Congestion Games with Failures - *CGFs*

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ABSTRACT

We introduce a new class of games, *congestion games with failures* (CGFs), which extends the class of congestion games to allow for facility failures. In a *basic* CGF (BCGF) agents share a common set of facilities (service providers), where each service provider (SP) may fail with some known probability. For reliability reasons, an agent may choose a subset of the SPs in order to try and perform his task. The cost of an agent for utilizing any SP is a function of the total number of agents using this SP. A main feature of this setting is that the cost for an agent for successful completion of his task is the minimum of the costs of his successful attempts. We show that although BCGFs do not admit a potential function, and thus are not isomorphic to classic congestion games, they always possess a pure-strategy Nash equilibrium. We also show that the SPs' congestion experienced in different Nash equilibria is (almost) unique. For the subclass of symmetric BCGFs we give a characterization of best and worst Nash equilibria. We extend the basic model by making task submission costly and define a model for *taxed* CGFs (TCGFs). We prove the existence of a pure-strategy Nash equilibrium for quasi-symmetric TCGFs, and present an efficient algorithm for constructing such Nash equilibrium in symmetric TCGFs.

Categories and Subject Descriptors

C.2.4 [**Computer-Communication Networks**]: Distributed Systems; I.2.11 [**Artificial Intelligence**]: Distributed Artificial Intelligence —*multiagent systems*

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Algorithms, Theory, Economics

Keywords

Congestion games, Failures, Pure-strategy Nash equilibrium

1. INTRODUCTION

Rosenthal [11] introduced the class of congestion games and proved that they always possess a Nash equilibrium in pure strategies. Congestion games are noncooperative games in which a collection of agents have to choose from a finite set of alternatives (facilities). The utility of an agent from using a particular facility depends only on the number of agents using it, and his total utility is the sum of the utilities obtained from the facilities he uses. Congestion games have been used to model traffic behavior in road and communication networks, competition among firms for production processes, migration of animals between different habitats, and received a lot of attention in the recent computer science and electronic commerce communities [6, 8, 10, 11, 13]. Rosenthal [11] studied games with a finite number of players. In addition, several authors have considered nonatomic congestion games with a continuum of players [6, 12].

However, the above settings do not take into consideration the possibility that facilities may fail to execute their assigned tasks. Typically, the facilities are machines, computers, service providers, communication lines etc. These kinds of facilities are obviously prone to failures because of breakage or for any other reasons. Thus, the issue of failures should not be ignored.

The notion of failures originates from the field of distributed systems. In a lot of situations failing components of the system may be viewed as playing against its correctly functioning parts. The issue of failures in game-theoretic setting is extensively discussed in [4]. In another line of research, Porter et. al. [9] introduced the notion of fault tolerant mechanism design which extends the standard game-theoretic framework of mechanism design to allow for uncertain executions. In the above settings failing components are self-motivated or malicious agents. In our work we initiate an investigation of failures in congestion games, where not the agents, but the facilities they share may fail.

As it turns out, such failures have significant implications on agent behavior, as illustrated by the following simple example. Consider a reliable network with two nodes s and t, and two parallel links. Assume an agent wishes to send a message from s to t. Then, he would send the message along one of the links. However, if the network links are not reliable then, for reliability reasons, the agent may decide to send his message along both links.

Suppose now that n agents share a reliable network with two parallel links, where the cost associated with each link is a (nondecreasing) function $l(x)$ of the congestion experienced by this link. Each agent has to send a message from

s to t, and his aim is to minimize his own cost. If n is even, then in an equilibrium, half of the agents would take one link and the other half would use the second link, and thus the cost to each agent is $l\left(\frac{n}{2}\right)$. If the network links are not reliable, the agents might send a message along both links. As a result of such behavior, the network might be overloaded, and the cost to each agent will be very high. Therefore, each agent wants to maximize the probability of successful delivery of his message and, simultaneously, to minimize his cost.

The above example illustrates the need for a careful study of the effects of failures in congestion settings. In order to address this challenge, we introduce a model for *congestion games with failures* (CGFs), and establish several basic results for this model. To the best of our knowledge, no attempt has been made so far to incorporate the issue of failures into congestion settings.

We begin by defining a basic model for congestion games with failures (BCGFs). In a BCGF agents share a common set of facilities (service providers), where each service provider (SP) may fail with some known probability. For reliability reasons, an agent may choose a subset of the service providers in order to try and perform his task. Therefore, each agent's set of pure strategies coincide with the power set of the set of SPs, and the total load on the system is not known in advance, but strategy-dependent. The cost for an agent for successful completion of his task is the minimum of the costs of his successful attempts. The cost function associated with each SP is not universal but agent-specific. That is, the utility to an agent depends not only on the number of agents using the same SP, but also on the identity of the agent in question. Congestion games with agent-specific cost functions were first studied by Milchtaich [5]. This generalization was, however, accompanied by the assumption that each agent chooses only one facility.

Our first result is that, although BCGFs do not admit a potential function, and thus are not isomorphic to classic congestion games, they always possess a pure-strategy Nash equilibrium. We also show that the SPs' congestion experienced in different Nash equilibria is (almost) unique. For the subclass of symmetric BCGFs we give a characterization of best and worst Nash equilibria with respect to the social disutility, present algorithms for their construction, and compare the social disutilities of the agents at these points.

We also consider the worst possible ratio between the social disutilities incurred by agents in an equilibrium and in an optimal outcome. This ratio (dubbed "the price of anarchy") was proposed by Koutsoupias and Papadimitriou [3] as a measure of the inefficiency of selfish behavior in noncooperative systems, and was extensively studied for nonatomic congestion games [2, 13, 12]. We show that in congestion games with failures the price of anarchy depends on the parameters of the game and cannot be bounded by a constant value, even for very simple (e.g., linear) cost functions.

A natural extension of the basic model is obtained by making task submission costly. There are two motivations for considering this setting: service providers may demand some fixed payment (cost) for task submission, or we can think about taxes that can be imposed by some central coordinator in order to achieve better social results. We define the *taxed congestion games with failures* (TCGFs) model which is obtained from the basic model by incorporating taxes as follows: each agent pays a fixed cost/tax for using each of the service providers he had chosen. Our main technical result is the existence of a pure-strategy Nash equilibrium in quasi-symmetric TCGFs. In a quasi-symmetric TCGF service costs are facility-dependent. Our proof is constructive and yields an efficient procedure for constructing such equilibria in these games. In addition, we develop a simpler algorithm for constructing Nash equilibrium in the special case of symmetric TCGFs.

The paper is organized as follows. In Section 2 we define our basic model. In Section 3 we show that BCGFs do not admit a potential. In Section 4 we provide a (constructive) proof of the existence of pure-strategy Nash equilibrium in BCGFs and consider its uniqueness properties. Section 5 is devoted to symmetric BCGFs. We characterize the best and worst Nash equilibria in symmetric BCGFs, present algorithms for their construction and provide an upper bound on the ratio between them. We also discuss the ratio between the social disutility in a Nash equilibrium and the optimal social disutility in these games. In Section 6 we define taxed congestion games with failures, and (constructively) prove the existence of a pure-strategy Nash equilibrium for quasi-symmetric TCGFs. We also provide an efficient procedure for computing Nash equilibrium in symmetric TCGFs. Some proofs are omitted from this paper due to lack of space and will appear in the full version.

2. THE BASIC MODEL

A basic CGF (BCGF) is defined as follows. Let $N = \{1, \ldots, n\}$ be a finite set of agents, and let $E = \{1, \ldots, m\}$ be a finite nonempty set of independent service providers, each associated with a *failure probability*. Each agent has a task which can be carried out by any of the service providers. Agent i's disutility from an uncompleted task is evaluated by his *incompletion cost* (denoted by W_i). The *service cost* (denoted by l_e^i) for agent i for utilizing service provider e consists of an *execution cost* (denoted by b_e^i) and a *fixed completion cost* (denoted by a).[1] The *disutility* π_i of agent i from a combination of strategies (one for each agent) is the expectation of the sum of his incompletion and service costs, where the service cost for an agent is the minimum of the service costs of the SPs he has chosen which did not fail. This is defined more precisely below.

The *success probability* of $e \in E$ is denoted by s_e ($0 < s_e < 1$). Similarly, $f_e = 1 - s_e$ stands for the *failure probability* of e. The set of pure strategies Σ_i for agent $i \in N$ is the power set of the set of SPs: $\Sigma_i = \mathcal{P}(E)$, and the set of pure-strategy profiles is defined to be $\Sigma = \Sigma_1 \times \cdots \times \Sigma_n = [\mathcal{P}(E)]^n$.

Let $\sigma = (\sigma_1, \ldots, \sigma_n) \in \Sigma$ be a combination of pure strategies. The ($|E|$-dimensional) *congestion vector* that corresponds to σ is $h^\sigma = (h_e^\sigma)_{e \in E}$, where $h_e^\sigma = \left|\{i \in N | e \in \sigma_i\}\right|$. The *execution cost* of service provider e for agent i is a function $b_e^i : \Sigma \to \mathbb{R}$ of the congestion experienced by e. The *disutility* function of agent i, $\pi_i : \Sigma \to \mathbb{R}$, is defined as follows. If agent i chooses strategy $\sigma_i = \varnothing$ (i.e., does not assign his task to any service provider) then his disutility equals his incompletion cost, $\pi_i(\sigma) = W_i$. For any strategy $\sigma_i \neq \varnothing$ of

[1] This models for example a payment to the network administrator for successful execution of a task, by one or more of the service providers. Our model can be extended, while leading to similar results, to the case where the completion cost is agent-dependent or facility dependent.

agent i,

$$\pi_i(\sigma) = W_i \prod_{e\in\sigma_i} f_e + \sum_{A\in\mathcal{P}(\sigma_i)\smallsetminus\{\varnothing\}} \min_{e\in A}\left(b_e^i(h_e^\sigma)+a\right) \times \prod_{e\in A} s_e \prod_{e\in\sigma_i\smallsetminus A} f_e = W_i \prod_{e\in\sigma_i} f_e + \sum_{A\in\mathcal{P}(\sigma_i)\smallsetminus\{\varnothing\}} \min_{e\in A} l_e^i(h_e^\sigma) \prod_{e\in A} s_e \prod_{e\in\sigma_i\smallsetminus A} f_e, \quad (1)$$

where a is the fixed completion cost, $b_e^i(h_e^\sigma)$ is the execution cost of service provider e for agent i, when its congestion is h_e^σ, and the sum of execution and fixed completion costs $l_e^i(h_e^\sigma) = b_e^i(h_e^\sigma) + a$ is the service cost of service provider e for agent i.

We assume that $b_e^i(\cdot)$ is a nonnegative nondecreasing function satisfying $b_e^i(x) \le W_i$ for all $i \in N$, $e \in E$ and integer $0 \le x \le n$. This means that the execution of a task does not cost more than its failure. W.l.o.g., we also assume that for any agent i his incompletion cost W_i is larger than the fixed completion cost a. Otherwise, the obvious dominant strategy of agent i is to avoid assigning his task to any service provider. Note that for all $0 \le x \le n$, $b_e^i(x) \le W_i$ and $a \le W_i$, but $l_e^i(x)$ might be larger than W_i. Obviously, if $l_e^i(1) > W_i$ for all $e \in E$, the dominant strategy of agent i is to avoid assigning a task, i.e. in this case agent i can be actually ignored. Therefore, w.l.o.g., we assume that such cases do not take place.

3. THE NON-EXISTENCE OF A POTENTIAL IN BCGFS

Monderer and Shapley [7] introduced the notion of *potential function* and defined a *potential game* to be a game which possesses a potential function. A potential function is a real-valued function over the set of pure-strategy profiles, with the property that the gain (or loss) of an agent shifting to another strategy while keeping the other agents' strategies unchanged is equal to the corresponding increment of the potential function. The authors showed that the classes of potential games and congestion games coincide.

In this section we show that the class of BCGFs does not possess a potential function, and therefore is not isomorphic to the class of congestion games. We show that even agent-symmetric BCGFs do not admit a potential function. Hence, the non-existence of a potential in BCGFs is a result of allowing facility failures. To prove this statement we employ Theorem 3.1 of Monderer and Shapley [7]. First, however, we need to present some definitions.

A *path* in Σ is a sequence $\gamma = (\sigma^0 \to \sigma^1 \to \cdots)$ such that for every $k \ge 1$ there exists a unique player, say player i, such that $\sigma^k = (\sigma_{-i}^{k-1}, x)$ for some $x \ne \sigma_i^{k-1}$ in Σ_i. σ^0 is called the *initial point* of γ, and if γ is finite, then its last element is called the *terminal point* of γ. A finite path $\gamma = (\sigma^0 \to \sigma^1 \to \cdots \to \sigma^K)$ is *closed* if $\sigma^0 = \sigma^K$. It is a *simple closed* path if in addition $\sigma^l \ne \sigma^k$ for every $0 \le l \ne k \le K-1$. The *length* of a simple closed path is defined to be the number of distinct points in it; that is, the length of $\gamma = (\sigma^0 \to \sigma^1 \to \cdots \to \sigma^K)$ is K. For a finite path $\gamma = (\sigma^0 \to \sigma^1 \to \cdots \to \sigma^K)$ and for a vector $U = (U_1, \ldots, U_n)$ of utility functions, let us define

$$U(\gamma) = \sum_{k=1}^{K} [U_{i_k}(\sigma^k) - U_{i_k}(\sigma^{k-1})],$$

where i_k is the unique deviator at step k. Then,

THEOREM 3.1. ***(Monderer—Shapley, [7])*** *Let Γ be a game in strategic form. Then, Γ is a potential game if and only if $U(\gamma) = 0$ for every finite simple closed path γ of length 4.*

By this theorem, if Γ is a game in strategic form with $U_i : \Sigma \to \mathbb{R}$ the utility function of agent i, then Γ is a potential game if and only if for every $i, j \in N$, for every $z \in \Sigma_{-\{i,j\}}$, and for every $x_i, y_i \in \Sigma_i$ and $x_j, y_j \in \Sigma_j$,

$$U_i(\beta) - U_i(\alpha) + U_j(\gamma) - U_j(\beta) + U_i(\delta) - U_i(\gamma) + U_j(\alpha) - U_j(\delta) = 0\,,$$

where $\alpha = (x_i, x_j, z)$, $\beta = (y_i, x_j, z)$, $\gamma = (y_i, y_j, z)$, $\delta = (x_i, y_j, z)$ (thus, $\alpha \to \beta \to \gamma \to \delta \to \alpha$ is a simple closed path of length 4).

PROPOSITION 3.2. *The class of BCGFs does not possess a potential function.*

Proof: A counterexample is the following symmetric game G in which two agents ($N = \{1, 2\}$) wish to assign a task to two independent SPs ($E = \{e_1, e_2\}$). The failure probability f of each SP is positive ($f > 0$). The failure cost to each of the agents is $W \ge 2$, and the service cost function of each SP to each agent is given by $l_e^i(x) = \min\{x, W\} + a$ ($\forall e, i$). Consider the simple closed path of length 4 which is formed by $\alpha = (\varnothing, \{e_2\})$, $\beta = (\{e_1\}, \{e_2\})$, $\gamma = (\{e_1\}, \{e_1, e_2\})$, $\delta = (\varnothing, \{e_1, e_2\})$:

$$\begin{aligned}
&\pi_1(\alpha) = W; \pi_2(\alpha) = fW + (1-f)(\min\{1, W\} + a);\\
&\pi_1(\beta) = \pi_2(\beta) = fW + (1-f)(\min\{1, W\} + a);\\
&\pi_1(\gamma) = fW + (1-f)(\min\{2, W\} + a); \pi_2(\gamma) = f^2W\\
&+(1-f)(\min\{1, W\} + a) + f(1-f)(\min\{2, W\} + a);\\
&\pi_1(\delta) = W; \pi_2(\delta) = f^2W + (1-f^2)(\min\{1, W\} + a).
\end{aligned}$$

Then,

$$\pi_1(\beta) - \pi_1(\alpha) + \pi_2(\gamma) - \pi_2(\beta) + \pi_1(\delta) - \pi_1(\gamma) + \pi_2(\alpha) - \pi_2(\delta) = -(1-f)^2 \ne 0.$$

Then, by Theorem 3.1, congestion games with failures do not possess a potential function. □

4. PURE-STRATEGY NASH EQUILIBRIA IN BCGFS

By Monderer and Shapley [7], every finite potential game possesses a pure-strategy Nash equilibrium. We have shown in Section 3 that BCGFs do not admit a potential function, but this fact, in general, does not contradict the existence of an equilibrium in pure strategies. In this section we prove that all basic congestion games with failures possess a Nash equilibrium in pure strategies, and present an efficient algorithm that finds such equilibrium points in a given BCGF. Moreover, we show that different Nash equilibrium profiles of a given BCGF correspond to (almost) the same congestion vector.

4.1 Existence and construction

We present below our first theorem.

THEOREM 4.1. *Congestion games with failures possess a Nash equilibrium in pure strategies.*

One point to notice is that the proof is constructive and makes use of the following efficient simple algorithm for finding a pure Nash equilibrium in a given BCGF.

4.1.1 NE-algorithm

Initialization: For all $1 \le i \le n$, set $\sigma_i := \varnothing$;

Main step: For all $e \in E$:
(1) Sort the agents in a non-increasing order of
$x_e^i = \max\{x | W_i > l_e^i(x),\, x = 0, 1, \ldots, n\}$;
Let $\varphi_e : N \to \{1, \ldots, n\}$
$i \mapsto i_e = \varphi_e(i)$
be the corresponding permutation function;
(2) For $i_e = 1$ to n:
if $i_e \le x_e^i$, then $\sigma_i := \sigma_i \cup \{e\}$.

4.2 (Almost) uniqueness

We consider uniqueness properties of Nash equilibria in BCGFs. We restrict our attention to games with strictly increasing service cost functions, and show that in such BCGFs the difference between the congestion experienced by any SP in two different Nash equilibria is bounded by 1.

Let $NE \subseteq \Sigma$ be a set of Nash equilibrium pure-strategy combinations, and let h_M represent the maximal congestion that may be experienced by any service provider at any Nash equilibrium, i.e. $h_M = \max\{h_e^\sigma | e \in E,\, \sigma \in NE\}$. Then,

PROPOSITION 4.2. *If for all $e \in E$ and $i \in N$, $l_e^i(x)$ is a strictly increasing monotone function on the interval $0 \le x \le h_M$, then for any pair of Nash equilibrium strategy profiles $\sigma^1, \sigma^2 \in NE$ the inequality $|h_e^{\sigma^1} - h_e^{\sigma^2}| \le 1$ holds for all $e \in E$.*

For the proof of Proposition 4.2 we need the following claim.

CLAIM 4.3. *Let $\sigma \in NE$ be a Nash equilibrium strategy profile. Then, for all $i \in N$,*

$$\begin{array}{lll} (i) & l_e^i(h_e^\sigma) \le W_i, & \forall e \in \sigma_i; \\ (ii) & l_e^i(h_e^\sigma + 1) \ge W_i, & \forall e \notin \sigma_i. \end{array}$$

Proof of Proposition 4.2: Let $\sigma^1, \sigma^2 \in NE$ be Nash equilibrium strategy profiles, and assume that $h_e^{\sigma^1} > h_e^{\sigma^2} + 1$ for some $e \in E$. Then, there is an agent i such that $e \in \sigma_i^1$, but $e \notin \sigma_i^2$. By Claim 4.3, for agent i we have $l_e^i(h_e^{\sigma^1}) \le W_i$ and $l_e^i(h_e^{\sigma^2} + 1) \ge W_i$. Therefore, $l_e^i(h_e^{\sigma^1}) \le l_e^i(h_e^{\sigma^2} + 1)$. Now, $h_e^{\sigma^1} > h_e^{\sigma^2} + 1$ coupled with the monotonicity of $l_e^i(x)$ lead to $l_e^i(h_e^{\sigma^1}) > l_e^i(h_e^{\sigma^2} + 1)$, in contradiction to $l_e^i(h_e^{\sigma^1}) \le l_e^i(h_e^{\sigma^2} + 1)$. □

It is easy to show that if in addition to the requirements of Proposition 4.2, the cost function $l_e^i(\cdot)$ satisfies $l_e^i(x) \ne W_i$ for $0 \le x \le h_M$, then all Nash equilibria of a given BCGF correspond to the same congestion vector, i.e. the congestion of any SP is fixed for all equilibrium points. In particular, all generic BCGFs have this uniqueness property.

5. SYMMETRIC BCGFS

In this subsection we give some additional characterization of Nash equilibria in symmetric BCGFs. In symmetric BCGFs, the agents and the SPs are symmetric, i.e. for all $i = 1, \ldots, n$ and $e \in E$ we have $W_i = W$, $f_e = f$, and $l_e^i(x) = l(x)$, for all $x \in \{0, 1, \ldots, n\}$. We also present efficient algorithms for finding best and worst Nash equilibria, and make a comparison between this equilibria.

PROPOSITION 5.1. *Let G be a symmetric BCGF. If $l(x)$ is a strictly increasing monotone function on the interval $0 \le x \le h_M$, then at any Nash equilibrium $\sigma \subseteq NE(G)$, the difference between the congestions of different SPs is bounded by 1, i.e. for all $\sigma \in NE$ and for all $a, b \in E$, the inequality $|h_a^\sigma - h_b^\sigma| \le 1$ holds.*

5.1 Best and worst equilibria

Given a strategy profile σ, define the *social disutility* $\pi(\sigma)$ as the sum of the agents' disutilities in this strategy profile: $\pi(\sigma) = \sum_{i \in N} \pi_i(\sigma)$. A strategy profile that minimizes the social disutility over the set of strategy profiles is called a *social optimum.* A *best (worst)* equilibrium is a strategy profile that minimizes (maximizes) the social disutility over the set of equilibrium strategies. The social disutility in a best equilibrium describes the best result that can be obtained in a system with noncooperative selfish agents. The ratio between the social disutilities in a worst equilibrium and in a social optimum serves as a measure of the inefficiency of Nash equilibrium. In this subsection we characterize, construct and compare best and worst Nash equilibria in symmetric BCGFs.

PROPOSITION 5.2. *Let $h^* = \max\{x | l(x) < W\}$. Then, there is a best Nash equilibrium strategy profile σ in which the congestion on each $e \in E$ is $h_e^\sigma = h^*$, and moreover, $\big||\sigma_i| - |\sigma_j|\big| \le 1$ for all $i, j \in N$.*

We prove below that the following algorithm (5.1.1), which is a (modified) version of the NE-algorithm (4.1.1), finds a best pure-strategy Nash equilibrium with the properties described by Proposition 5.2, in a given symmetric BCGF.

5.1.1 BNE-algorithm

Initialization: For all $1 \le i \le n$, set $\sigma_i := \varnothing$;

Main step: For all $e \in E$:
(1) Sort the agents in an order
$\varphi_e : N \to \{1, \ldots, n\}$
$i \mapsto i_e = \varphi_e(i)$
satisfying the following condition:
for all $i, j \in N$,
$|\sigma_i| < |\sigma_j| \Rightarrow i_e = \varphi_e(i) < \varphi_e(j) = j_e$;
(2) Let
$x_{\max} = \max\{x | W > l(x),\, x = 0, 1, \ldots, n\}$;
For $i_e = 1$ to n:
if $i_e \le x_{\max}$, then $\sigma_i := \sigma_i \cup \{e\}$.

Proof of Proposition 5.2: By Theorem 4.1, the combination of strategies constructed by BNE-algorithm is a Nash equilibrium strategy profile. One can check that the resulting combination of strategies satisfies the conditions of Proposition 5.2. More precisely, x agents choose $\lfloor \frac{mh^*}{n} \rfloor$ service providers, where m denote the number of SPs, and y agents choose $\lfloor \frac{mh^*}{n} \rfloor + 1$ service providers, where x and y satisfy the following equation:

$$\begin{cases} x\lfloor \frac{mh^*}{n} \rfloor + y(\lfloor \frac{mh^*}{n} \rfloor + 1) & = & mh^* \\ x + y & = & n. \end{cases}$$

The values of x and y are

$$x = n\left(\lfloor \frac{mh^*}{n} \rfloor + 1\right) - mh^*;$$
$$y = mh^* - n\lfloor \frac{mh^*}{n} \rfloor. \quad (2)$$

Note that if n divides mh^*, then $x = n$, $y = 0$. To complete the proof we need the following two claims.

Claim 5.3. *Let $\sigma \in NE$ be a combination of strategies at Nash equilibrium with two agents $i, j \in N$, such that $|\sigma_i| > |\sigma_j| + 1$. Then, the combination of strategies*

$$\hat{\sigma} = (\sigma_1, \ldots, \sigma_i \setminus \{b\}, \ldots, \sigma_j \cup \{b\}, \ldots, \sigma_n),$$

where $b \in \arg\max_{e \in \sigma_i \setminus \sigma_j} l(h_e^\sigma)$, is better than σ, i.e. $\sum_{k=1}^n \pi_k(\hat{\sigma}) \leq \sum_{k=1}^n \pi_k(\sigma)$.

Claim 5.4. *Let $\sigma \neq (E, \ldots, E)$ be a Nash equilibrium strategy profile and let i be an agent playing $\sigma_i \neq E$. Then, for all $k \in N$ and for all $e \in E \setminus \sigma_i$,*

$$\pi_k(\sigma) \leq \pi_k(\sigma_1, \ldots, \sigma_i \cup \{e\}, \ldots, \sigma_n).$$

By Claims 5.3 and 5.4, the combination of strategies constructed by the BNE-algorithm is a best Nash equilibrium profile. □

The BNE-algorithm provides an efficient procedure for constructing best Nash equilibria in symmetric BCGFs, as defined in Proposition 5.2. Next we identify some worst equilibria in symmetric BCGFs. These equilibrium points have very simple form and can be easily constructed, as follows from the next proposition.

Proposition 5.5. *Let $h^{**} = \arg\max\{x | l(x) \leq W\}$. Then, there is a worst Nash equilibrium strategy profile σ in which exactly h^{**} agents play E, $n - h^{**}$ agents play $\varnothing$ and $h_e^\sigma = h^{**}$ for all $e \in E$.*

Next we compare the best and the worst Nash equilibria. Let us denote the social disutility of a best Nash equilibrium strategy profile by π_B, and the worst one by π_W:

$$\begin{aligned}\pi_W &= h^{**}\left(W f^m + l(h^{**})(1 - f^m)\right) + (n - h^{**})W \\ &= h^{**}(1 - f^m)\left(l(h^{**}) - W\right) + nW; \quad (3)\\ \pi_B &= x\left(W f^{\lfloor \frac{mh^*}{n} \rfloor} + l(h^*)(1 - f^{\lfloor \frac{mh^*}{n} \rfloor})\right) \\ &\quad + y\left(W f^{\lfloor \frac{mh^*}{n} \rfloor + 1} + l(h^*)(1 - f^{\lfloor \frac{mh^*}{n} \rfloor + 1})\right) \\ &= f^{\lfloor \frac{mh^*}{n} \rfloor}(x + fy)\left(W - l(h^*)\right) + nl(h^*), \quad (4)\end{aligned}$$

where x and y are given by (2).

Therefore, the ratio between social disutilities in worst and best equilibria is

$$\frac{\pi_W}{\pi_B} = \frac{h^{**}(1 - f^m)\left(l(h^{**}) - W\right) + nW}{f^{\lfloor \frac{mh^*}{n} \rfloor}(x + fy)\left(W - l(h^*)\right) + nl(h^*)}. \quad (5)$$

Since $l(h^*) < W$ and $l(h^{**}) \leq W$, we have that

$$\frac{\pi_W}{\pi_B} < \frac{nW}{nl(h^*)} = \frac{W}{l(h^*)}. \quad (6)$$

This implies that the values of the social disutility in different Nash equilibrium points lie in a very narrow range. In the context of social performance of Nash equilibria, one has to ask how far these values are from the social optimum.

5.2 Nash equilibria and social optimum

In this subsection we discuss the social performance of Nash equilibrium in BCGFs. By Ashlagi [1], a best equilibrium strategy profile in classic congestion games with monotone concave cost functions is socially optimal. Simple examples (that were omitted from this paper) show that in BCGFs with such cost functions, best equilibrium strategy profiles are not always socially optimal.

Furthermore, we show below that in BCGFs the price of anarchy (the ratio between social disutilities in a worst Nash equilibrium and a social optimum) depends on the parameters of the game and cannot be bounded by a constant value, even for very simple (e.g., linear) cost functions.

Consider the following example. Suppose we have $n \geq 2$ agents sharing the set $E = \{1, \ldots, m\}$ of $m \geq 2$ independent SPs. Each service provider $e \in E$ has the failure probability f, and the service cost of each SP for each agent is $l(x) = \min\{x, W\} + a$, where a is a fixed completion cost. The failure cost of each agent is $W = n + a$.

The worst Nash equilibrium in this case corresponds to the combination of strategies σ in which each agent chooses to use each of the SPs. The disutility of agent i, $i \in N$, at this point is

$$\begin{aligned}\pi_i(\sigma) &= f^m W + (1 - f^m)\left(\min\{x, W\} + a\right) \quad (7)\\ &= f^m(n + a) + (1 - f^m)(n + a) = n + a,\end{aligned}$$

and the social disutility is

$$\pi(\sigma) = \sum_{i=1}^n \pi_i(\sigma) = n\,(n + a). \quad (8)$$

Consider the combination of strategies $\hat{\sigma}$ that corresponds to the following agents' behavior: each agent chooses only one SP and the agents divide up the SPs in a uniform way, i.e. each SP is chosen by $\frac{n}{m}$ agents (assume m divides n). The disutility of agent i, $i \in N$, at this point is

$$\begin{aligned}\pi_i(\hat{\sigma}) &= fW + (1 - f)\left(\min\{\frac{n}{m}, W\} + a\right) \quad (9)\\ &= f(n + a) + (1 - f)\left(\frac{n}{m} + a\right) = fn + (1 - f)\frac{n}{m} + a,\end{aligned}$$

and the social disutility is

$$\pi(\hat{\sigma}) = \sum_{i=1}^n \pi_i(\hat{\sigma}) = n\left(fn + (1 - f)\frac{n}{m} + a\right). \quad (10)$$

Then, the ratio between outcomes of the worst Nash equilibrium and the social optimum is

$$\begin{aligned}\frac{\pi(\sigma)}{\pi(OPT)} &\geq \frac{\pi(\sigma)}{\pi(\hat{\sigma})} = \frac{n\,(n + a)}{n\left(fn + (1 - f)\frac{n}{m} + a\right)} \quad (11)\\ &= \frac{n + a}{fn + (1 - f)\frac{n}{m} + a} = \frac{m(n + a)}{fmn + (1 - f)n + am} \\ &\xrightarrow{f \to 0} \frac{m(n + a)}{n + am} \xrightarrow{a \to 0} \frac{mn}{n} = m. \quad (12)\end{aligned}$$

This implies that the price of anarchy in congestion games with failures, unlike in classic congestion games, is not bounded by a constant value, but is game-dependent.

6. TAXED CONGESTION GAMES WITH FAILURES

A natural extension of the basic model is obtained by making task submission costly. We define the *taxed congestion games with failures* (TCGFs) model which is obtained from the basic model by incorporating fixed costs/taxes as follows: each agent pays a fixed cost/tax for using each of the service providers he had chosen. The disutility of an agent equals the sum of his disutility in the corresponding BCGF and the sum of taxes over the set of SPs selected by this agent. Let t_e be a fixed cost/tax for using service provider e. Then, the disutility of agent i is given by

$$\pi_i^{TCGF}(\sigma) = \pi_i^{BCGF}(\sigma) + \sum_{e \in \sigma_i} t_e. \tag{13}$$

Since BCGF is a special case of a TCGF, we can easily conclude that the class of taxed congestion games with failures does not admit a potential function. Nevertheless, in the following subsection we prove the existence of a pure-strategy Nash equilibrium for quasi-symmetric TCGFs as defined below, and develop a procedure for obtaining such equilibrium.

6.1 The existence of a pure-strategy Nash equilibrium in quasi-symmetric TCGFs

A *quasi-symmetric* TCGF is a TCGF in which service costs are not agent-specific ($\forall e \in E, i \in N, l_e^i(\cdot) = l_e(\cdot)$), and taxes and failure probabilities of all service providers are identical ($\forall e \in E, t_e = t, f_e = f$). Notice that in TCGFs service costs can be facility-dependent.

We now present our main technical result.

THEOREM 6.1. *Every quasi-symmetric TCGF possesses a Nash equilibrium in pure strategies.*

We say that a strategy profile is *stable* if there are no agents who wish to unilaterally drop an SP or exchange it for another one. We denote the set of all stable strategy profiles by Σ^0, and note that $(\varnothing, \dots, \varnothing)$ lies in Σ^0. Note that a stable strategy profile for which no agent would like to unilaterally add an SP to his strategy set is a Nash equilibrium. This leads us to prove the theorem by constructing an iterative algorithm having the following properties:

- The input and the output of each iteration of the algorithm lie in Σ^0.
- The congestion of each service provider $e \in E$ can only increase as the algorithm proceeds.
- The algorithm reaches a Nash equilibrium point after a finite number of iterations.

Given $\sigma \in \Sigma^0$, the algorithm selects, in a way described below, an agent i_{add} who wishes to add an SP to his strategy set. If after this addition the system is not stable, then a stabilization step closes this iteration of the process. We show that a Nash equilibrium is achieved if we initialize the iteration sequence using the empty strategy profile.

6.1.1 TNE-algorithm

Initialization: For all $i \in N$, set $\sigma_i := \varnothing$;
For all $e \in E$ set $h_e := 0$;

Main step:
(1) Set $\bar{E} := \{e \in E | h_e < n\}$;
(2) Order $\bar{E}$ according to the rule
$x \le y \Leftrightarrow l_x(h_x + 1) \le l_y(h_y + 1)$;
(3) For all $i \in N$, set $e_i := \min\{x | x \notin \sigma_i\}$;
(4) If for all $i \in N$,
$\pi_i(\sigma_1, \dots, \sigma_i \cup \{e_i\}, \dots, \sigma_n) > \pi_i(\sigma)$,
then QUIT. Otherwise, go to (5);
(5) Set $\bar{N} := \{i \in N | \pi_i(\sigma) \ge$
$\pi_i(\sigma_1, \dots, \sigma_i \cup \{e_i\}, \dots, \sigma_n)\}$;
(6) Set $e_{\min} := \min\{e_i | i \in \bar{N}\}$;
(7) Set $i_{add} := \min\{i | e_i = e_{\min}\}$;
$\sigma_{i_{add}} := \sigma_{i_{add}} \cup \{e_{\min}\}$;
$h_{e_{\min}} := h_{e_{\min}} + 1$;
(8) If $(\sigma_1, \dots, \sigma_{i_{add}} \cup \{e_{\min}\}, \dots, \sigma_n) \in \Sigma^0$,
then go to (1). Otherwise, go to (9);
(9) Set $\widetilde{N} := \{i \in N | \pi_i(\sigma) >$
$\pi_i(\sigma_1, \dots, \sigma_i \setminus \{e_{\min}\}, \dots, \sigma_n)$
$\vee\ \exists u \in \bar{E} \setminus \sigma_i : \pi_i(\sigma) >$
$\pi_i(\sigma_1, \dots, (\sigma_i \setminus \{e_{\min}\}) \cup \{u\}, \dots, \sigma_n)\}$;
(10) Set $i_{drop} := \min\{i | i \in \widetilde{N}\}$;
(11) Set $\sigma_{i_{drop}} := \sigma_{i_{drop}} \setminus \{e_{\min}\}$;
$h_{e_{\min}} := h_{e_{\min}} - 1$, and go to (3).

Each iteration of the above algorithm begins from a stable strategy set σ with its congestion vector h. First, the algorithm sorts the set of all $e \in E$ with $h_e < n$ in the non-decreasing order of $l_e(h_e + 1)$. For each agent i, let e_i be the smallest SP which is not included in the strategy set of i, according to the above order. If a unilateral addition of an SP to the strategy set of i does not deteriorate his payoff, then the most appropriate additional SP for agent i is e_i. If no agent wishes to change his strategy in this manner, we declare σ is a Nash equilibrium strategy profile and quit the algorithm. Otherwise, let $\bar{N}$ denote the set of agents who wish to add an SP to their strategy set, and let $e_{\min} = \min\{e_i \mid i \in \bar{N}\}$. The algorithm selects from $\bar{N}$ an agent $i_{add} := \min\{i \mid e_i = e_{\min}\}$, and adds the service provider $e_{\min}$ to his strategy set. If the resulting strategy profile σ' is stable, the algorithm proceeds to the next iteration. Otherwise, we need to stabilize σ'. We need the following lemma:

LEMMA 6.2. *Let $\sigma \in \Sigma^0$ and let σ' be obtained from σ by adding agent i to service provider x. Then, for all $j \in N$ and $z \in \sigma'_j \setminus \{x\}$:*

(i) $\pi_j(\sigma') \le \pi_j(\sigma'_1, \dots, \sigma'_j \setminus \{z\}, \dots, \sigma'_n)$;

(ii) $\pi_j(\sigma') \le \pi_j(\sigma'_1, \dots, (\sigma'_j \setminus \{z\}) \cup \{y\}, \dots, \sigma'_n)$, *where* $y \in E \setminus \sigma'_j$.

Proof: If $j = i$, then the proof is immediate.

Consider $j \ne i$. Since σ is a stable strategy profile,

$$\pi_j(\sigma) \le \pi_j(\sigma_1, \dots, \sigma_j \setminus \{z\}, \dots, \sigma_n).$$

Then,

$$W_j f^{|\sigma_j|} + \sum_{A \in P(\sigma_j)} \min_{e \in A} l_e(h_e) s^{|A|} f^{|\sigma_j| - |A|} + |\sigma_j| t$$
$$\le W_j f^{|\sigma_j| - 1} + \tag{14}$$
$$\sum_{B \in P(\sigma_j \setminus \{z\})} \min_{e \in B} l_e(h_e) s^{|B|} f^{|\sigma_j| - |B| - 1} + (|\sigma_j| - 1)t,$$

where $P(S)$ represent the set of all *nonempty* subsets of S, for any set S: $P(S) = \mathcal{P}(S) \setminus \{\varnothing\}$.

For every pair of sets S, Q, the next equality holds:

$$\begin{aligned} &P(S) = P(S \cap Q) \cup P(S \setminus Q) \cup \\ &\left\{\Omega \cup \Psi \middle| \Omega \in P(S \cap Q), \Psi \in P(S \setminus Q)\right\}. \end{aligned} \tag{15}$$

By (14) and (15) with $S = \sigma_j$, $Q = \sigma_j \setminus \{z\}$,

$$\begin{aligned} &W_j f^{|\sigma_j|} + f \sum_{B \in P(\sigma_j \setminus \{z\})} \min_{e \in B} l_e(h_e) s^{|B|} f^{|\sigma_j|-|B|-1} \\ &+ s \sum_{B \in P(\sigma_j \setminus \{z\})} \min_{e \in B \cup \{z\}} l_e(h_e) s^{|B|} f^{|\sigma_j|-|B|-1} \\ &+ l_z(h_z) s f^{|\sigma_j|-1} + t \qquad (16) \\ &\le W_j f^{|\sigma_j|-1} + \sum_{B \in P(\sigma_j \setminus \{z\})} \min_{e \in B} l_e(h_e) s^{|B|} f^{|\sigma_j|-|B|-1} \end{aligned}$$

$$\begin{aligned} &\Rightarrow \quad (1-f) \sum_{B \in P(\sigma_j \setminus \{z\})} \left(\min_{e \in B \cup \{z\}} l_e(h_e) - \min_{e \in B} l_e(h_e) \right) \\ &\qquad \times s^{|B|} f^{|\sigma_j|-|B|-1} \\ &\le (1-f)(W_j - l_z(h_z)) f^{|\sigma_j|-1} - t \end{aligned}$$

$$\begin{aligned} &\Rightarrow \sum_{B \in P(\sigma_j \setminus \{z\})} \left(\min_{e \in B \cup \{z\}} l_e(h_e) - \min_{e \in B} l_e(h_e) \right) s^{|B|} f^{-|B|} \\ &\le W_j - l_z(h_z) - \frac{t}{(1-f) f^{|\sigma_j|-1}}. \end{aligned} \tag{17}$$

By contrary, assume that adding agent i to service provider x causes agent j to drop service provider $z \neq x$. Then, by step (9) of the TNE-algorithm,

$$\begin{aligned} &\pi_j(\sigma') = \pi_j(\sigma_1, \ldots, \sigma_i \cup \{x\}, \ldots, \sigma_j \ldots, \sigma_n) \\ &> \pi_j(\sigma_1, \ldots, \sigma_i \cup \{x\}, \ldots, \sigma_j \setminus \{z\} \ldots, \sigma_n) \\ &= \pi_j(\sigma'_1, \ldots, \sigma'_i, \ldots, \sigma'_j \setminus \{z\} \ldots, \sigma'_n). \end{aligned}$$

That is,

$$\begin{aligned} &W_j f^{|\sigma_j|} + \sum_{A \in P(\sigma_j)} \min_{e \in A} l_e(h_e^x) s^{|A|} | f^{\sigma_j|-|A|} + |\sigma_j| t \\ &> W_j f^{|\sigma_j|-1} + \sum_{B \in P(\sigma_j \setminus \{z\})} \min_{e \in B} l_e(h_e^x) s^{|B|} f^{|\sigma_j|-|B|-1} \\ &+ (|\sigma_j| - 1) t, \end{aligned} \tag{18}$$

where for all $v \in \bar{E}$, h_e^v is defined to be

$$h_e^v = \begin{cases} h_e & e \neq v; \\ h_e + 1 & e = v. \end{cases} \tag{19}$$

From (18), by (15) we get

$$\begin{aligned} &\sum_{B \in P(\sigma_j \setminus \{z\})} \left(\min_{e \in B \cup \{z\}} l_e(h_e^x) - \min_{e \in B} l_e(h_e^x) \right) s^{|B|} f^{-|B|} \\ &> W_j - l_z(h_z) - \frac{t}{(1-f) f^{|\sigma_j|-1}}. \end{aligned} \tag{20}$$

By (19) and the monotonicity of $l_e(\cdot)$, for any $B \in P(\sigma_j \setminus \{z\})$ we have

$$\min_{e \in B \cup \{z\}} l_e(h_e^x) - \min_{e \in B} l_e(h_e^x) \le \min_{e \in B \cup \{z\}} l_e(h_e) - \min_{e \in B} l_e(h_e).$$

Then, by (17),

$$\begin{aligned} &\sum_{B \in P(\sigma_j \setminus \{z\})} \left(\min_{e \in B \cup \{z\}} l_e(h_e^x) - \min_{e \in B} l_e(h_e^x) \right) s^{|B|} f^{-|B|} \\ &\le \sum_{B \in P(\sigma_j \setminus \{z\})} \left(\min_{e \in B \cup \{z\}} l_e(h_e) - \min_{e \in B} l_e(h_e) \right) s^{|B|} f^{-|B|} \\ &\le W_j - l_z(h_z) - \frac{t}{(1-f) f^{|\sigma_j|-1}}, \end{aligned} \tag{21}$$

in contradiction to (20).

Now assume that adding agent i to service provider x causes agent j to deviate from service provider $z \neq x$ to service provider y. Then,

$$\begin{aligned} &\pi_j(\sigma') = \pi_j(\sigma_1, \ldots, \sigma_i \cup \{x\}, \ldots, \sigma_j, \ldots, \sigma_n) \\ &= W_j f^{|\sigma_j|} + \sum_{A \in P(\sigma_j)} \min_{e \in A} l_e(h_e^x) s^{|A|} f^{|\sigma_j|-|A|} \\ &+ |\sigma_j| t > W_j f^{|\sigma_j|} + |\sigma_j| t \qquad (22) \\ &+ \sum_{C \in P\left((\sigma_j \setminus \{z\}) \cup \{y\}\right)} \min_{e \in C} l_e(h_e^{x,y}) s^{|C|} f^{|\sigma_j|-|C|} \\ &= \pi_j(\sigma_1, \ldots, \sigma_i \cup \{x\}, \ldots, (\sigma_j \setminus \{z\}) \cup \{y\}, \ldots, \sigma_n) \\ &= \pi_j(\sigma'_1, \ldots, \sigma'_i, \ldots, (\sigma'_j \setminus \{z\}) \cup \{y\}, \ldots, \sigma'_n). \end{aligned}$$

From (22), by (15) with $S = \sigma_j$, $Q = \sigma_j \setminus \{z\}$ for the left hand side and $S = (\sigma_j \setminus \{z\}) \cup \{y\}$, $Q = \sigma_j \setminus \{z\}$ for the right hand side, we get

$$\begin{aligned} &\sum_{B \in P(\sigma_j \setminus \{z\})} \left(\min_{e \in B \cup \{z\}} l_e(h_e^x) - \min_{e \in B \cup \{y\}} l_e(h_e^{x,y}) \right) \\ &\times s^{|B|} f^{-|B|} > l_y(h_y + 1) - l_z(h_z) \end{aligned} \tag{23}$$

If $l_y(h_y + 1) \ge l_z(h_z)$, then by (23),

$$\begin{aligned} &0 \ge \sum_{B \in P(\sigma_j \setminus \{z\})} \left(\min_{e \in B \cup \{z\}} l_e(h_e^x) - \min_{e \in B \cup \{y\}} l_e(h_e^{x,y}) \right) \\ &\times s^{|B|} f^{-|B|} > l_y(h_y + 1) - l_z(h_z) \ge 0, \end{aligned} \tag{24}$$

a contradiction. Therefore,

$$(23) \Rightarrow \quad l_y(h_y + 1) < l_z(h_z). \tag{25}$$

Since σ is a stable strategy profile,

$$\begin{aligned} &\pi_j(\sigma) = W_j f^{|\sigma_j|} + \sum_{A \in P(\sigma_j)} \min_{e \in A} l_e(h_e) s^{|A|} f^{|\sigma_j|-|A|} \\ &+ |\sigma_j| t \le W_j f^{|\sigma_j|} + |\sigma_j| t \qquad (26) \\ &+ \sum_{C \in P\left((\sigma_j \setminus \{z\}) \cup \{y\}\right)} \min_{e \in C} l_e(h_e^y) s^{|C|} f^{|\sigma_j|-|C|} \\ &= \pi_j(\sigma_1, \ldots, (\sigma_j \setminus \{z\}) \cup \{y\}, \ldots, \sigma_n), \qquad (27) \end{aligned}$$

From (26), by (15) we get

$$\begin{aligned} &\sum_{B \in P(\sigma_j \setminus \{z\})} \left(\min_{e \in B \cup \{z\}} l_e(h_e) - \min_{e \in B \cup \{y\}} l_e(h_e^y) \right) \\ &\times s^{|B|} f^{-|B|} \le l_y(h_y + 1) - l_z(h_z) \end{aligned} \tag{28}$$

If $l_y(h_y+1) < l_z(h_z)$, then by (28),

$$0 \le \sum_{B\in P(\sigma_j \smallsetminus \{z\})} \left(\min_{e\in B\cup\{z\}} l_e(h_e) - \min_{e\in B\cup\{y\}} l_e(h_e^y)\right) \times s^{|B|} f^{-|B|} \le l_y(h_y+1) - l_z(h_z) < 0, \quad (29)$$

a contradiction. Therefore,

$$(28) \Rightarrow \quad l_y(h_y+1) \ge l_z(h_z), \quad (30)$$

in contradiction to (25). □

By lemma 6.2, the only potential cause for non-stability of σ' is the existence of an agent who wishes to drop service provider $e_{\min}$ or to exchange it for another one. Let $\widetilde{N}$ denote the set of agents who wish to make such a change in their strategies. The algorithm selects from $\widetilde{N}$ the agent $i_{drop} := \min\{i \mid i \in \widetilde{N}\}$, and remove the service provider $e_{\min}$ from his strategy set. The following lemma shows that the resulting strategy profile σ'' is stable. Therefore, now the algorithm can proceed to the next iteration.

Lemma 6.3. *Let $\sigma \in \Sigma^0$ and let σ' be obtained from σ by adding agent i to service provider x. If agent $j \ne i$ wants to drop service provider x or exchange it for another one, then the strategy profile σ'' obtained from σ' by removing service provider x from the strategy σ'_j of agent j, is stable.*

Proof: The case in which agent j wants to drop service provider x is trivial. If agent j wants to deviate from x to y, then

$$\pi_j(\sigma') = W_j f^{|\sigma'_j|} + \sum_{A\in P(\sigma'_j)} \min_{e\in A} l_e(h_e^x) s^{|A|} f^{|\sigma'_j|-|A|} + |\sigma'_j| t$$
$$> W_j f^{|\sigma'_j|} + \sum_{C\in P\left((\sigma'_j \smallsetminus \{x\})\cup\{y\}\right)} \min_{e\in C} l_e(h_e^y) s^{|C|} f^{|\sigma'_j|-|C|}$$
$$+ |\sigma_j| t = \pi_j(\sigma'_1,\ldots,(\sigma'_j \smallsetminus \{x\})\cup\{y\},\ldots,\sigma_n), \quad (31)$$

From (31), by (15) we get

$$\sum_{B\in P(\sigma_j \smallsetminus \{x\})} \left(\min_{e\in B\cup\{x\}} l_e(h_e) - \min_{e\in B\cup\{y\}} l_e(h_e^y)\right) \times s^{|B|} f^{-|B|} > l_y(h_y+1) - l_x(h_x+1) \quad (32)$$

$$\Rightarrow \quad l_y(h_y+1) < l_x(h_x+1). \quad (33)$$

It is clear that agent j does not wish to drop any SP in σ''_j, where $\sigma''_j = \sigma_j \smallsetminus \{x\}$. We show below that he does not wish to move from any SP in σ''_j to an SP in $E \smallsetminus \sigma''_j$. In contrary, assume that j wants to deviate from $\bar{v} \in \sigma''_j$ to $\bar{u} \in E \smallsetminus \sigma''_j$. Then,

$$\pi_j(\sigma'') > \pi_j(\sigma''_1,\ldots,(\sigma''_j \smallsetminus \{\bar{v}\})\cup\{\bar{u}\},\ldots,\sigma''_n). \quad (34)$$

By similar arguments used before,

$$(34) \Rightarrow \quad l_{\bar{u}}(h_{\bar{u}}+1) < l_{\bar{v}}(h_{\bar{v}}). \quad (35)$$

Since σ is a stable strategy profile, for all $v \in \sigma_j$ and for all $u \in E \smallsetminus \sigma_j$ we have

$$\pi_j(\sigma) \le \pi_j(\sigma_1,\ldots,(\sigma_j \smallsetminus \{v\})\cup\{u\},\ldots,\sigma_n).$$

Then, for all $v \in \sigma_j$ and for all $u \in E \smallsetminus \sigma_j$,

$$l_u(h_u+1) \ge l_v(h_v). \quad (36)$$

If $\bar{u} \ne x$, then (35) contradicts (36). Otherwise, by (33) and (36),

$$l_x(h_x+1) > l_{\bar{v}}(h_{\bar{v}}),$$

in contradiction to (35). □

Consider the k'th iteration of the algorithm, where adding agent i^k_{add} to service provider $e^k_{\min}$ destabilizes the system. If after adding i^k_{add} to $e^k_{\min}$, agent i^k_{drop} preferred to remove $e^k_{\min}$ from his strategy set, then he will not wish to add it to his strategy at the next iteration, i.e. $i^k_{drop} \notin \bar{N}^{k+1}$. If after adding i^k_{add} to $e^k_{\min}$, agent i^k_{drop} preferred to exchange $e^k_{\min}$ to another service provider $u \notin \sigma_{i^k_{drop}}$, then $l_u(h_u+1) < l_{e^k_{\min}}(h_{e^k_{\min}}+1)$. That is, $e_{i^k_{drop}} < e^k_{\min}$, and therefore, at the next iteration, i^k_{drop} will be the unique player in $\{i \in \bar{N}^{k+1} \mid e_i = e^{k+1}_{\min}\}$. Hence, at iteration $(k+1)$ this agent will be selected by the algorithm as i^{k+1}_{add} and will add the service provider u to his strategy set. Thus, breaking exchange move into two parts does not effect the process.

It remains to show that the TNE-algorithm halts. It is clear that the congestion of each service provider does not decrease as the algorithm proceeds. Therefore, in order to prove that the algorithm terminates after finitely many iterations, it suffices to show that every sequence of iterations with constant congestion is finite. This statement follows from the following lemma:

Lemma 6.4. *Let σ^k represent the input of the k'th iteration of the TNE-algorithm, and let h^k be the corresponding congestion vector. Then, for every $r > k$ such that $h^r = h^k$, $\sigma^r \ne \sigma^k$.*

Proof: Consider agent $p = i^k_{add}$ who adds service provider $e^k_{\min}$ to his strategy set σ^k_p at the beginning of the k'th iteration. We prove below that for all $r > k$ such that $h^r = h^k$ and for all $e \le e^k_{\min}$, $e \in \sigma^r_p$. Then, since $e^k_{\min} \notin \sigma^k_p$, we get $\sigma^r \ne \sigma^k$ for all such r.

In contrary, assume that agent p drops some service provider $e \le e^k_{\min}$ before or at the r'th iteration. Let $k < s \le r$ be the first iteration in which such a change occurs. Then, by Lemma 6.2, this change is caused by adding agent $q = i^s_{add}$ to service provider $e^s_{\min} \in \sigma^s_p$. Let $\sigma^{s+} = (\sigma^s_1,\ldots,\sigma^s_q \cup \{e^s_{\min}\},\ldots,\sigma^s_n)$. Since for all $e \le e^s_{\min}$, $e \in \sigma^s_p$, then agent p cannot improve his payoff by moving from $e^s_{\min}$ to another SP, but only by removing $e^s_{\min}$ from σ^s_p. Then,

$$\pi_p(\sigma^{s+}) > \pi_p(\sigma^{s-}), \quad (37)$$

where $\sigma^{s-} = (\sigma^{s+}_1,\ldots,\sigma^{s+}_p \smallsetminus \{e^s_{\min}\},\ldots,\sigma^{s+}_n)$.

By (15) and stability of σ^s,

$$(37) \Rightarrow \quad l_{e^s_{\min}}(h_{e^s_{\min}}+1) > W_p - \frac{t}{(1-f)f^{|\sigma^{s+}_p|-1}}. \quad (38)$$

Let $k \le l < s$ be the last iteration where agent p adds an SP to his strategy set, before dropping service provider $e^s_{\min}$. Then,

$$\pi_p(\sigma^{l+}) \le \pi_p(\sigma^l), \quad (39)$$

where $\sigma^{l+} = (\sigma^l_1,\ldots,\sigma^l_p \cup \{e^l_{\min}\},\ldots,\sigma^l_n)$.

By (15) and stability of σ^l,

$$(39) \Rightarrow \quad l_{e^l_{\min}}(h_{e^l_{\min}} + 1) \leq W_p - \frac{t}{(1-f)f^{|\sigma^l_p|}}. \qquad (40)$$

Since $|\sigma^{s+}_p| \leq |\sigma^l_p| + 1$, from (38) and (40) we have

$$l_{e^s_{\min}}(h_{e^s_{\min}} + 1) > W_p - \frac{t}{(1-f)f^{|\sigma^{s+}_p|-1}}$$
$$\geq W_p - \frac{t}{(1-f)f^{|\sigma^l_p|}} \geq l_{e^l_{\min}}(h_{e^l_{\min}} + 1), \qquad (41)$$

in contradiction to $l_{e^s_{\min}}(h_{e^s_{\min}} + 1) \leq l_{e^l_{\min}}(h_{e^l_{\min}} + 1)$. □

Proof of Theorem 6.1: By Lemmas 6.2, 6.3 and 6.4, the TNE-algorithm finds a Nash equilibrium strategy profile in any given quasi-symmetric TCGF. □

6.2 The construction of a pure-strategy Nash equilibrium in symmetric TCGFs

In this subsection we consider the special case of taxed congestion games with failures - *symmetric* TCGFs. In a symmetric TCGF, the agents and the SPs are symmetric, i.e. for all $i = 1, \ldots, n$ and $e \in E$ we have $W_i = W$, $f_e = f$, $t_e = t$ and $l^i_e(x) = l(x)$, for all $x \in \{0, 1, \ldots, n\}$. We present an efficient simple algorithm which easily finds a pure-strategy Nash equilibrium profile in the above class of games.

The algorithm is initialized with an empty strategy set for each agent. It orders the set $N \times E = \{(i, e) | i \in N, e \in E\}$ of pairs of the agents and the service providers, according to the rule described below. According to this order, it offers the agents to add an SP to their strategy set. If the addition of service provider e to the strategy set σ_i of agent i does not deteriorate the payoff of this agent, the algorithm updates the strategy set of agent i and proceeds to the next pair. The algorithm halts when it receives the first decline.

Let us denote $a(\text{mod } b)$ by $[a]_b$.

6.2.1 STNE-algorithm

Initialization: For all $i \in N$, set $\sigma_i := \varnothing$;
Set $k := 0$;

Main step: 1. Set $k := k + 1$.
If $k > \gcd(m, n)$, then QUIT;
2. Set $q := 1$;
(a) Let $e_q = [q + k - 1]_m$;
(b) If $\pi_{[q]_n}(\sigma_1, \ldots, \sigma_{[q]_n} \cup \{e_q\}, \ldots, \sigma_n) \leq \pi_{[q]_n}(\sigma)$, then set $\sigma_{[q]_n} := \sigma_{[q]_n} \cup \{e_q\}$;
Otherwise, QUIT;
(c) Set $q := q + 1$. If $q > \text{lcm}(m, n)$, then go to 1. Otherwise, go to (a).

The procedure of ordering the set $N \times E$ is illustrated by the following example. Suppose we have $n = 9$ agents and $m = 6$ service providers. We define an order in which we offer the agents to add an SP to their strategy set in the following way.

e_1	e_2	e_3	e_4	e_5	e_6
1	2	3	4	5	6
7	8	9	1	2	3
4	5	6	7	8	9
	1	2	3	4	5
6	7	8	9	1	2
3	4	5	6	7	8
9					
		1	2	3	4
5	6	7	8	9	1
2	3	4	5	6	7
8	9				

We assign the agents to SPs, beginning from the agent 1 assigned to service provider 1, agent 2 assigned to service provider 2, and so on. Agent 6 is assigned to the last service provider; then agent 7 goes to service provider 1. The last agent gets service provider 3, and we continue with assigning agent 1 to service provider 4. At the end of the first iteration, agent 9 is assigned to service provider 6. At the next iteration we move the agents by one step; that is, agent 1 is assigned to service provider 2, agent 2 is assigned to service provider 3, and at the end of the iteration, agent 9 is assigned to service provider 1. The length of each iteration is bounded by the least common multiplier of m and n, and the number of iterations is bounded by the greatest common divider of m and n.

THEOREM 6.5. *The STNE-algorithm finds a pure-strategy Nash equilibrium in a given symmetric TCGF.*

7. DISCUSSION & FUTURE WORK

In this paper we studied congestion games in which facilities may fail to complete their assigned tasks. We have shown that these games do not admit a potential function, and therefore are not isomorphic to classic congestion games. However, we were able to prove the existence of pure-strategy Nash equilibrium for these games, and to find an efficient algorithm for its construction. We also showed that the congestion experienced by each of the facilities in different Nash equilibria is (almost) unique. For symmetric BCGFs we provided a characterization of the best and worst Nash equilibria, presented algorithms for their construction, and made a comparison of agents' payoffs at these equilibrium points. We defined a model for taxed congestion games with failures and proved the existence of a pure-strategy Nash equilibrium in quasi-symmetric TCGFs. We also provided an efficient algorithm for computing Nash equilibrium in symmetric TCGFs.

Since it is known that Nash equilibria do not optimize the overall welfare, the social performance of Nash equilibria should be studied. In this context, we outline the following two directions: (i) evaluation of the inefficiency of Nash equilibria; (ii) developing methods for improving the outcome of Nash equilibria. In both directions we have some partial results for the games presented in this paper. For instance, the price of anarchy in BCGFs is a function of the parameters of the game and cannot be bounded by constant value, even for very simple (e.g., linear) cost functions. The inefficiency of Nash equilibria motivates the study of methods for improving the social outcome obtained by selfish agents. In this context, we have some positive results (that were omitted from this paper) showing that economic incentives, e.g. taxation, can improve the outcome of Nash equilibria in congestion games with failures. That is, we can price the facilities to reduce the total social disutility of Nash equilibrium - the sum of the agents' disutilities plus taxes paid. We are interested in formulating meaningful conditions under which taxes can reduce the total cost of Nash equilibrium in games with failures.

As part of our research we plan to take further look at the modelling of noncooperative games with failures. The models we presented here could be extended or modified. In particular, the facility failures might be congestion-dependent or unknown to the agents.

Overall, we believe this work tackles a fundamental connection between distributed computing and game theory. While congestion is substantial to both disciplines (and indeed is extensively studied by both communities), the notion of selfish behavior pertains to game theory and the notion of failures originates from distributed computing. However, there is a natural connection between these topics which to the best of our knowledge is first explored in this work.

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