Discrete Mathematics 310 (2010) 3432–3435

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Note

Walking in circles

Noga Alon [a,b], Michal Feldman [c,a], Ariel D. Procaccia [d,*,1], Moshe Tennenholtz [a,e]

[a] Microsoft Israel R&D Center, 13 Shenkar Street, Herzeliya 46725, Israel
[b] Schools of Mathematics and Computer Science, Tel Aviv University, Tel Aviv, 69978, Israel
[c] School of Business Administration and Center for the Study of Rationality, The Hebrew University of Jerusalem, Jerusalem 91904, Israel
[d] School of Engineering and Applied Sciences, Harvard University, Cambridge, MA 02138, USA
[e] Technion, IIT, Haifa 32000, Israel

ARTICLE INFO

Article history:
Received 26 April 2010
Received in revised form 11 August 2010
Accepted 13 August 2010
Available online 9 September 2010

Keywords:
Geometry
Game theory

ABSTRACT

Consider the unit circle S^1 with distance function d measured along the circle. We show that for every selection of $2n$ points $x_1, \ldots, x_n, y_1, \ldots, y_n \in S^1$ there exists $i \in \{1, \ldots, n\}$ such that $\sum_{k=1}^{n} d(x_i, x_k) \leq \sum_{k=1}^{n} d(x_i, y_k)$. We also discuss a game theoretic interpretation of this result.

1. Introduction

Let $x_1, \ldots, x_n, y_1, \ldots, y_n \in \mathbb{R}$, and denote $N = \{1, \ldots, n\}$. We claim that there exists $i \in N$ such that

$$\sum_{k \in N} |x_i - x_k| \leq \sum_{k \in N} |x_i - y_k|. \tag{1}$$

To see this, assume without loss of generality that $x_1 \leq x_2 \leq \cdots \leq x_n$. For every $k \in N$

$$|x_1 - x_k| + |x_n - x_k| = |x_1 - x_n| \leq |x_1 - y_k| + |x_n - y_k|,$$

and by summing over all $k \in N$ we get

$$\sum_{k \in N} |x_1 - x_k| + \sum_{k \in N} |x_n - x_k| \leq \sum_{k \in N} |x_1 - y_k| + \sum_{k \in N} |x_n - y_k|.$$

It immediately follows that (1) holds with respect to $i = 1$ or $i = n$, that is, with respect to one of the extreme points.

Next, let $x_1, \ldots, x_n, y_1, \ldots, y_n \in S^1$, where S^1 is the unit circle. Let $d : S^1 \times S^1 \to \mathbb{R}_+$ be the distance on S^1, i.e., the distance between two points is the length of the shorter arc between them. If $x_1, \ldots, x_n$ cannot be placed on one semicircle then there are no longer points that can easily be identified as "extreme". Is it still true that there exists $i \in N$ such that

$$\sum_{k \in N} d(x_i, x_k) \leq \sum_{k \in N} d(x_i, y_k)?$$

Put another way, if n people walk on a circle from the starting points $x_1, \ldots, x_n$ to the destination points $y_1, \ldots, y_n$ respectively, is it true that they cannot jointly move closer (in terms of the sum of distances) to *every* starting point?

* Corresponding author.
E-mail addresses: nogaa@tau.ac.il (N. Alon), mfeldman@huji.ac.il (M. Feldman), arielpro@gmail.com, arielpro@seas.harvard.edu (A.D. Procaccia), moshet@microsoft.com (M. Tennenholtz).

[1] This work was done while the author was at Microsoft Israel R&D Center.

0012-365X/$ – see front matter
doi:10.1016/j.disc.2010.08.007

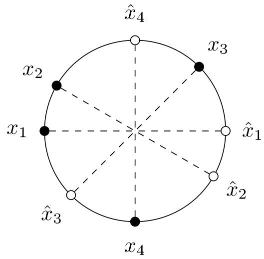

Fig. 1. An illustration of the construction in the proof of Theorem 1, for $n = 4$. The nearly antipodal pairs are $A = \{\{x_1, x_3\}, \{x_2, x_4\}, \{x_3, x_4\}\}$.

In Section 2 we answer this question in the affirmative. Although our main result is formulated with respect to S^1, it clearly also holds for any closed curve that is homeomorphic to S^1. On the other hand it is easy to see that it does not hold for any graph embedded in the plane that contains a vertex v of degree at least 3. This is shown by taking $n = 3$ and letting x_1, x_2 and x_3 be three points that lie on different edges incident with v, each being of equal distance from v, with $y_1 = y_2 = y_3 = v$. In Section 3 we briefly discuss a game theoretic implication of this result.

2. Main theorem

We first introduce some notations. Let $x, y \in S^1$; we denote the shorter open arc between x and y by (x, y), and the shorter closed arc between x and y by $[x, y]$.[2] For every $x \in S^1$ we let $\hat{x}$ be the antipodal point of x on S^1, i.e., the diametrically opposite point. Finally, given $x, y \in S^1$ we denote the "clockwise operator" by $\succeq$, and its strong version by $\succ$. Without being very formal, $x \succeq y$ means that x is clockwise from y on the circle; this operator is well defined in the context of an arc of length less than π.

We are now ready to formulate and prove our main result.

Theorem 1. *Let $x_1, \ldots, x_n, y_1, \ldots, y_n \in S^1$. Then there exists $i \in N$ such that*

$$\sum_{k \in N} d(x_i, x_k) \le \sum_{k \in N} d(x_i, y_k). \tag{2}$$

Proof. Let $x_1, \ldots, x_n \in S^1$, and define a multiset X by $X = \{x_1, \ldots, x_n\}$. We first note that we can assume that there are no $x_i, x_j \in X$ such that $x_j = \hat{x}_i$. Indeed, in this case the claim holds trivially with respect to either i or j, since for all $z \in S^1$,

$$d(x_i, z) + d(x_j, z) = \pi.$$

In particular, for every $x_i, x_j \in X$, (x_i, x_j) and $(\hat{x}_i, \hat{x}_j)$ are well-defined.

We say that two points $x_i, x_j \in X$ are *nearly antipodal* if there is no point $x_k \in X$ such that $x_k \in (x_i, \hat{x}_j)$ or $x_k \in (x_j, \hat{x}_i)$. Let A be the set of all unordered pairs of nearly antipodal points. Given a nearly antipodal pair $\{x_i, x_j\} \in A$, let the *critical arc of* $\{x_i, x_j\}$, denoted $\mathrm{crit}(x_i, x_j)$, be the *long* open arc between $\hat{x}_i$ and $\hat{x}_j$, that is,

$$\mathrm{crit}(x_i, x_j) = S^1 \setminus [\hat{x}_i, \hat{x}_j] = (x_i, \hat{x}_j) \cup [x_i, x_j] \cup (x_j, \hat{x}_i).$$

See Fig. 1 for an illustration of the construction given above.

Let $y_1, \ldots, y_n \in S^1$, and define a multiset Y by $Y = \{y_1, \ldots, y_n\}$. It is sufficient to prove that there exists a pair of nearly antipodal points $\{x_i, x_j\} \in A$ such that

$$\sum_{k \in N} d(x_i, x_k) + \sum_{k \in N} d(x_j, x_k) \le \sum_{k \in N} d(x_i, y_k) + \sum_{k \in N} d(x_j, y_k).$$

Indeed, in this case we get that Eq. (2) holds with respect to either x_i or x_j. Therefore, assume for the purpose of contradiction that for every pair of nearly antipodal points $\{x_i, x_j\} \in A$,

$$\sum_{k \in N} d(x_i, x_k) + \sum_{k \in N} d(x_j, x_k) > \sum_{k \in N} d(x_i, y_k) + \sum_{k \in N} d(x_j, y_k). \tag{3}$$

We claim that Eq. (3) implies that for every pair of nearly antipodal points $\{x_i, x_j\} \in A$, the number of points from Y on $\mathrm{crit}(x_i, x_j)$ is strictly greater than the number of points from X on the same arc. Formally, for $\{x_i, x_j\} \in A$, let

$$\alpha_{ij}^X = |\{x_k \in X : \; x_k \in \mathrm{crit}(x_i, x_j)\}|,$$

[2] If x and y are antipodal then these arcs are ambiguously defined.

and

$$\alpha_{ij}^{Y} = |\{y_k \in Y : y_k \in \text{crit}(x_i, x_j)\}|.$$

We have the following claim.

Lemma 1. *Let* $\{x_i, x_j\} \in A$. *Then* $\alpha_{ij}^{Y} > \alpha_{ij}^{X}$.

Proof. For every point $z \in [x_i, x_j]$, we have that

$$d(x_i, z) + d(x_j, z) = d(x_i, x_j).$$

Let $d'(x_i, x_j)$ be the length of the longer arc $S^1 \setminus [x_i, x_j]$ between x_i and x_j, namely

$$d'(x_i, x_j) = d(x_i, \hat{x}_j) + d(\hat{x}_j, \hat{x}_i) + d(\hat{x}_i, x_j) = d(x_i, x_j) + 2 \cdot d(x_i, \hat{x}_j) > d(x_i, x_j).$$

For every point $z \in [\hat{x}_i, \hat{x}_j]$ it holds that

$$d(x_i, z) + d(x_j, z) = d'(x_i, x_j).$$

Finally, it holds that for every $z \in (x_i, \hat{x}_j) \cup (x_j, \hat{x}_i)$,

$$d(x_i, x_j) < d(x_i, z) + d(x_j, z) < d'(x_i, x_j).$$

Since x_i and x_j are nearly antipodal, there are no points from X in $(x_i, \hat{x}_j)$ and $(x_j, \hat{x}_i)$. Therefore,

$$\sum_{k \in N} d(x_i, x_k) + \sum_{k \in N} d(x_j, x_k) = \alpha_{ij}^{X} \cdot d(x_i, x_j) + (n - \alpha_{ij}^{X}) \cdot d'(x_i, x_j). \tag{4}$$

On the other hand,

$$\sum_{k \in N} d(x_i, y_k) + \sum_{k \in N} d(x_j, y_k) \geq \alpha_{ij}^{Y} \cdot d(x_i, x_j) + (n - \alpha_{ij}^{Y}) \cdot d'(x_i, x_j). \tag{5}$$

Using Eqs. (4) and (5), we get that (3) directly implies that $\alpha_{ij}^{Y} > \alpha_{ij}^{X}$, as claimed. □

From Lemma 1, we immediately get that

$$\sum_{\{x_i, x_j\} \in A} \alpha_{ij}^{X} < \sum_{\{x_i, x_j\} \in A} \alpha_{ij}^{Y}. \tag{6}$$

In order to derive a contradiction, we also need the following lemma.

Lemma 2. *There exists* $r \in \mathbb{N}$ *such that*

$$\sum_{\{x_i, x_j\} \in A} \alpha_{ij}^{X} = r \cdot n, \tag{7}$$

whereas

$$\sum_{\{x_i, x_j\} \in A} \alpha_{ij}^{Y} \leq r \cdot n. \tag{8}$$

Proof. It is easy to see that $|A|$ is odd (e.g., by induction on n); let $|A| = 2s + 1$, for some $s \in \mathbb{N}$. We first wish to claim that every $x_i \in X$ is a member of exactly $s + 1$ critical arcs, which directly proves Eq. (7) with $r = s + 1$.

Without loss of generality we prove the claim with respect to $x_1 \in X$. Consider the clockwise closed arc between x_1 and $\hat{x}_1$. Let $Z = \{z_1, \ldots, z_t\}$ be all the points x_i or $\hat{x}_i$ on this arc, where for all k, $z_{k+1} \succeq z_k$. In particular, $z_1 = x_1$ and $z_t = \hat{x}_1$. For instance, in Fig. 1 we have that $Z = \{x_1, x_2, \hat{x}_4, x_3, \hat{x}_1\}$.

Now, we have that the set of nearly antipodal pairs A is exactly the set of pairs $\{x_i, x_j\}$ such that z_k is a point x_i and z_{k+1} is an antipodal point $\hat{x}_j$ (this is a *type* 1 nearly antipodal pair), or z_k is an antipodal point $\hat{x}_i$ and z_{k+1} is a point x_j (this is a *type* 2 nearly antipodal pair). If $\{x_i, x_j\}$ is a nearly antipodal pair of type 1, we have that $x_1 \in [x_i, x_j]$, and hence $x_1 \in \text{crit}(x_i, x_j)$. On the other hand, if $\{x_i, x_j\}$ is a nearly antipodal pair of type 2, then $x_1 \notin \text{crit}(x_i, x_j)$. Since $z_1 = x_1$ is a point from X and $x_{n+1} = \hat{x}_1$ is an antipodal point, the number of nearly antipodal pairs of type 1 is exactly $s + 1$, which proves the claim.

In order to prove Eq. (8), let $y \in S^1$. It is sufficient to prove that there exists $x_i \in X$ such that y appears in at most as many critical arcs as x_i, since we already know that x_i is a member of exactly $s + 1$ critical arcs. We consider the two points or antipodal points that are adjacent to y, and briefly examine four cases.

1. $x_i \preceq y \preceq x_j$: y appears in exactly the critical arcs that contain x_i (these are also exactly the critical arcs that contain x_j).
2. $x_i \preceq y \prec \hat{x}_j$: y appears in exactly the critical arcs that contain x_i.

3. $\hat{x}_i \prec y \preceq x_j$: y appears in exactly the critical arcs that contain x_j.
4. $\hat{x}_i \preceq y \preceq \hat{x}_j$: When walking counterclockwise from $\hat{x}_i$, let $x_k \in X$ be the first point from X, and let $\hat{x}_l$ be the last antipodal point such that $x_k \prec \hat{x}_l \preceq \hat{x}_i$. Then y is contained in exactly the critical arcs that contain x_k, except for $\mathrm{crit}(x_k, x_l)$, that is, in exactly s critical arcs.

We deduce that every y_i is contained in at most $r = s + 1$ critical arcs, which implies the validity of Eq. (8). □

It follows from Lemma 2 that

$$\sum_{\{x_i,x_j\}\in A} \alpha_{ij}^X \geq \sum_{\{x_i,x_j\}\in A} \alpha_{ij}^Y,$$

in contradiction to Eq. (6). □

3. A game theoretic interpretation

Consider a facility location setting where the facility is to be located on a network. Each player $i \in N$ has an *ideal location* for the facility on the network; the player's *cost* is the distance between its ideal location and the location that was selected for the facility. A *mechanism* is a function that receives the reported ideal locations of the players as input, and returns the location of the facility.

From the game theoretic point of view it is desirable that mechanisms be immune to manipulation by rational players. A mechanism is *strategyproof* if players can never benefit by misreporting their ideal location, regardless of the reports of the other players. In other words, by misreporting his location a player cannot influence the facility location in a way that it becomes closer to his ideal location. Schummer and Vohra [2] establish a characterization of deterministic strategyproof facility location mechanisms on networks. In particular, they show that if the network is a circle then the only deterministic strategyproof and onto mechanism is a *dictatorship* of one of the players, i.e., given any constellation of ideal locations the mechanism selects the ideal location of a fixed player.

Randomization provides a way around this negative result. Indeed, under the *random dictator* mechanism the ideal location of one of the agents is selected uniformly at random. This mechanism is strategyproof: if an agent was chosen as the dictator then it could not have gained from lying, whereas if it was not chosen then it could not have affected the outcome. Random dictator is also "fair" compared to a deterministic dictatorship, and in particular produces an outcome that yields a good approximation to the optimal facility location in terms of minimizing the sum of players' costs.[3]

Taking our game theoretic requirements a step further, we say that a mechanism is *group strategyproof* if even a coalition of agents cannot all benefit by lying, that is, for every joint deviation by a coalition there is a member of the coalition whose expected distance from the facility does not decrease. Group strategyproofness is a highly desirable property, but is rarely satisfied by nontrivial mechanisms. We can derive the following result as an immediate corollary of Theorem 1.

Corollary 1. *Assume that the network is a circle. Then the random dictator mechanism is group strategyproof.*

To see this, note that we can assume without loss of generality that the deviating coalition contains all the players. Indeed, the expected cost of a player given that a nondeviating player is selected by the mechanism, and the probability that a nondeviating player is selected by the mechanism, are both independent of the reports of the deviating players. The corollary follows after scaling by a factor of $1/n$. For more details, including the formal facility location model, the reader is referred to [1].

Acknowledgements

The first author's research was supported in part by a USA Israeli BSF grant, by an ERC advanced grant and by the Hermann Minkowski Minerva Center for Geometry at Tel Aviv University. The second author's research was supported in part by the Israel Science Foundation (grant number 1219/09) and by the Leon Recanati Fund of the Jerusalem school of business administration.

References

[1] N. Alon, M. Feldman, A.D. Procaccia, M. Tennenholtz, Strategyproof approximation of the minimax on networks, Mathematics of Operations Research 35 (3) (2010) 513–526. A more exhaustive version appears in arXiv:0907.2049.
[2] J. Schummer, R.V. Vohra, Strategy-proof location on a network, Journal of Economic Theory 104 (2) (2002) 405–428.

[3] Specifically, it gives a $2 - 2/n$ approximation, where approximation is defined in the usual sense by looking at the worst-case ratio between the expected cost of the mechanism's solution and the cost of the optimal solution [1].