathbf{V}, \mathbf{O}, \mathbf{w})$ *is the environment. A* mediator for the pre-Bayesian game *is a vector* $\mathbf{m} = (\mathbf{m}_s)_{S\subseteq N}$*, where* $\mathbf{m}_S : \mathbf{V}_S \to \mathbf{B}_S$*.*

If the set of players that give the mediator the right of play is S, and the members of S send the mediator the profile of types $\mathbf{v}_S = (v_i)_{i\in S} \in \mathbf{V}_S$, the mediator plays $\mathbf{m}_S(\mathbf{v}_S) \in \mathbf{B}_S$ on behalf of the players in S.

Every mediator $\mathbf{m}$ for the pre-Bayesian game $\mathbb{G} = (\mathbb{E}, \mathbf{B}, \psi)$ defines a new pre-Bayesian game, $\mathbb{G}^{\mathbf{m}} = (\mathbb{E}, \mathbf{B}^{\mathbf{m}}, \psi^{\mathbf{m}})$, which is called the *mediated game*. This game shares the same environment $\mathbb{E}$ with the original game. The set of actions of i is $B_i^{\mathbf{m}} = B_i \cup V_i$, where without loss of generality we assume that $B_i \cap V_i = \emptyset$. Choosing $b_i^m = b_i \in B_i$ means that i is not using the mediator's services. Choosing $b_i^m = v_i \in V_i$ means reporting the value v_i to the mediator, as well as the permission to play on behalf of i. Let $\mathbf{b}^{\mathbf{m}} \in \mathbf{B}^{\mathbf{m}}$. Denote by $N(\mathbf{b}^{\mathbf{m}})$ the set of players that use the mediator; $N(\mathbf{b}^{\mathbf{m}}) = \{i \in N | b_i^{\mathbf{m}} \in V_i\}$. The action-to-outcome function in the mediated game, $\psi^{\mathbf{m}} : \mathbf{B}^{\mathbf{m}} \to \mathbf{O}$, is defined as follows:

$$\psi^{\mathbf{m}}(\mathbf{b}^{\mathbf{m}}) = \psi(\mathbf{m}_{N(\mathbf{b}^m)}(\mathbf{b}^{\mathbf{m}}_{N(\mathbf{b}^{\mathbf{m}})}), \mathbf{b}^{\mathbf{m}}_{-N(\mathbf{b}^{\mathbf{m}})}),$$

where $-N(\mathbf{b}^{\mathbf{m}}) = N \setminus N(\mathbf{b}^{\mathbf{m}})$. The utility function of i in the mediated game $\mathbb{G}^{\mathbf{m}}$ is denoted by $u_i^{\mathbf{m}}$.

Definition 2 (Mediated equilibrium) *Let* $\mathbb{G}$ *be a pre-Bayesian game, and let* $\varphi : \mathbf{V} \to \mathbf{O}$ *be an outcome function. We say that* φ *is a mediated equilibrium in* $\mathbb{G}$ *if there exists a mediator for* $\mathbb{G}$*,* $\mathbf{m}$*, and an ex post equilibrium* $\mathbf{g} = (g_1, \cdots, g_n)$ *in* $\mathbb{G}^{\mathbf{m}}$ *such that* $g_i(v_i) \in V_i$ *for every* $i \in N$ *and for every* $v_i \in V_i$*, and the following holds:*

$$\varphi(\mathbf{v}) = \psi(\mathbf{m}_N(\mathbf{g}(\mathbf{v}))), \quad \mathbf{v} \in \mathbf{V}.$$

[10] In general, $\mathbf{b}_S$ can be a correlated strategy.

The strategy of i at the mediated game in which he gives the right of play to the mediator and reports his true type is called the $T-$strategy. The strategy profile in which every player uses the $T-$strategy is called the $T-$strategy profile. We denote by $\varphi^{\mathbf{m}} : \mathbf{V} \rightarrow \mathbf{O}$ the outcome function generated by the mediator, when every player is using the $T-$ strategy. That is,

$$\varphi^{\mathbf{m}}(\mathbf{v}) = \psi(\mathbf{m}_N(\mathbf{v})), \quad \mathbf{v} \in \mathbf{V}.$$

When the $T-$ strategy profile is an ex post equilibrium in $\mathbb{G}^{\mathbf{m}}$ we say that $\mathbf{m}$ *implements* $\varphi^{\mathbf{m}}$ *by truthful mediation.* The well known revelation principle[11] applies to our setting:

Observation 1 (Revelation principle) *Let* $\mathbb{G}$ *be a pre-Bayesian game, and Let* $\varphi : \mathbf{V} \rightarrow \mathbf{O}$ *be an outcome function.* φ *is a mediated equilibrium if and only if there exists a mediator* $\mathbf{m}$ *that implements* φ *by truthful mediation.* [12]

3 Position Auctions

In a position auction there is a seller who sells a finite number of positions $j \in K = \{1, ..., m\}$. There is a finite number of (potential) bidders $i \in N = \{1, ..., n\}$. We assume that there are more bidders than positions, i.e. $n > m$. The positions are sold for a fixed period of time. For each position j there is a commonly-known number $\alpha_j > 0$, which is interpreted as the expected number of visitors at that position. For every $j \in K$ α_j is called the *click-through rate* of position j. We assume that $\alpha_1 > \alpha_2 > \cdots > \alpha_m > 0$. If i holds a position then every visitor to this position gives i a revenue of $v_i > 0$, where v_i is called the *valuation* of i. The set of possible valuations of i is $V_i = (0, \infty)$.

We assume that the players' payoff functions are quasi-linear. That is, if player i is assigned to position j and pays p_j per click then his payoff is $\alpha_j(v_i - p_j)$.

Every player is required to submit a bid, $b_i \in B_i = [0, \infty)$. We assume that bidding 0 is a symbol for non-participation. Therefore, a player with a bid 0 is not assigned to any position, and pays 0.

[11] See e.g. [16] (page 871).

[12] The revelation principle can be generalized as in mechanism design for mediators that enable players to report any message and not just a type.

In all position auctions we consider, the player with the highest positive bid receives the first position, the player with the second highest positive bid receives the second position, and so on. Ties will be considered later. It is useful to define for every position auction two dummy positions $m+1$ and -1, which more than one player may be "assigned" to. All players, who participate in the auction but do not get a position in K are assigned to position $m+1$ and all players who choose not to participate are assigned to position -1. We also define $\alpha_{m+1} = \alpha_{-1} = 0$.

An assignment of players to positions is called an *allocation*. Hence, an allocation is a vector $\mathbf{s} = (s_1, s_2, \cdots, s_n)$ with $s_i \in K \cup \{-1, m+1\}$ such that if $s_i \in K$ then $s_i \neq s_l$ for every $l \neq i$; s_i is the position of player i. The set of all allocations is denoted by $\mathbf{A}$. Given the above, a position auction is defined by its tie breaking rule, which determines the allocation in case of ties, and by its payment scheme. These are discussed below.

3.1 Tie breaking rules

In practice, the most commonly used tie breaking rule is the First-Arrival rule: if a set of players submit the same bid, their priority in receiving the positions is determined by the times their bids were recorded; An earlier bid receives a higher priority. In auction theory this tie breaking rule is typically modelled by assuming that the auctioneer is using a random priority rule. More specifically, let Γ be the set of all permutations, $\gamma = (\gamma_1, ..., \gamma_n)$ of $N = \{1, \cdots, n\}$. Every such γ defines a priority rule as follows: Player i has a higher priority than k if and only if $\gamma_i < \gamma_k$. Every vector of bids $\mathbf{b}$ and a permutation γ uniquely determine an allocation. An auctioneer who is using the random priority rule chooses a fixed priority rule γ by randomizing uniformly over Γ. However, the resulting priority rule is not told to the players before they make their bids. When the priority rule γ is told to the players before they make their bids, the tie breaking rule is called a *fixed priority rule*. Dealing with a fixed priority rule simplifies notations and proofs, and in most cases, and in particular in this paper, results that are obtained with this tie breaking rule are identical to the results obtained with the random priority rule. Therefore, unless otherwise specified, we will assume this tie breaking rule.

Without loss of generality we assume that the fixed priority rule is defined by the natural order, $\tilde{\gamma} = (1, 2, ..., n)$. That is, bidder i has a higher priority than bidder k if and only if $i < k$. Given this fixed priority rule we can make the following definitions, which apply to all position auctions:

We denote by $s(\mathbf{b}, i)$ the position player i is assigned to when the bid profile is $\mathbf{b}$. The allocation determined by $\mathbf{b}$ is denoted by

$$\mathbf{s}(\mathbf{b}) = (s(\mathbf{b}, 1), s(\mathbf{b}, 2), \cdots, s(\mathbf{b}, n)).$$

For every $j \in K \cup \{-1, m+1\}$ we denote by $\delta(\mathbf{b}, j)$ the set of players assigned to position j. Note that for $j \in K$, $\delta(\mathbf{b}, j)$ contains at most one player. In case $\delta(\mathbf{b}, j)$ is a singleton for $j \in K$, we will also let it be the player (and not the set with this player) that is assigned to position j.

3.2 The payment schemes

Let $\mathbf{B} = B_1 \times B_2 \times \cdots \times B_n$ be the set of bid profiles. Each position $j \in K \cup \{-1, m+1\}$ is associated with a payment function, $p_j : \mathbf{B} \to \mathbb{R}_+$, where $p_j(\mathbf{b})$ is the payment per click for position j when the bid profile is $\mathbf{b}$. Naturally we assume that p_{-1} is identically zero. However, we also assume that p_{m+1} is identically zero. Hence, a participant who is not assigned a real position pays nothing.

We call the vector of payment functions $\mathbf{p} = (p_j)_{j \in K}$ *the position payment scheme*.

We deal with anonymous position payment schemes, i.e. the players' payments to the auctioneer are not influenced by their identities. This is modeled as follows: Let $\mathbf{b} \in \mathbf{B}$ be a bid profile. We denote by $b_{(j)}$ the j^{th} highest bid in $\mathbf{b}$. For $j > n$ we let $b_{(j)} = 0$. For example if $\mathbf{b} = (3, 7, 3, 0, 2)$ then $b_{(1)} = 7, b_{(2)} = 3, b_{(3)} = 3, b_{(4)} = 2, b_{(5)} = 0$. We let $\mathbf{b}^* = (b_{(1)}, \cdots, b_{(n)})$. Anonymity is modeled by the requirement that for every two bid profiles $\mathbf{b}, \mathbf{d} \in \mathbf{B}$, $\mathbf{p}(\mathbf{b}) = \mathbf{p}(\mathbf{d})$ whenever $\mathbf{b}^* = \mathbf{d}^*$. That is, for every position j there exists a real-valued function $\tilde{p}_j$ defined over all ordered vectors of bids such that for every $\mathbf{b} \in \mathbf{B}$ $p_j(\mathbf{b}) = \tilde{p}_j(\mathbf{b}^*)$.

We further assume that a player never pays more than his bid. That is, $p_j(\mathbf{b}) \leq b_{(j)}$ for every $\mathbf{b} \in \mathbf{B}$ and for every $j \in K$.

It is convenient in certain cases to describe the payment functions indexed by the players. Let G be a position auction with a position payment scheme $\mathbf{p}$.

For every player i we denote

$$q_i(\mathbf{b}) = p_{s(\mathbf{b}, i)}(\mathbf{b}),$$

and

$$\mathbf{q}(\mathbf{b}) = (q_1(\mathbf{b}), q_2(\mathbf{b}), \cdots q_n(\mathbf{b})).$$

Note that the correspondence $p \rightarrow q$ is one-to-one. We call $\mathbf{q}$ the *player payment scheme*. All our assumptions about the position payment schemes can be transformed to analogous assumptions about the player payment schemes. We will describe position auctions by their position payment schemes, $\mathbf{p}$, but we will freely use the associated player payment scheme, $\mathbf{q}$ in the definitions and proofs.

3.3 Central position auctions

We next describe the payment schemes of three central position auctions.

Self-price position auctions: Each player who is assigned to a position with a positive click-through rate pays his own bid. That is, for every $j \in K$ and every $\mathbf{b} \in \mathbf{B}$

$$p_j(\mathbf{b}) = b_{(j)} \tag{1}$$

Next-price position auctions: In this auction (run with a slight variation by Google), every player who is assigned to a position with a positive click-through rate pays the bid of the player assigned to the position right after him if there is such a player, and zero otherwise. That is, for every $j \in K$ and for every $\mathbf{b} \in B$

$$p_j(\mathbf{b}) = b_{(j+1)} \tag{2}$$

VCG position auctions: In a Vickrey-Clarke-Groves (VCG) position auction the payment function for position $j \in K$ is defined as follows.[13] For every $\mathbf{b} \in \mathbf{B}$

$$p_j^{vcg}(\mathbf{b}) = \frac{\sum_{k=j+1}^{m+1} b_{(k)}(\alpha_{k-1} - \alpha_k)}{\alpha_j} \tag{3}$$

Note that the VCG position auction is not the next-price position auction unless there is only one position.

[13] We use the standard payment function of the VCG mechanism. A general VCG mechanism may be obtained from the standard one by adding an additional payment function to each player, which depends only on the types of the other players. Some authors (see e.g., [11]) call the standard VCG mechanism, the VC mechanism. According to this terminology we actually deal with VC position auctions. However, we decided to use the more common terminology.

3.4 Position auctions as pre-Bayesian games

We denote by $\mathbb{G} = \mathbb{G}(\alpha, \mathbf{p})$ the position auction with the click-through rate vector α and the position payment scheme $\mathbf{p}$. We denote by $\mathbf{q}$ the associated player payment scheme.

$\mathbb{G}$ is a pre-Bayesian game as defined in Section 2: $N = \{1, \cdots, n\}$ is the set of players. $V_i = (0, \infty)$ is the set of types of i, and $\mathbf{V} = V_1 \times V_2 \times \cdots \times V_n$ is the set of states. $B_i = [0, \infty)$ is the set of actions of i, and $\mathbf{B} = B_1 \times B_2 \times \cdots \times B_n$ is the set of action profiles. $\mathbf{O} = \mathbf{A} \times \mathbb{R}_+^n$ is the set of outcomes, where $\mathbf{A}$ is the set of allocations; A typical outcome is $(s_1, \cdots, s_n, c_1, \cdots, c_n)$, where s_i is the slot of i, and c_i is the cost per click paid by i. Position auctions are not as general as the pre-Bayesian games defined in Section 2: The payoff function of i depends only on v_i and not on the vector of types $\mathbf{v} = (v_1, \cdots, v_n)$. Hence, $w_i : V_i \times \mathbf{O} \to \mathbb{R}$ is defined as follows:

$$w_i(v_i, (\mathbf{s}, \mathbf{c})) = w_i(v_i, (s_1, \cdots, s_n, c_1, \cdots, c_n)) = \alpha_{s_i}(v_i - c_i).$$

The function that maps profiles of actions to outcomes, $\psi : \mathbf{B} \to \mathbf{O}$ is

$$\psi(\mathbf{b}) = (s(\mathbf{b}), \mathbf{q}(\mathbf{b})).$$

The utility function for player i, $u_i : V_i \times \mathbf{B} \to \mathbb{R}$ is therefore defined as follows:

$$u_i(v_i, \mathbf{b}) = \alpha_{s(\mathbf{b},i)}(v_i - q_i(\mathbf{b})).$$

3.5 Mediators for position auctions

We next recall the definitions from Section 2, and apply them to the context of position auctions. Let $\mathbb{G}$ be a position auction. A *mediator* for $\mathbb{G}$ is a vector of functions $\mathbf{m} = (\mathbf{m}_S)_{S \subseteq N}$, where $\mathbf{m}_S : \mathbf{V}_S \to \mathbf{B}_S$. The mediator $\mathbf{m}$ generates the pre-Bayesian *mediated game* $\mathbb{G}^{\mathbf{m}}$. In this game every player i receives his type v_i and can either send a type, $\hat{v}_i$ (not necessarily the true type) to the mediator, or submit a bid directly to the auction. If S is the set of players that send a type to the mediator, the mediator bids on their behalf $\mathbf{m}_S(\hat{\mathbf{v}}_S)$. Hence, the action set of player i in the mediated game is $B_i \cup V_i$, where in this case V_i is a copy $(0, \infty)$ which is disjoint from B_i.[14]

[14] A player either chooses a bid, when participating directly in the auction, or a type, when using the mediator services.

Recall that the *T-strategy* for a player in the mediated game is the strategy, in which the player uses the mediator's services and reports to her his true type. The *T-strategy profile* is the profile of strategies in which every player is using the T-strategy.

The T-strategy profile is an ex post equilibrium in the mediated game if for every player i and type v_i, and for every vector of types of the other players, $\mathbf{v}_{-i}$. the following two conditions hold:
E1: i is not better off when he gives the mediator the right of play and report a false type. That is, for every $\hat{v}_i \in V_i$

$$u_i(v_i, \mathbf{m}_N(v_i, \mathbf{v}_{-i})) \geq u_i(v_i, \mathbf{m}_N(\hat{v}_i, \mathbf{v}_{-i})).$$

E2: i is not better off when he bids directly. That is, for every $b_i \in B_i$,

$$u_i(v_i, \mathbf{m}_N(v_i, \mathbf{v}_{-i})) \geq u_i(v_i, b_i, \mathbf{m}_{N\setminus\{i\}}(\mathbf{v}_{-i})).$$

Whenever the T-strategy profile is an ex post equilibrium in $\mathbb{G}^{\mathbf{m}}$, the mediator $\mathbf{m}$ *implements* an *outcome function* in $\mathbb{G}$. This outcome function is denoted by $\varphi^{\mathbf{m}} : \mathbf{V} \to \mathbf{O}$, and it is defined as follows:

$$\varphi^{\mathbf{m}}(\mathbf{v}) = \psi(\mathbf{m}_N(\mathbf{v})) = (\mathbf{s}(\mathbf{m}_N(\mathbf{v})), \mathbf{q}(\mathbf{m}_N(\mathbf{v}))).$$

3.6 Implementing the VCG Outcome Function by Truthful Mediation

As is discussed in the introduction, in this paper we deal with implementation of the outcome of the VCG position auction by truthful mediation. In this section we demonstrate this type of implementation by an example. This example also shows the need for a definition of individually rational mediators.

Given a position auction $\mathbb{G}$ our goal is to construct a mediator that would implement the outcome function of the VCG position auction. This outcome function, $\varphi^{vcg} : \mathbf{V} \to \mathbf{O}$ is defined as follows:

$$\varphi^{vcg}(\mathbf{v}) = (\mathbf{s}(\mathbf{v}), q^{vcg}(\mathbf{v})).$$

Definition 3 *Let* $\mathbb{G}$ *be a position auction . Let* $\mathbf{m}$ *be a mediator for* $\mathbb{G}$*. We say the* $\mathbf{m}$ implements the VCG outcome function *in* $\mathbb{G}$*, or that it implements* φ^{vcg} *in* $\mathbb{G}$ *if the T-strategy profile is an ex post equilibrium in* $\mathbb{G}^{\mathbf{m}}$*, and* $\varphi^{\mathbf{m}} = \varphi^{vcg}$*.*

We demonstrate implementing the VCG outcome function by a simple example:

Example 1 Consider a self-price position auction $\mathbb{G} = \mathbb{G}(\alpha, \mathbf{p})$ with 2 players and one position, and with $\alpha_1 = 1$. That is, $\mathbb{G}$ is a standard two-person first-price auction. The corresponding VCG position auction is a standard second-price auction. We define a family of mediators $\mathbf{m}^c$, $c \geq 1$, each of them implements the VCG position auction. Assume both players use the mediator's services and send it the types $\hat{\mathbf{v}} = (\hat{v}_1, \hat{v}_2)$, then the $c-$mediator acts as follows: If $\hat{v}_1 \geq \hat{v}_2$ the mediator makes the following bids on behalf of the players: $b_1 = \hat{v}_2$, and $b_2 = \frac{\hat{v}_2}{2}$. If $\hat{v}_2 > \hat{v}_1$, the mediator makes the bids $b_1 = \frac{\hat{v}_1}{2}$, $b_2 = \hat{v}_1$. If only one player uses the mediator services, say player i, then the mediator bids $b_i = c\hat{v}_i$ on behalf of i. We claim that for every $c \geq 1$, the T-strategy profile is an ex post equilibrium in the mediated game generated by $\mathbf{m}^c$.

Indeed, assume player 2 uses the T-strategy and reports his type v_2 to the mediator, and consider player 1.

If $v_1 \geq v_2$ then by using the T-strategy player 1 receives the position and pays v_2. Hence, 1's utility is $v_1 - v_2$. If player 1 deviates by using the mediator's services and reporting $\hat{v}_1 \geq v_2$ his utility is still $v_1 - v_2$. If he reports $\hat{v}_1 < v_2$ his utility will be 0. If player 1 does not use the mediator, he should bid at least cv_2 in order to get the position, and therefore his utility cannot exceed $v_1 - v_2$.

If $v_1 < v_2$, then the T-strategy yields 0, and any other strategy yields a non-positive utility.

Hence, the $T-$strategy profile is an ex post equilibrium for every $c-$mediator, $c \geq 1$.[15]$\diamond$

While each of the mediators $\mathbf{m}^c$, $c \geq 1$, in Example 1 implements the VCG outcome function, the mediator with $c = 1$ has a distinct characteristic: a player who uses the T-strategy cannot get a negative utility. In contrast, for every $c > 1$, if say player 2 does not use the mediator services, participates directly and bids less than cv_1, then the T-strategy yields a negative utility

[15]However, note that the T-strategy is not a dominant strategy; e.g., for $c > 1$, if $v_1 > v_2$ and player 2 bids directly v_2 (without using the mediator services), then by bidding directly $\frac{v_1+v_2}{2}$ is better for player 1, than using the T-strategy: in the former case player 1's utility is $\frac{v_1-v_2}{2}$ and in the latter case her utility is non-positive. A similar argument shows that the $T-$strategy is not dominant for $c = 1$ as well.

of $(1-c)v_1$ to player 1. This motivates our definition of individually rational mediators:

Definition 4 (Individually rational mediator) *Let $\mathbb{G}$ be a position auction. A mediator, $\mathbf{m}$ for $\mathbb{G}$ is individually rational, if for every player, using the T-strategy guarantees a non-negative level of utility. That is, for every $S \subseteq N$ and every player $i \in S$*

$$u_i(v_i, \mathbf{m}_S(\mathbf{v}_S), \mathbf{b}_{-S}) \geq 0, \quad \forall \mathbf{b}_{-S} \in \mathbf{B}_{-S} \quad \textit{and} \quad \forall \mathbf{v}_s \in \mathbf{V}_S.$$

Hence, in Example 1 the mediator $\mathbf{m}^c$, with $c = 1$, is a individually rational mediator that implements the VCG outcome function.[16]

4 Mediators in Next-Price Position Auctions

We now show that there exists an individually rational mediator, which implements the VCG outcome function in next-price position auctions. Although in the following section we prove a more general result, we present this result first, given the importance of next-price position auctions in the literature and in practice.

Theorem 2 *Let $\mathbb{G}$ be a next-price position auction. There exists an individually rational mediator that implements φ^{vcg} in $\mathbb{G}$.*

In order to prove Theorem 2 and other theorems we need the following lemma:

Lemma 3 *Let $\mathbf{p}^{vcg}$ be the VCG payment scheme.*

1. *$p_j^{vcg}(\mathbf{b}) \leq b_{(j+1)}$ for every $j \in K$.*
2. *$p_j^{vcg}(\mathbf{b}) \geq p_{j+1}^{vcg}(\mathbf{b})$ for every $j = 1, ..., m-1$ and for every $\mathbf{b} \in B$, where for every j, equality holds if and only if $b_{(j+1)} = b_{(j+2)} = \cdots = b_{(m+1)}$.*

The proof of Lemma 3 (which part of it can also be deduced from a recursive definition of the VCG payments given in [26] and [7]) is given in Section 8.

In the proof of Theorem 2 we will show that the following mediator implements φ^{vcg} in a next-price position auction:[17]

[16] It is interesting to note that this simple example cannot be extended to general (more than one slot) self-price position auctions, as will be discussed in Section 6.

[17] For the case in which all players choose the mediator, the mediator is similar to the algorithm given in [7].

Mediator 1 (A mediator for next-price position auctions)

(a) For every $\mathbf{v} \in V$ let $\mathbf{m}_N(\mathbf{v}) = \mathbf{b}(\mathbf{v})$, where $\mathbf{b}(\mathbf{v})$ is defined as follows:

- $\mathbf{b}_i(\mathbf{v}) = p^{vcg}_{s(\mathbf{v},i)-1}(\mathbf{v})$ for every player i such that $2 \le s(\mathbf{v}, i) \le m$.[18]
- $\mathbf{b}_{i(\mathbf{v})}(\mathbf{v}) = p^{vcg}_m(\mathbf{v})$, where $i(\mathbf{v})$ is the player with the lowest index in $\delta(\mathbf{v}, m+1)$.
- $\mathbf{b}_i(\mathbf{v}) = \frac{\mathbf{b}_{i(\mathbf{v})}(\mathbf{v})}{1+\rho_i}$ for every $i \in \delta(\mathbf{v}, m+1) \setminus \{i(\mathbf{v})\}$. and some positive ρ_i's.
- $\mathbf{b}_{\delta(\mathbf{v},1)}(\mathbf{v}) = \epsilon + p^{vcg}_1(\mathbf{v})$ for some $\epsilon > 0$.[19]

(b) For every strict subset $S \subset N$ $\mathbf{m}_S(\mathbf{v}) = \mathbf{v}_S$ for every $\mathbf{v}_S \in V_S$.

We now provide some intuition for the reason that Mediator 1 implements the VCG outcome function. Suppose there are no ties. [20] If player i is suppose to be assigned to position $K \setminus \{1\}$, then Mediator 1 bids on behalf of i the VCG payment of the player with the highest valuation which is smaller than player i's valuation. For the player with the highest value Mediator 1 assigns the highest bid, and for all players which are not supposed to be assigned to any position $j \in K$ in the VCG position auction (except one player) Mediator 1 assigns low enough bids. By the first part of Lemma 1 the allocation will be as in the VCG position auction. Therefore, given the next-price position payment functions, the payments are as in the VCG position auction. Furthermore, assigning valuations as bids when only a subset of the players use Mediator 1, ensures that any unilateral deviation in which an agent plays directly in the auction is not beneficial. This follows from the second part of Lemma 3. Notice that such a mediator is individually rational.

Proof of Theorem 2:

Let $\mathbf{m}$ be Mediator 1 defined above. We show that $\varphi^{\mathbf{m}}(\mathbf{v}) = \varphi^{vcg}(\mathbf{v})$ for every $\mathbf{v} \in V$. Let $\mathbf{v} \in V$ be an arbitrary valuation vector.

We have to show that $s(\mathbf{b}(\mathbf{v})) = s(\mathbf{v})$ and that $q(\mathbf{b}(\mathbf{v})) = q^{vcg}(\mathbf{v})$.

We begin by showing that $s(\mathbf{b}(\mathbf{v})) = s(\mathbf{v})$. It is sufficient to show that whenever $1 \le s(\mathbf{v}, i) < s(\mathbf{v}, l) \le m+1$ for some $i \ne l$, then $s(\mathbf{b}(\mathbf{v}), i) < s(\mathbf{b}(\mathbf{v}), l)$.

[18]Recall that $s(\mathbf{b}, i)$ denotes the position of player i under the bid profile $\mathbf{b}$.

[19]Recall that whenever $\delta(\mathbf{b}, j)$ is a singleton for $j \in K$, $\delta(\mathbf{b}, j)$ also denotes the player (and not the set of players) assigned to position j.

[20]We deal with ties in the definition and the proof.

We first show it for $s(\mathbf{v}, i) = 1$, that is $\delta(\mathbf{v}, 1) = i$. Since $b_{\delta(\mathbf{v},1)}(\mathbf{v}) > b_j(\mathbf{v})$ for every $j \neq i$, $s(\mathbf{b}(\mathbf{v}), i) = 1$. Therefore $s(\mathbf{b}(\mathbf{v}), i) < s(\mathbf{b}(\mathbf{v}), l)$. Suppose that $s(\mathbf{v}, l) = m + 1$ and $l \neq i(\mathbf{v})$. By the second part of Lemma 3 and by the definition of $\mathbf{b}(\mathbf{v})$, $\mathbf{b}_l(\mathbf{v}) < \mathbf{b}_i(\mathbf{v})$. Therefore $s(\mathbf{b}(\mathbf{v}), i) < s(\mathbf{b}(\mathbf{v}), l)$. If $s(\mathbf{v}, i) > 1$, and $l \leq m$ or $l = i(\mathbf{v})$, we distinguish between two cases:

1. $v_i = v_l$. Since $s(\mathbf{v}, i) < s(\mathbf{v}, l)$, the fixed priority rule implies that $i < l$. By the second part of Lemma 3, $p^{vcg}_{s(\mathbf{v},i)-1}(\mathbf{v}) \geq p^{vcg}_{s(\mathbf{v},l)-1}(\mathbf{v})$. Therefore $\mathbf{b}_i(\mathbf{v}) \geq \mathbf{b}_l(\mathbf{v})$, which yields $s(\mathbf{b}(\mathbf{v}), i) < s(\mathbf{b}(\mathbf{v}), l)$.

2. $v_i > v_l$. Let $j + 1 = s(\mathbf{v}, i)$. That is, $v_{(j+1)} = v_i$, and therefore by the second part of Lemma 3, $p^{vcg}_{s(\mathbf{v},i)-1}(\mathbf{v}) > p^{vcg}_{s(\mathbf{v},i)}(\mathbf{v})$. Since $s(\mathbf{v}, i) \leq s(\mathbf{v}, l) - 1$, by the second part of Lemma 3, $p^{vcg}_{s(\mathbf{v},i)}(\mathbf{v}) \geq p^{vcg}_{s(\mathbf{v},l)-1}(\mathbf{v})$. Therefore $p^{vcg}_{s(\mathbf{v},i)-1}(\mathbf{v}) > p^{vcg}_{s(\mathbf{v},l)-1}(\mathbf{v})$, which yields $\mathbf{b}_i(\mathbf{v}) > \mathbf{b}_l(\mathbf{v})$. Therefore $s(\mathbf{b}(\mathbf{v}), i) < s(\mathbf{b}(\mathbf{v}), l)$.

This completes the proof that $s(\mathbf{b}(\mathbf{v})) = s(\mathbf{v})$ for all $\mathbf{v} \in V$.

Observe that for every player i such that $s(\mathbf{b}(\mathbf{v}), i) \in K$

$$p_{s(\mathbf{b}(\mathbf{v}),i)}(\mathbf{b}(\mathbf{v})) = p^{vcg}_{s(\mathbf{v},i)}(\mathbf{v}).$$

Therefore $q_i(\mathbf{b}(\mathbf{v})) = q^{vcg}_i(\mathbf{v})$ for every $i \in N$. This shows that $q(\mathbf{b}(\mathbf{v})) = q^{vcg}(\mathbf{v})$ for all $\mathbf{v} \in V$. Hence, $\varphi^{\mathbf{m}} = \varphi^{vcg}$.

We proceed to prove that the T-strategy is an ex post equilibrium. Note that by the truthfulness of VCG, it is not beneficial for any player i to miss report her value to the mediator, given that all other players use the T-strategy. Next we show that it is not beneficial for a single player $i \in N$ to participate in the auction directly, given that all other players use the T-strategy. Fix some $\mathbf{v} \in V$. Assume that player i is the only player that participates directly in the auction. Hence, $\mathbf{v}_{-i}$ is the vector of bids submitted by the mediator. Let b_i be player i's bid. Let $k = s(\mathbf{v}, i)$. Therefore, since $\varphi^{\mathbf{m}} = \varphi^{vcg}$, $s(\mathbf{b}(\mathbf{v}), i) = k$. Let j be player i's position in the deviation. Hence $j = s((\mathbf{v}_{-i}, b_i), i)$. If $j \notin K$ then player i's utility is zero and therefore deviating is not worthwhile for i. Suppose $j \in K$. Let $\tilde{\mathbf{b}} = (\mathbf{v}_{-i}, b_i)$. Then

$$\alpha_k(v_i - p_k(\mathbf{b}(\mathbf{v}))) = \alpha_k(v_i - p^{vcg}_k(\mathbf{v})) \geq$$

$$\alpha_j(v_i - p^{vcg}_j(\tilde{\mathbf{b}})) \geq \alpha_j(v_i - \tilde{b}_{(j+1)}),$$

where the first equality follows from $\varphi^{\mathbf{m}} = \varphi^{vcg}$, the first inequality follows since VCG is truthful, and the second inequality follows from the first part of

Lemma 3. Since p_j is position j's payment function in the next-price position auction, $\alpha_j(v_i - \tilde{b}_{(j+1)}) = \alpha_j(v_i - p_j(\tilde{\mathbf{b}}))$. Therefore

$$\alpha_k(v_i - p_k(\mathbf{b}(\mathbf{v}))) \geq \alpha_j(v_i - p_j(\tilde{\mathbf{b}})).$$

Hence, player i does not gain from participating directly in the auction.

Finally we show that $\mathbf{m}$ is individually rational. If all players choose the mediator then by the first part of Lemma 3 each player which uses the T-strategy will not pay more than his value. Consider the situation in which a subset of players, S, participate directly in the auction. Since the mediator submits the reported values on behalf of the other players, these other players will not pay more than their reported values. Hence a player which used the T-strategy will not pay more than his value. □

Remark: As shown in the proof part (b) is important for showing both that the Mediator 1 implements the VCG outcome function and that Mediator 1 is individually rational.

5 Implementing the VCG Outcome Function in General Position Auctions

In the previous section we discussed the implementation of the VCG outcome function in the next price position auction. In this section we provide a much larger class of position auctions in which it is possible to implement φ^{vcg} by individually rational mediators. We are about to give sufficient conditions for implementing φ^{vcg} by an individually rational mediator in position auctions. We need the following definitions:

Definition 5 (GLP position auctions) *A position auction, $\mathbb{G}$ is a generalized lower price (GLP) position auction if the payment of each player who has been assigned a position in K, is a function of the bids of players assigned to "lower" positions than his own. Formally, for every $j \in K$ and for every couple of bid profiles $\mathbf{b}^1, \mathbf{b}^2 \in \mathbf{B}$ such that $b^1_{(l)} = b^2_{(l)}$ for every $l > j$,*

$$p_j(\mathbf{b}^1) = p_j(\mathbf{b}^2).$$

Definition 6 (*VCG* cover) *A position auction* $\mathbb{G}$ *is a VCG cover if for every* $\mathbf{v} \in \mathbf{V}$ *there exists* $\mathbf{b} \in \mathbf{B}$ *such that* $\psi^{\mathbb{G}}(\mathbf{b}) = \varphi^{vcg}(\mathbf{v})$*, where* $\psi^{\mathbb{G}}(\mathbf{b}) = (s(\mathbf{b}), q(\mathbf{b}))$.

Definition 7 (Monotone position auctions) *A position auction* $\mathbb{G}$ *is monotone if* $p_j(\mathbf{b}) \geq p_j(\mathbf{b}')$ *for every* $j \in K$ *and for every* $\mathbf{b} \geq \mathbf{b}'$*, where* $\mathbf{b} \geq \mathbf{b}'$ *if and only if* $b_i \geq b'_i$ *for every* $i \in N$.

We are now able to show:

Theorem 4 *Let* $\mathbb{G}$ *be a position auction.*

(i) If the following conditions hold then there exists an individually rational mediator that implements φ^{vcg} *in* $\mathbb{G}$*:*

1. $\mathbb{G}$ *is a GLP position auction.*
2. $\mathbb{G}$ *is a VCG cover.*
3. $\mathbb{G}$ *is monotone.*

(ii) The set of Conditions 1-3 is minimal. That is, if any one of the conditions 1-3 is dropped, there exists a position auction, which satisfies the other two conditions, but φ^{vcg} *cannot be implemented by an individually rational mediator.*

The proof of part (ii) is given below. The proof of part (i) is given in Section 8. However, the following remarks about part (i) are important:
Remarks:

1. Theorem 4 applies in particular to next-price position auctions discussed in Section 4. However, it applies to many other interesting position auctions as will be shown later. Moreover, the mediator constructed for this general case is different from Mediator 1 used in the proof of Theorem 2.

2. The monotonicity condition is only used when proving that Mediator 2 is individually rational.

3. In this paper we assume that the allocation rule ranks players in positions in decreasing order of their bids. We actually prove a stronger result than stated in Theorem 4:all we need from the allocation rule of the position auction G in Theorem 4 is that if a single player changes her bid the relative order between the other players remain.

We now prove part (ii) of Theorem 4.

Proof of Part (ii) of Theorem 4:

It is easy to see that if $\mathbb{G}$ is not a VCG cover, then implementing φ^{vcg} by truthful mediation is impossible. We prove the necessity of conditions 1 and 3 by Examples 2 and 3 respectively.

Example 2 (Condition 1 is necessary) Let $\mathbb{G} = \mathbb{G}(\alpha, \mathbf{p})$ be the following position auction. Let $N = \{1, 2, 3\}$, $K = \{1, 2\}$ and $\alpha = (2, 1)$. Let $p_1(\mathbf{b}) = \frac{b_{(1)}}{4}$ and $p_2(\mathbf{b}) = b_{(2)}$. It is immediate to see that the monotonicity condition is satisfied. We next show that $\mathbb{G}$ is a VCG cover. Let $\mathbf{v} \in \mathbf{V}$ be an arbitrary valuation vector. We need to find a bid profile $\mathbf{b}(\mathbf{v})$ such that $\psi^{\mathbb{G}}(\mathbf{b}(\mathbf{v})) = \varphi^{vcg}(\mathbf{v})$. Note that $p_1^{vcg}(\mathbf{v}) = \frac{v_{(2)}+v_{(3)}}{2}$ and $p_2^{vcg}(\mathbf{v}) = v_{(3)}$. We define the bid profile $\mathbf{b}(\mathbf{v})$ as follows.

Let $b_{\delta(\mathbf{v},3)}(\mathbf{v}) = \frac{v_{(3)}}{2}$, $b_{\delta(\mathbf{v},2)}(\mathbf{v}) = v_{(3)}$ and $b_{\delta(\mathbf{v},1)}(\mathbf{v}) = 2v_{(2)} + 2v_{(3)}$.

By the construction of $\mathbf{b}(\mathbf{v})$, $s(\mathbf{b}(\mathbf{v}), i) = s(\mathbf{v}, i)$ for $i = 1, 2, 3$. In addition observe that $p_j(\mathbf{b}(\mathbf{v})) = p_j^{vcg}(\mathbf{v})$ for $j = 1, 2$. Therefore $\psi^{\mathbb{G}}(\mathbf{b}(\mathbf{v})) = \varphi^{vcg}(\mathbf{v})$. Since $\mathbf{v}$ is arbitrary, $\mathbb{G}$ is a VCG cover. Obviously $\mathbb{G}$ is not a GLP position auction. Suppose in negation that there exists an individually rational mediator $\mathbf{m}$, which implements the VCG outcome function in $\mathbb{G}$. Consider the following vector of valuations $\mathbf{v} = (12, 10, 8)$. If all players use the mediator, then player 2 (with valuation 10) gets position 2, pays 8, and therefore her utility is $1(10 - 8) = 2$. Player 2 can always bid more than the other players, and by that cause some other player to be positioned second; therefore, since the mediator is required to be individually rational it must be that the mediator submits not more than 12 on behalf of both players 1 and 3. But then player 2 can bid 13, and win the first position; therefore, if both players, 1 and 3, use the T-strategy, by bidding 13 player 2's utility will be $2(10 - \frac{13}{4}) > 8$. This contradicts that $\mathbf{m}$ is an individually rational mediator that implements the VCG outcome function in $\mathbb{G}$.$\diamond$

Example 3 (Condition 3 is necessary) Let $\mathbb{G} = \mathbb{G}(\alpha, \mathbf{p})$ be the following position auction. Let $n = 4$, $m = 3$, $\alpha = (100, 10, 1)$, $p_1(\mathbf{b}) = b_{(2)} - b_{(3)}$ and $p_2(\mathbf{b}) = \frac{b_{(3)}+b_{(4)}}{2}$, and $p_3(\mathbf{b}) = b_{(4)}$. Notice that $\mathbb{G}$ is not monotone. Observe that $\mathbb{G}$ is a GLP position auction.

We next show that $\mathbb{G}$ is a VCG cover. Let $\mathbf{v} \in V$ be an arbitrary valuation vector. We need to find a bid profile $\mathbf{b}(\mathbf{v})$ such that $\psi^{\mathbb{G}}(\mathbf{b}(\mathbf{v})) = \varphi^{vcg}(\mathbf{v})$. Note that $p_1^{vcg}(\mathbf{v}) = \frac{90v_{(2)}+9v_{(3)}+v_{(4)}}{100}$, $p_2^{vcg}(\mathbf{v}) = \frac{9v_{(3)}+v_{(4)}}{10}$ and $p_3^{vcg}(\mathbf{v}) = v_{(4)}$. We define the bid profile $\mathbf{b}(\mathbf{v})$ recursively:

Let $b_{\delta(\mathbf{v},4)}(\mathbf{v}) = p_3^{vcg}(\mathbf{v})$, $b_{\delta(\mathbf{v},3)}(\mathbf{v}) = 2p_2^{vcg}(\mathbf{v}) - b_{\delta(\mathbf{v},4)}(\mathbf{v})$, $b_{\delta(\mathbf{v},2)}(\mathbf{v}) = b_{\delta(\mathbf{v},3)} + p_1^{vcg}(\mathbf{v})$ and $b_{\delta(\mathbf{v},1)}(\mathbf{v}) = b_{\delta(\mathbf{v},2)}(\mathbf{v}) + 1$.

By Lemma 3 and by the construction of $\mathbf{b}(\mathbf{v})$, $s(\mathbf{b}(\mathbf{v}), i) = s(\mathbf{v}, i)$ for $i = 1, ..., 4$. In addition observe that $p_j(\mathbf{b}(\mathbf{v})) = p_j^{vcg}(\mathbf{v})$ for $j = 1, 2, 3$. Therefore $\psi^{\mathbb{G}}(\mathbf{b}(\mathbf{v})) = \varphi^{vcg}(\mathbf{v})$. Since $\mathbf{v}$ is arbitrary, $\mathbb{G}$ is a VCG cover.

Suppose in negation that there exists an individually rational mediator $\mathbf{m}$, that implements the VCG outcome function in $\mathbb{G}$. Consider the following vector of valuations $\mathbf{v} = (14, 12, 14, 1)$. Suppose players 1,3 and 4 use the T-strategy. We will show that player 2 is better off participating directly in the auction, which will contradict that $\mathbf{m}$ implements the VCG outcome function in $\mathbb{G}$. If player 2 (with valuation 12) uses the T-strategy, then she is assigned to position 3 and pays 1, and therefore her utility is $1(12-1) = 11$. Suppose player 2 is the only player that participates directly in the auction. Let b_i, $i = 1, 2, 3, 4$ be the bids submitted to the auction (b_i for $i = 1, 3, 4$ are the bids submitted by the mediator). If $b_i = 0$ for some $i \in \{1, 3, 4\}$, then by bidding $b_2 = 0.5$, player 2 gains at least position 3, pays at most 0.5, and therefore strictly improves his utility; this contradicts that $\mathbf{m}$ implements φ^{vcg}. Suppose $b_i > 0$ for every $i \in \{1, 3, 4\}$. We distinguish between the following cases, while in each case the idea is to derive constraints about the possible bids an individually rational mediator can submit on behalf of at least one of the players that use the T-strategy, and show that under these constraints player 2 has a beneficial deviation:

1. $b_1 \geq b_3 \geq b_4$. We first show that in order for $\mathbf{m}$ to be individually rational it must be the case that $b_4 \leq 1$. If $b_4 > 1$, then by bidding $b_2 = \frac{b_4+1}{2}$, $q_4(\mathbf{b}) > 1$, which contradicts that $\mathbf{m}$ is individually rational. If $b_4 < 1$, then by letting $b_2 = b_4$, $q_2(\mathbf{b}) < 1$ and since $s(\mathbf{b}, 2) \in K$, player 2's utility is at least $1(12 - q_2(\mathbf{b})) > 11$. Suppose $b_4 = 1$. If $b_2 < b_4$ then $q_1(\mathbf{b}) = b_3 - b_4$. Therefore since $\mathbf{m}$ is individually rational, $b_3 - b_4 \leq 14$. Hence, $b_3 \leq 15$. Let $b_2 = b_3$. Therefore, by the priority rule $\tilde{\gamma}$, $s(\mathbf{b}, 2) = 2$; hence, $q_2(\mathbf{b}) \leq \frac{15+1}{2}$. Therefore player 2's utility is at least $10(12 - 8) > 11$.

2. $b_1 \geq b_4 > b_3$. In order for $\mathbf{m}$ to be individually rational it must be the case that $b_3 \leq 1$. If $b_3 > 1$, then by letting $b_2 = 1$ we have that $q_4(\mathbf{b}) = \frac{b_3+b_2}{2} > 1$, which contradicts that $\mathbf{m}$ is individually rational. If $b_2 < b_3$ then $q_1(\mathbf{b}) = b_4 - b_3$. Therefore, since $\mathbf{m}$ is individually rational, $b_4 - b_3 \leq 14$. Hence, $b_4 \leq 15$. Let $b_2 = b_4$. Therefore, by the

priority rule $\tilde{\gamma}$, $s(\mathbf{b}, 2) = 2$; hence, $q_2(\mathbf{b}) \leq \frac{15+1}{2}$. Therefore player 2's utility is at least $10(12-8) > 11$.

3. $b_4 > b_1 \geq b_3$. In order for $\mathbf{m}$ to be individually rational it must be the case that $b_3 \leq 1$. If $b_3 > 1$, then by letting $b_2 > b_4$, $s(\mathbf{b}, 4) = 2$; therefore $q_4(\mathbf{b}) = \frac{b_1+b_3}{2} > 1$, which contradicts that $\mathbf{m}$ is individually rational. Suppose $b_2 < b_3$. Therefore $q_4(\mathbf{b}) = b_1 - b_3$. Hence, since $\mathbf{m}$ is individually rational, $b_1 \leq 2$. Let $b_2 = 3$. Therefore, by the priority rule $\tilde{\gamma}$, $s(\mathbf{b}, 2) \in \{1, 2\}$; since $q_2(\mathbf{b}) \leq 3$, player 2's utility is at least $10(12-3) > 11$.

4. $b_3 > b_1 \geq b_4$. From the arguments of the first case and the symmetry of players 1 and 3 we obtain that $b_1 \leq 15$ and $b_4 \leq 1$. Let $b_2 = \frac{b_3+b_1}{2}$. Hence $s(\mathbf{b}, 2) = 2$; therefore $q_2(\mathbf{b}) \leq \frac{15+1}{2} = 8$. Hence, player 2's utility is at least $10(12-8) > 12$.

The cases $b_3 \geq b_4 > b_1$ and $b_4 > b_3 \geq b_1$ are similar to cases 2 and 3 respectively. We obtained that player 2 benefits from participating directly in the auction, which contradicts that $\mathbf{m}$ is an individually rational mediator which implements φ^{vcg} in $\mathbb{G}$.$\diamond$

This completes the proof of part 2 of Theorem 4.$\square$

To summarize, we have given a minimal set of conditions, which is sufficient for transforming a large class of position auctions to the VCG position auction by truthful mediation.

Conditions 1 and 3 in Theorem 4 are easy to verify. However verifying condition 2 is more challenging. In Theorem 5 we derive two sufficient conditions for a GLP position auction to be a VCG cover. We need a few notations.

Let $\mathbf{Z} = \{(z_1, ..., z_n) : z_1 \geq z_2 \geq \cdots \geq z_n \geq 0\}$ be the set of all ordered vector of bid profiles. Let p_j be a position payment function for some $j \in K \cup \{m+1\}$. Recall that $\tilde{p}_j$ is the real valued function over $\mathbf{Z}$ such that for every bid profile $\mathbf{b}$ $p_j(\mathbf{b}) = \tilde{p}_j(\mathbf{b}^*)$. Let

$$T(j) = \{k \in N | \forall \mathbf{z} \in \mathbf{Z} \quad \tilde{p}_j(\mathbf{z}) = \tilde{p}_j(\hat{z}_k, \mathbf{z}_{-k}) \quad \forall \hat{z}_k \quad \text{such that} \quad (\hat{z}_k, \mathbf{z}_{-k}) \in \mathbf{Z}\},$$

and let $D(j) = N \setminus T(j)$. In addition let $D(m+1) = \phi$.

$D(j)$ can be interpreted as the set of coordinates for which $\tilde{p}_j$ is a function of. For example, in the next-price auction $D(j) = \{j+1\}$, and in the VCG position auction $D(j) = \{j+1, j+2, ..., m+1\}$. For any finite set of integers $S \subseteq \{1, ..., n\}$ let $l(S)$ and $h(S)$ denote the smallest and largest integer in S respectively. For $S = \phi$ we define $l(S) = h(S) = n+1$. The proof of the following is given in Section 8.

Theorem 5 *Let $G = G(\alpha, \mathbf{p})$ be a GLP position auction. If the following conditions holds then G is a VCG cover.*

1. *For every $\mathbf{z} \in Z$, for every $j \in K$, and for every $c \in [\tilde{p}_{j+1}(\mathbf{z}), \infty)$ there exists $\mathbf{z}' \in Z$, in which $z'_k = z_k$ for every $k \geq l(D(j+1))$, such that $\tilde{p}_j(\mathbf{z}') = c$.*
2. *For every $j \in K$ $\tilde{p}_j(\cdot)$ is continuous and strictly increasing in every coordinate k such that $k \in D(j) \setminus \cup_{t=j+1}^{m} D(t)$.*

In the following examples we apply Theorems 4 and 5 to interesting classes of position auctions, *Generalized next-price position auctions*[21], and *Weighted next-price position auctions*. In addition, note that it follows immediately from Theorem 5 that the position auction in Example 3 is a VCG cover.

Definition 8 (Generalized next-price position auctions) *A position auction $\mathbb{G}(\alpha, \mathbf{p})$ is called a generalized next-price position auction if the payment scheme $\mathbf{p}$ is of the following form. For every $j \in K$ and for every $\mathbf{b} \in B$ $p_j(\mathbf{b}) = b_{(l(j))}$, where $l(j)$ is an integer such that $l(j) > j$.*[22]

We show:

Theorem 6 *Let $\mathbb{G}$ be a generalized next-price position auction. There exists an individually rational mediator that implements φ^{vcg} in $\mathbb{G}$ if and only if the following two conditions hold:*
(i) $l(j+1) > l(j)$ for $j = 1, ..., m-1$.
(ii) $l(m) \leq n$.[23]

Proof. Let $\mathbb{G} = \mathbb{G}(\alpha, \mathbf{p})$ be a generalized next-price position auction. We first show that if conditions (i) or (ii) are not satisfied then $\mathbb{G}$ is not

[21] Generalized next-price position auctions are not used today in practice but are recommended to sellers under certain circumstances [20, 25]

[22] Recall that $b_{(j)} = 0$ for every $j > n$.

[23] The requirement that $l(m) \leq n$, is that the payment will depend on some player's bid, and not be identically zero.

a VCG cover, and therefore the only if part immediately holds. Suppose that (ii) is not satisfied, i.e. $l(m) > n$. Let $\mathbf{v} \in V$. By (3) we have that $p_m^{vcg}(\mathbf{v}) = v_{(m+1)} > 0$. However $p_m(\mathbf{b}) = 0$ for every $\mathbf{b} \in B$ since $l(m) > n$. Therefore, $\mathbb{G}$ is not a VCG cover. Suppose that only (i) is satisfied. That is $l(m) \leq n$ but $l(j+1) \leq l(j)$ for some $j \in \{1, ..., m-1\}$. Let $\mathbf{v} \in V$ be a valuation profile such that $v_{(1)} > v_{(2)} > \cdots > v_{(n)}$. Therefore by Lemma 3, $p_j^{vcg}(\mathbf{v}) > p_{j+1}^{vcg}(\mathbf{v})$. Suppose in negation that $\mathbb{G}$ is a VCG cover. Therefore there exists a bid profile $\mathbf{b}$ such that $\psi^{\mathbb{G}}(\mathbf{b}) = \varphi^{vcg}(\mathbf{v})$. We obtained $b_{(l(j))} = p_j(\mathbf{b}) > p_{j+1}(\mathbf{b}) = b_{(l(j+1))}$ which contradicts that $l(j+1) \leq l(j)$.

To prove the if part, suppose conditions (i) and (ii) hold. Notice that conditions 1 and 3 in Theorem 4 are satisfied. Note that conditions 1 and 2 in Theorem 5 are also satisfied and therefore condition 2 in Theorem 4 is also satisfied. This completes the proof.□

Definition 9 (Weighted next-price position auctions) *A position auction $\mathbb{G}(\alpha, \mathbf{p})$ is called a weighted next-price position auction if the payment scheme $\mathbf{p}$ is of the following form. For every $j \in K$ and for every $\mathbf{b} \in B$, $p_j(\mathbf{b}) = \frac{b_{(j+1)}}{c_j}$, where $c_j \geq 1$.*[24]

We show:

Theorem 7 *Let $\mathbb{G}$ be a weighted next-price position auction with the weights $c_1, c_2, ..., c_m$. There exists an individually rational mediator that implements φ^{vcg} in $\mathbb{G}$ if and only if $c_1 \geq \cdots \geq c_m$.*

Proof. Let $\mathbb{G} = \mathbb{G}(\alpha, \mathbf{p})$ be a weighted next-position auction with the weights $c_1, c_2, ..., c_m$ where $c_j \geq 1$ for every $j = 1, ..., m$. We first prove the only if part. Let $\hat{j} < m$ be such that $c_{\hat{j}} < c_{\hat{j}+1}$. It is enough to show that $\mathbb{G}$ is not a VCG cover. Suppose in negation that $\mathbb{G}$ is a VCG cover. Let $\mathbf{v}$ be a vector profile in which $v_1 = v_2 = \cdots = v_n$. Therefore $p_j^{vcg}(\mathbf{v}) = p_{j+1}^{vcg}(\mathbf{v})$ for every $j = 1, ..., m-1$ by Lemma 3. In addition $s(v, i) = i$ for every $i \leq m$ by the priority rule. Since $\mathbb{G}$ is a VCG cover there exists $\mathbf{b}$ such that $\psi^{\mathbb{G}}(\mathbf{b}) = \varphi^{vcg}(\mathbf{v})$. Therefore $b_1 \geq b_2 \geq \cdots \geq b_m$ and $b_m \geq b_j$ for every $j > m$, since the allocations under $\mathbf{b}$ and under $\mathbf{v}$ are identical. In addition $p_j(\mathbf{b}) = p_{j+1}(\mathbf{b})$ for every $j = 1, ..., m-1$. Therefore

$$p_{\hat{j}}(\mathbf{b}) = \frac{b_{(\hat{j}+1)}}{c_{\hat{j}}} > \frac{b_{(\hat{j}+2)}}{c_{\hat{j}+1}} = p_{\hat{j}+1}(\mathbf{b}),$$

[24]The requirement that $c_j \geq 1$ is consistent with our assumption that players do not pay more than they bid.

which contradicts that $p_{\hat{j}}(\mathbf{b}) = p_{\hat{j}+1}(\mathbf{b})$.

Notice that conditions 1 and 3 in Theorem 4 are satisfied. Note that conditions 1 and 2 in Theorem 5 are also satisfied and therefore condition 2 in Theorem 4 is also satisfied. This completes the proof. □.

6 Self-Price Position Auctions

Let $\mathbb{G}$ be a self-price position auction as described in section 2. In example 1 we showed that when there is one position and two players, the VCG outcome function is implemented by an individually rational mediator in this auction. The proof in this example can be easily generalized to show that the VCG outcome function can be implemented by an individually rational mediator in a self-price position auction, in which there is one position and an arbitrary number of players, $n \geq 2$.

Next we show that it is impossible to implement the VCG outcome function, even by a non individually rational mediator, in a self-price position auction which has more than one position ($m > 1$).

Theorem 8 *Let $\mathbb{G}$ be a self-price position auction with more than one position. There is no mediator that implements the VCG outcome function in $\mathbb{G}$.*

Proof. Let $\mathbf{v} \in V$ be the following valuation profile. $v_n = 10$ and $v_1 = v_2 = \cdots = v_{n-1} = 5$. The VCG outcome function assigns to this v an allocation, in which player n receives position 1 and player 1 receives position 2. The VCG payments of players n and 1 are both equal to 5. In order to implement such an outcome, a mediator must bid 5 on behalf of player n (so that this player pays 5), and it must bid less than 5 on behalf of any other player, otherwise, by the priority rule, player n does not receive position 1. In particular, even if player 1 is assigned to position 2, she will pay less than 5. Hence, no mediator can implement the VCG outcome function in $\mathbb{G}$.□

The impossibility result in Theorem 8 follows from the ties. It remains true even for random priority rules. In contrast we give below a positive result for self-price position auctions with the first arrival rule, in which ties are impossible.

As discussed in Section 3.1, the fixed and random priority rules are just convenient ways to model the first-arrival rule, which is common in practice.

When one attempts to directly model position auctions that use the first-arrival rule without these modeling choices he tackles a lot of modeling problems. In particular, it is not clear how to model a position auction with the first-arrival rule as a game with incomplete information. To do this, one has to allow a player not only to submit a bid but also to decide about the time of the bid. This raises a lot of additional modeling problems, such as determining the relationship between the time a player decides to submit a bid and the time in which this bid is actually recorded. Nevertheless, we next analyze mediators in position auctions, which use the first-arrival rule. We will define ex post equilibrium and the notion of implementation by truthful mediation without explicitly modeling well-defined games. We will show that in this case there is a way to implement the VCG outcome function in a self-price position auction. In particular, we will find an individually rational mediator that does the job.

Let $\mathbb{G}$ be a position auction with the first-arrival rule. Every mediator for $\mathbb{G}$ has the ability to determine the order in which the bids he submits on behalf of the players are recorded; He can just submit the bids sequentially, waiting for a confirmation before submitting the next bid. We need the following notations.

Every order of bidding can be described by some $\gamma \in \Gamma$; i bids before k if and only if $\gamma_i < \gamma_k$. Hence, an order of bids can serve as a priority rule. For every order of bids γ and a vector of bids $\mathbf{b}$ we define $s(\mathbf{b}, \gamma, i)$ as the position assigned to i. We denote the payment of i when the vector of bids is b and the order of bidding is γ by $q_i(\mathbf{b}, \gamma) = p_{s(\mathbf{b},\gamma,i)}(b)$, and we denote $u_i(v_i, \mathbf{b}, \gamma)$ the utility of i.

A mediator for $\mathbb{G}$ should determine the bids of the players who use its services and also the order of bids as a function of the reported types. However, all mediators discussed in this section will use the same rule to determine the order of bids: If all players report the vector of types $\hat{\mathbf{v}}$, the mediator uses the order of bids $\gamma^{\hat{\mathbf{v}}}$, which is defined as follows: $\gamma_i^{\hat{\mathbf{v}}} < \gamma_k^{\hat{\mathbf{v}}}$ if and only if $\hat{\mathbf{v}}_i > \hat{\mathbf{v}}_k$, or $\hat{\mathbf{v}}_i = \hat{\mathbf{v}}_k$ and $i < k$. For example, if $n = 3$ and the reported types are $\hat{\mathbf{v}} = (6, 7, 6)$, then $\gamma^{\hat{\mathbf{v}}} = (2, 1, 3)$. If only a strict subset of the players use the mediator's services, the mediator applies the same order of bids rule to this subset. A mediator for a position auction with the first arrival rule is therefore defined by a vector $\mathbf{m} = (\mathbf{m}_S)_{S \subseteq N}$. However, such a mediator is called a *s-mediator* (scheduling mediator) in order to stress the fact that it determines not only the bids but also the order of bids. To summarize: If all players use the directed mediator $\mathbf{m}$, and the reported bids are $\hat{\mathbf{v}}$, then the directed mediator bids $\mathbf{m}_N(\hat{\mathbf{v}})_i$ on behalf of i, i receives the position $s(\hat{\mathbf{v}}, \gamma^{\hat{\mathbf{v}}}, i)$, and pays $q_i(m_N(\hat{\mathbf{v}}), \gamma^{\hat{\mathbf{v}}})$. If only the subset S uses the mediator's services, the reported types are $\hat{\mathbf{v}}_S$, and the other players bid directly $\mathbf{b}_{-S}$ then the actual order of bids is not uniquely determined. If this order is γ then the position of $i \in N$ is $s(\mathbf{b}, \gamma, i)$, and its payment is $q_i(\mathbf{b}, \gamma)$, where $\mathbf{b} = (\mathbf{m}_S(\hat{\mathbf{v}}_S), \mathbf{b}_{-S})$. In particular, if every player is using the T-strategy and the players' profile of types is $\mathbf{v}$, then the outcome generated by the directed mediator is

$$\psi^{\mathbf{m}}(\mathbf{v}) = (\mathbf{s}(\mathbf{v}, \gamma^{\mathbf{v}}), \mathbf{q}(\mathbf{m}_N(\mathbf{v}), \gamma^{\mathbf{v}})).$$

But why should the players use the T-strategy? Assume all players but i use the T-strategy. If player i deviates from the T-strategy by reporting a false type to the directed mediator, the resulting outcome is well-defined. On the other hand, when this player sends a bid directly to the auctioneer, the resulting outcome is not clear, because the order of bids is not clear. [25]

A desired directed mediator would be one that no player would want to deviate from the T-strategy independently of the order in which the bids are recorded because of his deviation. More specifically:

Definition: Let $\mathbb{G}$ be a position auction with the first-arrival rule, and let $\mathbf{m}$ be a directed mediator for $\mathbb{G}$. The T-strategy profile is an *ex post equilibrium* with respect to $\mathbf{m}$ if for every player i and type v_i, and for every vector of types of the other players, $\mathbf{v}_{-i}$. the following two conditions hold:

F1: i is not better off when he gives the directed mediator the right of play and reports a false type. That is, for every $\hat{v}_i \in V_i$

$$u_i(v_i, \mathbf{m}_N(v_i, \mathbf{v}_{-i}), \gamma^{(v_i, v_{-i})}) \geq u_i(v_i, \mathbf{m}_N(\hat{v}_i, \mathbf{v}_{-i}), \gamma^{(\hat{v}_i, v_{-i})}).$$

F2: i is not better off when he bids directly independently of the resulting order of recorded bids. That is, for every $b_i \in B_i$, and for every $\gamma \in \Gamma$, which is consistent with the order of bids of members of $N \setminus \{i\}$ resulting from the vector of types v_{-i},

$$u_i(v_i, \mathbf{m}_N(v_i, \mathbf{v}_{-i}), \gamma^{(v_i, \mathbf{v}_{-i})}) \geq u_i(v_i, b_i, \mathbf{m}_{N \setminus \{i\}}(\mathbf{v}_{-i}), \gamma).$$

The notion of individually rational directed mediators is analogously defined:

Definition: Let $\mathbb{G}$ be a position auction with the first-arrival rule. A directed mediator for $\mathbb{G}$ is *individually rational*, if for every player, using the T-strategy guarantees a non-negative level of utility.

[25] It is clear however, that the resulting order γ is consistent with the well-defined order of bids of $N \setminus \{i\}$.

Formally, a directed mediator $\mathbf{m}$ for $\mathbb{G}$ is *individually rational*, if for every player i, for every subset $S \subseteq N$ such that $i \in S$, for every $\mathbf{v}_S$, and for every $\mathbf{b}_{-S}$, $u_i(v_i, \mathbf{m}_S(\mathbf{v}_S), \mathbf{b}_{-S}, \gamma) \geq 0$ for every $\gamma \in \Gamma$, which is consistent with the standard order of bids of S determined by the mediator when the reported types are $\mathbf{v}_S$.

The notion of implementation by truthful mediation remains as before: The directed mediator $\mathbf{m}$ implements the VCG outcome function in $\mathbb{G}$ if $\psi^{\mathbf{m}} = \varphi^{vcg}$.

Our previous results remain true for directed mediators for position auctions with the first arrival rule. Next we show that in contrast to Theorem 8, it is possible to implement the VCG outcome function in every self-price position auction with the first-arrival rule.

Theorem 9 *Let $\mathbb{G} = \mathbb{G}(\alpha, \mathbf{p})$ be the self-price position auction with the first arrival rule. There exists an individually rational directed mediator that implements the VCG outcome function in $\mathbb{G}$.*

Proof. We define a directed mediator $\mathbf{m}$ as follows:

For every $\mathbf{v} \in V$, $\mathbf{m}_N(\mathbf{v}) = \mathbf{b}(\mathbf{v})$, where $\mathbf{b}(\mathbf{v})$ is the bid profile defined as follows: $\mathbf{b}_i(\mathbf{v}) = p^{vcg}_{s(\mathbf{v},\gamma^{\mathbf{v}},i)}(\mathbf{v})$ for every i such that $1 \leq s(\mathbf{v}, \gamma^{\mathbf{v}}, i) \leq m$, and $\mathbf{b}_i(\mathbf{v}) = \frac{p^{vcg}_{s(\mathbf{v},\gamma^{\mathbf{v}},m)}(\mathbf{v})}{2}$ for every i such that $s(\mathbf{v}, \gamma^{\mathbf{v}}, i) = m + 1$.

For every strict subset, $S \subset N$ let $\mathbf{m}_S(\mathbf{v}_S) = \mathbf{v}_S$.

It is easily shown now, that because the order of bids is determined by $\mathbf{v}$ and not by the bids, $\psi^{\mathbf{m}}(\mathbf{v}) = \varphi^{vcg}(\mathbf{v})$ for every $\mathbf{v} \in V$.

This implies that it is not beneficial to report a false value to the mediator because of the fact that telling the truth is a dominant strategy in the VCG position auction with the fixed priority rule. Suppose player i bids directly in the auction and all other players but i choose the T-strategy. Let v_{-i} be the values of all other players. Let b_i be player i's bid, and let $\tilde{b} = (b_i, v_{-i})$. Let $\tilde{\gamma} \in \Gamma$ be a priority order consistent with order of bids sent by the directed mediator $\mathbf{m}$. Let $j = s(\tilde{b}, \tilde{\gamma}, i)$. W.l.o.g. we assume that $j \in K$. Since the mediator bids on behalf of all other players their reported values, player i will pay at least $v_{(j+1)}$. If player i would have deviated to position j in the VCG position auction with the fixed order $\tilde{\gamma}$, then by Lemma 3 part 1, her payment would have been at least $v_{(j+1)}$. Therefore direct participation in the position auction is not beneficial for i.

The mediator is individually rational since $p^{vcg}_j(\mathbf{b}) \leq b_{(j)}$ for every j.□

7 Position Auctions with Quality Factors

Every position auction is player-anonymous except for ties. However, companies like Google express preferences over players by introducing quality factors (see e.g., [26]); Player i has a fixed *quality factor* $\beta_i > 0$. Let $\mathbb{G} = \mathbb{G}(\alpha, \mathbf{p})$ be a position auction , and let $\beta = (\beta_1, \cdots, \beta_n)$ be a vector of quality factors. we define a new auction, $\mathbb{G}(\beta, \alpha, \mathbf{p})$ as follows:

If the players send the vector of bids $\mathbf{b} = (b_1, b_2, \cdots, b_n)$ the auctioneer choose the allocation $\mathbf{s}_\beta(\mathbf{b}) = \mathbf{s}(\beta \mathbf{b})$, where $\beta \mathbf{b} = (\beta_1 b_1, \beta_2 b_2, \cdots, \beta_n b_n)$. If i receives position j, that is $s(\beta \mathbf{b}, i) = j$, then i pays $\frac{1}{\beta_i} p_j(\beta \mathbf{b})$ per click. Every such auction, $\mathbb{G}(\beta, \alpha, p)$ is called a β-position auction. The β-VCG position auction is $\mathbb{G}(\beta, \alpha, \mathbf{p}^{vcg})$. This mechanism chooses the allocation of the weighted VCG mechanism (see e.g., [23]) with the vector of weights β. Every player pays his standard weighted VCG payment. As observed by Varian [26], every position auction with quality factors can be reformulated as a position auction without quality factors by redefining bidders' valuations to be the product of their original valuations and their quality factors, i.e. $v_i \beta_i$ is considered to be player i's valuation. Based on this the following observation holds:

Observation 10 *Let $\mathbb{G} = \mathbb{G}(\alpha, \mathbf{p})$ be a position auction, and let $\beta = (\beta_1, \beta_2, \cdots, \beta_n)$ be a vector of quality factors. There exists an individually rational mediator that implements the β-VCG outcome function in $\mathbb{G}(\beta, \alpha, \mathbf{p})$ if and only if there exists an individually rational mediator that implements the VCG outcome function in $\mathbb{G}(\alpha, \mathbf{p})$.*

Hence, the β versions of all the theorems proved in previous sections hold.

8 The Remaining Proofs

Proof of Lemma 3:
1. Let $j \in K$. Note that by (3) $p_j^{vcg}(\mathbf{b})$ is a convex combination of $b_{(j+1)}, b_{(j+2)}, ..., b_{(m+1)}$. Therefore it never exceeds the maximal element in the sequence, $b_{(j+1)}$.
2. If $j = m$ then for every $\mathbf{b} \in B$

$$p_j^{vcg}(\mathbf{b}) = b_{(j+1)} \geq 0 = p_{j+1}^{vcg}(\mathbf{b}).$$

Suppose $j < m$. Hence, $j + 1 \in K$. Since $b_{(j+1)} \geq b_{(j+2)}$,

$$p_j^{vcg}(\mathbf{b}) \geq$$

$$\frac{b_{(j+2)}(\alpha_j - \alpha_{j+1})}{\alpha_j} + \frac{\sum_{k=j+2}^{m+1} b_{(k)}(\alpha_{k-1} - \alpha_k)}{\alpha_j}. \tag{4}$$

Note that the right-hand-side of (4) equals to

$$b_{(j+2)} - \sum_{k=j+2}^{m} \frac{\alpha_k}{\alpha_j}(b_{(k)} - b_{(k+1)}).$$

Since $\alpha_j > \alpha_{j+1}$,

$$p_j^{vcg}(\mathbf{b}) \geq$$

$$b_{(j+2)} - \sum_{k=j+2}^{m} \frac{\alpha_k}{\alpha_{j+1}}(b_{(k)} - b_{(k+1)}) = p_{j+1}^{vcg}(\mathbf{b}). \tag{5}$$

Obviously if $b_{(j+1)} = b_{(j+2)} = \cdots = b_{(m+1)}$ then $p_j^{vcg}(\mathbf{b}) = p_{j+1}^{vcg}(\mathbf{b})$. If $b_{(j+1)} = b_{(j+2)} = \cdots = b_{(m+1)}$ doesn't hold it implies that there is a strong inequality in either (4) or (5). This implies that $p_j^{vcg}(\mathbf{b}) > p_{j+1}^{vcg}(\mathbf{b})$. □

Proof of Part (i) of Theorem 4:

We will show that the following mediator, Mediator 2, is an individually rational mediator that implements the VCG outcome function in position auctions which satisfy conditions 1-3 in the theorem:

Mediator 2

(a) $\mathbf{m}_N(\mathbf{v}) = \mathbf{b}(\mathbf{v})$ *for every* $\mathbf{v} \in V$, *where* $\mathbf{b}(\mathbf{v})$ *is some bid profile such that* $\psi^{\mathbb{G}}(\mathbf{b}(\mathbf{v})) = \varphi^{vcg}(\mathbf{v})$

(b) *For every* i *and for every* $\mathbf{v}_{-i} \in \mathbf{V}_{-i}$, *let* $\mathbf{v}^i = (\mathbf{v}_{-i}, M(\mathbf{v}_{-i}))$,*where* $M(\mathbf{v}_{-i}) = \epsilon + \max_{j \neq i} v_j$ *for some* $\epsilon > 0$.

$\mathbf{m}_{N\setminus\{i\}}(\mathbf{v}_{-i}) = \mathbf{b}_{-i}(\mathbf{v}^i)$ *for every* $i \in N$ *and every* $\mathbf{v}_{-i} \in \mathbf{V}_{-i}$, *where* $\mathbf{b}(\mathbf{v}^i)$ *is some bid profile such that* $\psi^{\mathbb{G}}(\mathbf{b}(\mathbf{v}^i)) = \varphi^{vcg}(\mathbf{v}^i)$.

(c) *For every* $S \subseteq N$ *such that* $1 \leq |S| \leq n-2$, $\mathbf{m}_S(\mathbf{v}_S) = \mathbf{v}_S$ *for every* $\mathbf{v}_S \in \mathbf{V}_S$.

Let $\mathbb{G} = \mathbb{G}(\alpha, \mathbf{p})$ be a position auction which satisfies conditions 1-3. Let $\mathbf{m}$ be the mediator defined above. First note that since $\mathbb{G}$ is a VCG cover, (a) and (b) are well defined. We have to show that $\mathbf{m}$ implements φ^{vcg} in $\mathbb{G}$. By (a), $\varphi^{\mathbf{m}} = \varphi^{vcg}$. We proceed to show that the T-strategy profile is an ex post equilibrium. Note that by the truthfulness property of the VCG position auction, it is not beneficial for a player i to miss report her value to the mediator, given that all other players are using the T-strategy. Next we show that it is not beneficial for a single player $i \in N$ to participate in the auction directly, given that all other players are using the T-strategy. Assume player i is the only player that participates directly in the auction, that player i's type is v_i, and that he bids b_i. Let $\mathbf{v}_{-i}$ be a fixed profile of types of all other players, and set $\mathbf{v} = (v_i, \mathbf{v}_{-i})$.

At this case, $\mathbf{b}_{-i}(\mathbf{v}^i)$ is the vector of bids submitted by the mediator. Let $\tilde{\mathbf{b}} = (b_{-i}(\mathbf{v}^i), b_i)$ be the profile of bids reported to the auctioneer, and let $j = s(\tilde{\mathbf{b}}, i)$ be the position of i resulting from his deviation. Let $h = s(\mathbf{b}(\mathbf{v}), i)$ be the position of i if he does not deviate. We have to show that

$$\alpha_h(v_i - p_h(\mathbf{b}(\mathbf{v})) \geq \alpha_j(v_i - p_j(\tilde{\mathbf{b}})).$$

Because $\varphi^{\mathbf{m}} = \varphi^{vcg}$, it suffices to show that:

$$\alpha_h(v_i - p_h^{vcg}(\mathbf{v})) \geq \alpha_j(v_i - p_j(\tilde{\mathbf{b}})). \tag{6}$$

Because in the VCG position auction, the utility of a truth-telling agent is always non-negative, the above inequality, (6) holds trivially if $j = -1$ or $j = m + 1$, since in both cases $\alpha_j = 0$. Therefore, we can assume without loss of generality that $1 \leq j \leq m$. Let $b'_i \in \mathbf{B}_i$ be some bid for player i such that $b'_i = (\mathbf{v}_{-i}, b'_i)_{(j)}$.

Before we continue we need the following discussion: The allocation rule in the VCG auction is defined by the priority rule $\tilde{\gamma} = (1, ..., n)$. Let $VCG(\gamma)$ be the VCG position auction, in which the allocation rule is γ. The (position) payment scheme in $VCG(\gamma)$ is the same as in $VCG(\tilde{\gamma})$. We define $s(\mathbf{b}, i, \gamma)$ to be player i's position, when the bid profile is $\mathbf{b}$, and the priority rule is γ. Note that $s(\mathbf{b}, i) = s(\mathbf{b}, i, \tilde{\gamma})$.

Since $b'_i = (\mathbf{v}_{-i}, b'_i)_{(j)}$, there exists $\gamma \in \Gamma$ such that $s((\mathbf{v}_{-i}, b'_i), i, \gamma) = j$. Let $\hat{h} = s(\mathbf{v}, i, \gamma)$. Observe that $v_{(\hat{h})} = v_{(h)}$. Since players with identical values, which report truthfully have the same utility in the VCG position auction,

$$\alpha_h(v_{(h)} - p_h^{vcg}(\mathbf{v})) = \alpha_{\hat{h}}(v_{(\hat{h})} - p_{\hat{h}}^{vcg}(\mathbf{v})).$$

Since $v_{(h)} = v_i$,

$$\alpha_h(v_i - p_h^{vcg}(\mathbf{v})) = \alpha_{\hat{h}}(v_i - p_{\hat{h}}^{vcg}(\mathbf{v})).$$

Because of the truth-telling property of $VCG(\gamma)$

$$\alpha_{\hat{h}}(v_i - p_{\hat{h}}^{vcg(\gamma)}(\mathbf{v})) \geq \alpha_j(v_i - p_j^{vcg(\gamma)}(\mathbf{v}_{-i}, b'_i)).$$

Hence, since the payment schemes in VCG and $VCG(\gamma)$ are identical,

$$\alpha_h(v_i - p_h^{vcg}(\mathbf{v})) \geq \alpha_j(v_i - p_j^{vcg}(\mathbf{v}_{-i}, b'_i)).$$

Since the VCG position auction is a GLP position auction, and $(v_{-i}, b'_i)_{(l)} = v^i_{(l)}$ for every $l > j$,

$$\alpha_j(v_i - p_j^{vcg}(\mathbf{v}_{-i}, b'_i)) = \alpha_j(v_i - p_j^{vcg}(\mathbf{v}^i)).$$

Because $\mathbf{b}(\mathbf{v}^i) \in O(\mathbf{v}^i)$,

$$\alpha_j(v_i - p_j^{vcg}(\mathbf{v}^i)) = \alpha_j(v_i - p_j(\mathbf{b}(\mathbf{v}^i))).$$

Because $\mathbb{G}$ is a GLP position auction,

$$\alpha_j(v_i - p_j(\mathbf{b}(\mathbf{v}^i))) = \alpha_j(v_i - p_j(\tilde{\mathbf{b}})).$$

Hence, we proved (6).

Finally we show that $\mathbf{m}$ is individually rational. We have to show that each player, which uses the T-strategy doesn't pay more than his valuation. Let l be a player which uses the T-strategy. Assume player l's value is v_l. We will show that player l never pays more than v_l. If all other players choose the mediator, and report $\mathbf{v}_{-l}$, then since $\varphi^{\mathbf{m}} = \varphi^{vcg}$, player l's payment is $p_l^{vcg}(v_l, \mathbf{v}_{-l}) \leq v_l$. Assume not all other players are using the mediator. First consider the case, where $n-1$ players are using the mediator. Let $i \neq l$ be the only player i that participates directly in the auction. Let $\mathbf{v}_{-i}$ be the values reported to the mediator. Let b_i be player i's bid. Let $\tilde{\mathbf{b}} = (b_{-i}(\mathbf{v}^i), b_i)$ be the profiles of bids reported to the auctioneer, and let $j = s(\tilde{\mathbf{b}}, i)$ be the position of i resulting from his deviation. Let $h = s(\tilde{\mathbf{b}}, l)$ be the position of l resulting from i's deviation. We need to show that $p_h(\tilde{\mathbf{b}}) \leq v_l$. If $h \in \{m+1, -1\}$, $p_h(\tilde{\mathbf{b}}) = 0 < v_l$. Therefore we can assume without loss of generality that $1 \leq h \leq m$. We distinguish between the following cases:

- $h > j$. Observe that $s(\mathbf{b}(\mathbf{v}^i), l) = s(\tilde{\mathbf{b}}, l)$. Therefore, $s(\mathbf{v}^i, l) = h$. Hence, $\mathbf{v}^i_{(h)} = v_l$. We now show that $p_h(\tilde{\mathbf{b}}) \leq v_l$. Since $\mathbb{G}$ is a GLP position auction,

$$p_h(\tilde{\mathbf{b}}) = p_h(\mathbf{b}(\mathbf{v}^i)).$$

Because $\mathbf{b}(\mathbf{v}^i) \in O(\mathbf{v}^i)$,

$$p_h(\mathbf{b}(\mathbf{v}^i)) = p_h^{vcg}(\mathbf{v}^i).$$

From the first part of Lemma 3,

$$p_h^{vcg}(\mathbf{v}^i) \leq v^i_{(h+1)} \leq v^i_{(h)}.$$

Hence, $p_h(\tilde{\mathbf{b}}) \leq v^i_{(h)} = v_l$.

- $h < j$. Observe that $s(\mathbf{b}(\mathbf{v}^i), l) = h + 1$. Therefore $v^i_{(h+1)} = v_l$. Since $h < j$, $\tilde{\mathbf{b}} \leq \mathbf{b}(\mathbf{v}^i)$. Therefore, by the monotonicity condition

$$p_h(\tilde{\mathbf{b}}) \leq p_h(\mathbf{b}(\mathbf{v}^i)).$$

Because $\mathbf{b}(\mathbf{v}^i) \in O(\mathbf{v}^i)$,

$$p_h(\mathbf{b}(\mathbf{v}^i)) = p_h^{vcg}(\mathbf{v}^i).$$

From the first part of Lemma 3

$$p_h^{vcg}(\mathbf{v}^i) \leq v^i_{(h+1)} = v_l.$$

Therefore

$$p_h(\tilde{\mathbf{b}}) \leq v_l.$$

We showed that any player l, which uses the T-strategy does not pay more than his value, when a single player participates directly in the auction.

Consider the situation in which a subset of players, which contains more than a single player, participate directly in the auction. The mediator submits the reported values on behalf of the other players. Therefore, by our assumption that players never pay more than their own bid, each of these other players will not pay more than his reported values. In particular a player l will not pay more than his value. This completes the proof of Part 1 of Theorem 4. $\square$

Proof of Theorem 5:

First we prove the following lemma:

Lemma 11 *Let* $S \subset \{1, 2, ..., n\}$. *Let* a_1, a_2, c_1, c_2 *be real numbers such that* $a_1 < a_2$ *and* $c_1 < c_2$. *Let* $X_1 = X_2 = \cdots = X_n = [a_1, a_2]$. *Let* $f : X_1 \times X_2 \times \cdots \times X_n \to [c_1, c_2]$ *be a function with the following properties:* (*i*) f *is increasing and continuous in every coordinate* $i \in \{1, 2, ..., n\} \setminus S$. (*ii*) *For every* $i \in S$ *and for every* x_{-i}, $f(x_i, x_{-i}) = f(y_i, x_{-i})$ *for every* $x_i, y_i \in X_i$. (*iii*) f *is on* $[c_1, c_2]$. *Then for every* $c \in (c_1, c_2)$ *there exists* $\mathbf{x} = (x_1, x_2, ..., x_n) \in X$ *such that* $a_2 > x_n > x_{n-1} > \cdots > x_1 > a_1$ *and* $f(\mathbf{x}) = c$.

Proof. Note that w.l.o.g it is enough to prove the Lemma for $S = \phi$, since one can renumber the indices in $\{1, 2, ..., n\} \setminus S$ from 1 to $|\{1, 2, ..., n\} \setminus S|$, and in addition f is not a function of x_j for every $j \in S$.

Let $c \in (c_1, c_2)$. By the monotonicity and continuity in every coordinate of f there exists $0 < \epsilon < \frac{a_2 - a_1}{n+1}$ such that $f(a_1 + \epsilon, a_1 + 2\epsilon, ..., a_1 + n\epsilon) < c$ and $f(a_2 - n\epsilon, a_2 - (n-1)\epsilon, ..., a_2 - \epsilon) > c$. Let $g(\delta) = f(a_1 + \epsilon + \delta, a_1 + 2\epsilon + \delta, ..., a_1 + n\epsilon + \delta)$. Because g is a continuous function in $[0, a_2 - a_1 - (n+1)\epsilon]$, it obtains any value in $[g(0), g(a_2 - a_1 - (n+1)\epsilon)]$, and in particular it obtains the value c. Let δ' be such that $g(\delta') = c$. Therefore $f(a_1 + \epsilon + \delta', a_1 + 2\epsilon + \delta', ..., a_1 + n\epsilon + \delta') = c$, which completes the proof. □

Continue of the proof of Theorem 5:

Let $G = G(\alpha, \mathbf{p})$ be a GLP position auction, which satisfies conditions 1 and 2 in the hypothesis. We need to show that G is a VCG cover. Let $\mathbf{v} \in V$ be some valuation profile. It is enough to show that there exists $\mathbf{b}(\mathbf{v}) \in B$ such that $\varphi^G(\mathbf{b}(\mathbf{v})) = \varphi^{vcg}(\mathbf{v})$.

First notice by condition 1 that for every $j \in K$ $D(j) \neq \phi$ and $l(D(j)) < l(D(j+1))$.

We need the following notations. Let $L(j) = \{k \geq 1 : k < l(D(j))\}$ and let $\tilde{D}(j) = \{k : l(D(j)) \leq k < l(D(j+1))\}$. Note by the above that $\tilde{D}(j) \neq \phi$, and since G is a GLP position auction $L(j) \neq \phi$.

We begin by describing a process, which constructs a sequence of vectors $\mathbf{z}^m, \mathbf{z}^{m-1}, ..., \mathbf{z}^1 \in \mathbf{Z}$ satisfying for every $j \in K$ the following:

(i) $\tilde{p}_l(\mathbf{z}^j) = p_l^{vcg}(\mathbf{v})$ for every $l \geq j$, and

(ii) For every $j \in K$, if $p_j^{vcg}(\mathbf{v}) > p_{j+1}^{vcg}(\mathbf{v})$ then $\mathbf{z}_l^j > \mathbf{z}_{l+1}^j$ for every $l \in \tilde{D}(j)$, and

(iii) $\mathbf{z}_k^j > 0$ for every k.

Hence, for every $j \in K$, we will obtain that $\tilde{p}_j(\mathbf{z}^1) = p_j^{vcg}(\mathbf{v})$, and if $\tilde{p}_j(\mathbf{z}^1) > \tilde{p}_{j+1}(\mathbf{z}^1)$ then $\mathbf{z}_l^1 > \mathbf{z}_{l+1}^1$ for every $l \in \tilde{D}(j)$. In addition $\mathbf{z}_k^1 > 0$ for every $k = 1, \ldots, n$. From $\mathbf{z}^1$ we will eventually construct $\mathbf{b}(\mathbf{v})$.

First we construct $\mathbf{z}^m$. By (3) $p_m^{vcg}(\mathbf{v}) = \mathbf{v}_{(m+1)}$. Hence, $p_m^{vcg}(\mathbf{v}) > 0$. Let $\mathbf{z}' \in Z$ be such that $\tilde{p}_m(\mathbf{z}') = p_m^{vcg}(\mathbf{v})$. There exists such a $\mathbf{z}'$ by condition 1. Notice that there exists at least one coordinate $k \in D(m)$ such that $z'_k > 0$; Otherwise, consider the bid profile, in which m players bid some $\epsilon > 0$ such that $\epsilon < p_m^{vcg}(\mathbf{v})$, and all other players bid 0. In this bid profile the player in position m also pays $p_m^{vcg}(\mathbf{v})$ which is more than his bid contradicting the assumption that a player never pays more than his bid. By condition 2 there exists $\mathbf{z}'' \in \mathbf{Z}$ for which $z_k^{''} > 0$ for every $k \in D(m)$, $z_k^{''} \neq z_l^{''}$ for every $k, l \in D(m)$ and $\tilde{p}_m(\mathbf{z}^{''}) = \tilde{p}_m(\mathbf{z}')$. Let $\mathbf{z}^m \in \mathbf{Z}$ be the following vector: $z_k^m = z_k^{''}$ for every $k \leq h(D(m))$, and $z_k^m = \frac{z^{''}_{h(D(m))}}{2}$ for every $k > h(D(m)$. Hence, $\tilde{p}_m(\mathbf{z}^m) = p_m^{vcg}(\mathbf{v})$ and for every $k = 1, \ldots, n$ $\mathbf{z}_k^m > 0$.

For $j = m-1, ..., 1$ we construct $\mathbf{z}^j$ from $\mathbf{z}^{j+1}$ in the following way. We distinguish between two cases:

1. $p_j^{vcg}(\mathbf{v}) = p_{j+1}^{vcg}(\mathbf{v})$. Let $\mathbf{z}^j \in Z$ be a vector which satisfies: (i) $\tilde{p}_j(\mathbf{z}^j) = p_j^{vcg}(\mathbf{v})$, (ii) $z_k^j = z_k^{j+1}$ for every $k \geq l(D(j+1)$. Such a vector $\mathbf{z}^j$ exists by condition 1.

2. $p_j^{vcg}(\mathbf{v}) > p_{j+1}^{vcg}(\mathbf{v})$. Let $\mathbf{w} = (M, M, ..., M) \in Z$ for large enough M, such that the vector $\mathbf{y}^j = (\mathbf{w}_{L(D(j+1))}, \mathbf{z}^{j+1}_{-L(D(j+1))})$ satisfies $p_j(\mathbf{y}^j) > p_j^{vcg}(\mathbf{v})$. Such an M exists by conditions 1 and 2. Let $a^j = z^{j+1}_{l(D(j+1))}$, $b^j = M$, $c^j = p_{j+1}(\mathbf{z}^{j+1})$ and $d^j = p_j(\mathbf{y}^j)$. Let $f^j : [a^j, b^j]^{|\tilde{D}(j)|} \rightarrow [c^j, d^j]$ be defined by $f^j(x) = p_j(\mathbf{y}^j_{-\tilde{D}(j)}, x)$. By Lemma 11 there exist a vector $x^j = (x_1^j, ..., x^j_{|\tilde{D}(j)|})$ such that $b^j > x_1^j > x_2^j > \cdots > x^j_{|\tilde{D}(j)|} > a^j$ and $f^j(x^j) = p_j^{vcg}(\mathbf{v})$. Let $\mathbf{z}^j$ be defined as follows. $z_k^j = y_k^j$ for every $k \in L(D(j))$, $z^j_{k+l(D(j))-1} = x_k^j$ for every $k = 1, ..., |\tilde{D}(j)|$ and $z_k^j = z_k^{j+1}$ for every $k > h(D(j))$.

Let $\mathbf{z}$ be the following vector. $z_k = z_k^1$ for every $k \geq l(D(1))$. For every $k \in L(D(1))$ let z_k be a distinct coordinate larger than $z^1_{l(D(1))}$.

Let $\mathbf{b} = \mathbf{b}(\mathbf{v}) \in B$ be the following bid profile. For every i such that $s(\mathbf{v}, i) \in K$ let $b_i = z_{s(\mathbf{v},i)}$. For every i such that $s(\mathbf{v}, i) = m+1$ b'_i is a distinct coordinate in $\mathbf{z}_{\{m+1,\ldots,n\}}$. It remains to show that for every $j \in K \setminus m$, if $v_{(j)} > v_{(j+1)}$ then $\mathbf{b}_{\delta(\mathbf{b},j)} > \mathbf{b}_{\delta(\mathbf{b},j+1)}$. Suppose that $v_{(j)} > v_{(j+1)}$ for some $j \in K \setminus m$. To complete the proof we distinguish between the following two cases:

1. $j < l(D(1))$. By the construction of $\mathbf{z}$ from $\mathbf{z}^1$ and since $L(D(1)) \neq \phi$ we obtain that $\mathbf{b}_{\delta(\mathbf{b},j)} > \mathbf{b}_{\delta(\mathbf{b},j+1)}$.

2. $j \geq l(D(1))$. By the second part of Lemma 1 $p_1^{vcg}(\mathbf{v}) > p_2^{vcg}(\mathbf{v}) > \cdots > p_{j-1}^{vcg}(\mathbf{v}) > p_j^{vcg}(\mathbf{v})$. By condition 1 there exists some $1 \leq \hat{j} < j$ such that $j \in \tilde{D}(\hat{j})$. By the construction of $\mathbf{z}^{\hat{j}}$ from $\mathbf{z}^{\hat{j}+1}$, we obtained that $z_j^{\hat{j}} > z_{j+1}^{\hat{j}}$. Observe that $z_j = z_j^{\hat{j}}$ and $z_{j+1} = z_{j+1}^{\hat{j}}$. Therefore $\mathbf{b}_{\delta(\mathbf{b},j)} > \mathbf{b}_{\delta(\mathbf{b},j+1)}$.

We obtained that $\mathbf{b}$ satisfies $\varphi^G(\mathbf{b}) = \varphi^{vcg}(\mathbf{v})$, which completes the proof. □

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