Partition Equilibrium

Michal Feldman[*] Moshe Tennenholtz[†]

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Abstract

We introduce partition equilibrium and study its existence in resource selection games (RSG). In partition equilibrium the agents are partitioned into coalitions, and only deviations by the prescribed coalitions are considered. This is in difference to the classical concept of strong equilibrium according to which any subset of the agents may deviate. In resource selection games, each agent selects a resource from a set of resources, and its payoff is an increasing (or non-decreasing) function of the number of agents selecting its resource. While it has been shown that strong equilibrium exists in resource selection games, these games do not possess super-strong equilibrium, in which a fruitful deviation benefits at least one deviator without hurting any other deviator, even in the case of two identical resources with increasing cost functions. Similarly, strong equilibrium does not exist for that restricted two identical resources setting when the game is played repeatedly. We prove that for any given partition there exists a super-strong equilibrium for resource selection games with identical resources with increasing cost functions; we also show similar existence results for a variety of other classes of resource selection games. For the case of repeated games we provide characterizations for the partitions under which strong equilibrium exists. Together, our work introduces a natural concept, which turns out to lead to positive and applicable results in one of the basic domains studied in the literature.

Keywords: strong equilibrium, coalitions, congestion games, resource selection games

1 Introduction

When considering a prescribed behavior in a multi-agent system, it makes little sense to assume that an agent will stick to its part of that behavior, if deviating from it can increase its payoff. This leads to much interest in the study of Nash equilibrium in games. A Nash equilibrium is an action profile of the agents for which unilateral deviations are not beneficial. When agents are allowed to use mixed actions, a Nash equilibrium always exists. Moreover, in the context of congestion games [14, 12], there always exists a pure action equilibrium. However, Nash equilibrium does not take into account deviations by non-singleton sets of agents. While stability against deviations by subsets of the agents, captured by the notion of strong equilibrium [3], is a most natural requirement, it is well-known that obtaining such stability is possible only in rare situations. In the context of congestion

[*]School of Business Administration and Center for the Study of Rationality, Hebrew University of Jerusalem, and Microsoft Israel R&D Center. *E-mail*: `mfeldman@huji.ac.il`.

[†]Microsoft Israel R&D Center and Technion. *E-mail*: `moshet@microsoft.com`.

games, Holzman and Law-Yone [8, 9] characterized the networks where strong equilibrium always exist. From pragmatic perspective the most important part of their results is the existence of strong equilibrium in *resource selection games*. In a resource selection game (RSG) we have a set of n players, and a set of m resources. Each player chooses a resource from among the set of resources, and his cost is a non-decreasing function of the number of players who have chosen his selected resource. Needless to say, resource selection games are fundamental and central to work in various communities, such as operations research, computer science, game theory and economics. However, a closer look at the above fundamental result shows severe limitations to its applicability. In particular, the following issues arise:

(1) In the original definition of strong equilibrium a deviation is considered profitable only if it is strictly beneficial to all players. However, it makes much sense to consider super-strong equilibrium, in which a beneficial deviation improves the payoff of at least one of the deviator without hurting any other deviator.

(2) The results on existence of strong equilibrium are obtained for one-shot games, while it makes sense to consider a repeated play, with the desire to have stability against deviations in that game.

As it turns out, the important basic results about resource selection games fail to generalize to either super-strong equilibrium or to repeated resource selection games. Consider the basic setting of two identical resources with (strictly) increasing cost functions. This setting is fundamental to many studies in electronic commerce, operations research, communication networks, and economics. Apparently, there are simple instances of that setting in which there is no super-strong equilibrium, and simple instances of that setting in which there is no strong equilibrium when the game is played repeatedly. In order to deal with these issues, we introduce in this paper the study of *partition equilibrium*, and apply it in the context of resource selection games. Partition equilibrium introduces a social context into the study of group deviations by explicitly stating a partition over the players, allowing only for deviations in which the set of deviators constitutes an element of the partition. Needless to say that partition equilibrium makes much sense in the context of games that take into account the social structure of the set of participants.

One way to view partition equilibrium is as an extension of work on social context games [2]. In a social context game, an agent's utility is effected by the payoffs of its friends, where friends are defined using some topological or graph-theoretic structure. However, unlike previous work on social context games, dealing with single agent deviations, in partition equilibrium we consider the situation where members of a coalition coordinate their activity and potential deviations, as in strong equilibrium. Notice that partition equilibrium suggests a novel solution to *non-cooperative games*; in particular, no side payments are considered or allowed.

Previous work on coalitional congestion games [7, 10] has considered side payments in the context of congestion games; in this context each player is a set of agents, each of which is a participant in the resource selection game, and the utility of the player is the sum of his agents' utilities. Side payments however deviate from the non-transferable utility assumption which is the basic assumption in work on strong equilibrium [4, 1, 5, 11, 13, 6]. Our work on partition equilibrium re-considers deviations by coalitions in the classical non-transferable utility setting. Notice that in the context of one-shot games, a positive result showing the existence of equilibrium when monetary transfers are allowed implies the existence of super-strong partition equilibrium. Indeed, one of our results can be deduced from these relationships. In most cases however the existence of monetary transfers yields negative results; in fact, even if we have two identical resources with increasing cost functions it has been shown that if coalitions are not restricted to have size of at most two then no equilibrium exists when monetary transfers are allowed; our work shows positive results about the existence of super-strong partition equilibria in this setting, and in much wider sets of resource selection games.

The paper is structured as follows. Section 2 presents some definitions. In particular we define T-SE, strong equilibrium for a partition T, and T-SSE, super-strong equilibrium for a partition T. In section 3 we consider T-SSE for one shot games, and in section 4 we consider T-SE for repeated games. Together, our analysis addressed the above mentioned two basic issues.

In section 3 we first concentrate on resource selection games with increasing cost functions; this is a most classical type of games. Recall that even in the case of two identical resources there is no super-strong equilibrium. We show the existence of T-SSE for any T, and arbitrary number of resources, in that setting. We then extend our results to dealing with the case of two non-identical resources with increasing cost functions, and to the case of two identical resources with non-decreasing (rather than increasing) cost functions. In both cases we provide subtle analysis, yielding positive results about equilibrium existence. Notice that in all related cases these are the first positive results on equilibrium existence when group deviations are considered, and deviations are not required to strictly benefit all agents. We also consider the case of general resource selection games with non-decreasing resources and coalitions bounded by size 2. Since this restricted case is the only one for which a positive result is known in games with monetary transfers (see [10]) we know that a super-strong equilibrium exists in that setting; however, we provide a simpler proof for that case. Unfortunately, as we will illustrate, our results do not scale to arbitrary congestion games.

In section 4 we consider repeated resource selection games. In that setting, strong equilibrium (in the classical sense of Aumann [3]) does not exist even if we have two identical resources with increasing cost functions and we allow deviations of size two. We consider general repeated resource selection games, with non-decreasing cost functions, and show that there exists a T-SE when all elements in the partition are of size at most 2, as well as when all elements in the partition are of size at least 2. The above conditions are in a sense complete: we show the existence of a repeated resource selection game, where the society consists of a singleton and a triplet under which there is no T-SE. While this provides a characterization for the general case, we provide a stronger characterization for a restricted case where the resources are identical and there is a majority of singletons in the partition. In this case we show that if the number of players is odd there is a T-SE if all coalitions are of size at most 3, and that when there is a different coalition structure we can find a resource selection game with no T-SE. If the numbers of players is even there is a T-SE if all coalitions are of size at most 2, and when there is a different coalition structure we can find a resource selection game with no T-SE.

2 Model and Preliminaries

A game is denoted by a tuple $G = < N, \{S_i\}_{i=1}^n, \{c_i\}_{i=1}^n >$, where N is the set of players, S_i is a finite action space for player $i \in N$, and $c_i(\cdot)$ is a cost function of player i. We denote by $n = |N|$ the number of players. The action profile space of the players is $S = \times_{i=1}^n S_i$. For an action profile $s \in S$ we denote by s_{-i} the actions of players $j \neq i$, i.e., $s_{-i} = (s_1, \dots, s_{i-1}, s_{i+1}, \dots, s_n)$. Similarly, for a set of players Γ (also called a *coalition*) we denote by s_Γ and $s_{-\Gamma}$ the actions of players $j \in \Gamma$ and $j \notin \Gamma$, respectively. The cost function of player i maps an action profile $s \in S$ to a real number, i.e., $c_i : S \to \mathbb{R}$. Throughout this paper we restrict attention to *pure* actions.

Nash Equilibrium (NE): An action profile $s \in S$ is a *pure* Nash Equilibrium if no player $i \in N$ can benefit from unilaterally deviating from his action to another action, i.e., $\forall i \in N\ \forall a \in S_i$: $c_i(s_{-i}, a) \geq c_i(s)$.

Resilience to coalitions: A *pure action profile* of a set of players $\Gamma \subseteq N$ specifies an action for each player in the coalition, i.e., $\gamma \in \times_{i\in\Gamma} S_i$. An action profile $s \in S$ is not resilient to a pure *strong* deviation of a coalition Γ if there is a pure action profile γ of Γ such that $c_i(s_{-\Gamma}, \gamma) < c_i(s)$ for every $i \in \Gamma$ (i.e., the players in the coalition can deviate in such a way that *each* player reduces its cost). In this case we say that the coalition Γ has a strongly-profitable deviation.

Definition 2.1. A *strong equilibrium (SE)* is a profile that is resilient to a pure strongly-profitable deviation of any coalition $\Gamma \subseteq N$.

An action profile $s \in S$ is not resilient to a pure *weak* deviation of a coalition Γ if there is a pure action profile γ of Γ such that $c_i(s_{-\Gamma}, \gamma) \leq c_i(s)$ for every $i \in \Gamma$, and $\exists i \in \Gamma$ s.t. $c_i(s_{-\Gamma}, \gamma) < c_i(s)$ (i.e., the players in the coalition can deviate in such a way that none of the players increases its cost, and at least one player strictly reduces its cost). In this case we say that the coalition Γ has a weakly-profitable deviation.

Definition 2.2. A *super strong equilibrium (SSE)* is a profile that is resilient to a pure weakly-profitable deviation of any coalition $\Gamma \subseteq N$.

Note that the set of super strong equilibria is contained in the set of strong equilibria.

Suppose the coalitional structure is exogenously given. That is, the finite set of coalitions is given by a partition $T = (T_1, \ldots, T_k)$ of the set of players. Given a partition T, we shall define the following.

Definition 2.3. A *T-strong equilibrium (T-SE)* is a profile that is resilient to a pure strongly-profitable deviation of any coalition $T_i \in T$.

Definition 2.4. A *T-super strong equilibrium (T-SSE)* is a profile that is resilient to a pure weakly-profitable deviation of any coalition $T_i \in T$.

Obviously for any T, a T-SSE is a T-SE.

Observation 2.5. *Every SE is also a T-SE for any T, and every SSE is a T-SSE for any T.*

It is important to note that while the set of SE is contained in the set of NE, the set of T-SE (or T-SSE) is not necessarily contained in the set of NE (nor does the set of NE contained in the set of T-SE (or T-SSE)). It might be the case that a single player can deviate unilaterally and strictly improve his own payoff, but if such a deviation reduces the payoff of a member of his coalition (or does not improve it), it will not be considered as a beneficial deviation.

We identify two extreme cases:

Single coalition case: where there is a single coalition that contains all of the players; i.e., $T = \{N\}$. In the single-coalition case the set of T-SSE outcomes coincides with the set of Pareto-optimal outcomes; thus there always exists a T-SSE.

Claim 2.6. *Every finite game admits a T-SSE if $T = \{N\}$.*

Fully distributed case: This is the case in which each individual player constitutes a coalition; i.e., $T = \{\{1\}, \ldots, \{n\}\}$. In the fully-distributed case the set of T-SE coincides with the set of T-SSE and with the set of NE. Thus, any game that admits a pure NE admits a T-SSE as well.

A direct corollary of the above observations is that every 2-player game that admits a pure NE admits a T-SSE for any T. This is interesting, for example, in the context of potential (or congestion) games, where many of the counter examples refuting the existence of a SE are 2-player games (see, e.g., SE in cost-sharing connection games [4]). Yet, these games always admit some T-SSE.

2.1 Resource Selection Games

A resource selection setting is characterized by the tuple $< M, N, \{b_i(\cdot)\}_{i=1}^{m} >$, where $M = \{M_1, \ldots, M_m\}$ is the set of resources, $N = \{1, \ldots, n\}$ is the set of players (jobs) and $b_i(l) \in \mathbb{R}$ is the cost of resource M_i under a load of l players. We also denote the cost function of resource M_i by a vector $b_i = (b_i(1), b_i(2), \ldots, b_i(n))$. A resource selection setting has identical resources if $\forall i, i' \in \{1, \ldots, m\}$ $\forall l \in \{1, \ldots, n\}$ $b_i(l) = b_{i'}(l)$. In identical resources settings we will use the vector $b = (b(1), \ldots, b(n))$ to denote the cost vector of all the resources.

A *one-shot resource selection game* (RSG) has N as the set of players, and we identify the set of resources with the set of actions; i.e., the action space S_J of player $J \in N$ are all the individual resources, i.e., $S_J = M$ $\forall J \in N$. The action profile space is $S = \times_{J=1}^{n} S_J$. In an action profile $s \in S$ player J selects resource s_J as its action. The load of a resource M_i in the action profile $s \in S$, denoted $l_i(s)$, is the number of players that chose resource M_i. The cost of a player J who chose resource M_i under profile s is $c_J(s) = b_i(l_i(s))$.

We assume that the cost function $b_i(\cdot)$ of all resources is non decreasing; thus, if $b_i(l) < b_i(l')$ then $l < l'$. In some cases, we will assume a strictly-increasing cost function; i.e., $\forall i \forall l$ $b_i(l) < b_i(l+1)$. In this case, $b_i(l) \leq b_i(l')$ implies $l \leq l'$

Every RSG is a congestion game, thus admits a NE in pure actions. In addition, it has been shown in [8] that every RSG with non-decreasing cost functions admits a SE. Therefore, by Observation 2.5 it also admits a T-SE for any T. Yet, as we shall see, an RSG with non-decreasing cost functions might not admit a T-SSE, nor shall a repeated RSG necessarily admit a T-SE. These two matters shall be our focus in the following two sections, respectively.

3 T-Super Strong Equilibrium (T-SSE) Existence

Every RSG admits a SE [8], and by Claim 2.5 admits a T-SE as well. However, an RSG might not admit any SSE. This non-existence may occur even for an RSG with 2 identical strictly-increasing resources, as the following observation shows.

Observation 3.1. *There exists a one-shot RSG with* 2 *identical strictly increasing resources that does not admit any SSE.*

Proof. Consider an RSG with 2 identical resources of cost function $b = (1, 2, 3)$ and 3 players $N = \{1, 2, 3\}$. If all three players share the same resource, this is obviously not a SSE (and not even a NE). Suppose WLOG that players $1, 2$ are assigned to M_1 and player 3 is assigned to M_2. Then, players 1 and 2 can deviate such that player 1 migrates to M_2, incurring the same cost as before, while player 2 reduces its cost from 2 to 1. Therefore, this game does not admit a SSE. □

3.1 The case of identical, strictly-increasing resources

While a SSE might not exist even under the restricted setting of 2 identical strictly increasing resources, as we shall soon show, every RSG with identical strictly-increasing resources admits a T-SSE for any T. Before formulating the theorem, we introduce the following lemma and definition we shall use in the sequel.

Lemma 3.2. *Let G be an RSG with m identical strictly increasing resources, and let s be a NE of G. Suppose there is a coalition Γ that can weakly improve by deviating to a profile $s' = (s'_\Gamma, s_{-\Gamma})$. It holds that $l_i(s') \leq l_i(s) + 1$ $\forall i \in \{1, \ldots, m\}$.*

Definition 3.3. Let $l(s) = (l_1(s), \ldots, l_m(s))$ be the congestion vector of a profile s, sorted in non-increasing order. A *T-spread-out-s* assignment is an assignment obtained by filling out the resources by spreading out the members of each coalition by non-increasing order of $|T_i|$ on the resources, according to the sorted vector $l(s)$.

Theorem 3.4. *Every RSG with identical strictly increasing resources admits a T-SSE for any T.*

3.2 The case of 2 strictly-increasing non-identical resources

In the following few paragraphs we consider the case of 2 resources, but move to the more general case of non-identical resources. We begin with several characteristics of RSG's with 2 resources.

We distinguish between two types of deviations by a coalition Γ on 2 resources, namely *uni-directional* and *bi-directional* deviations. In a uni-directional deviation, some jobs in Γ deviate from one resource to the second one. In a bidirectional deviation, a set Γ_1 ($|\Gamma_1| > 0$) deviates from M_1 to M_2 and a set Γ_2 ($|\Gamma_2| > 0$) deviates from M_2 to M_1, s.t. $\Gamma = \Gamma_1 \cup \Gamma_2$ and $\Gamma_1 \cap \Gamma_2 = \emptyset$.

Lemma 3.5. *Let G be an RSG with 2 strictly increasing resources with cost functions $b_1(\cdot)$ and $b_2(\cdot)$, and let s be a NE of G. Suppose WLOG that $b_1(l_1(s)) \leq b_2(l_2(s))$. Suppose there is a* bidirectional *coalition Γ that has a weakly-profitable deviation to a profile $s' = (s'_\Gamma, s_{-\Gamma})$. Then, the coalition must be of the following structure: it should include the set $S = \{J | s_J = M_1, J \in \Gamma\}$ (which deviate to M_2) and $|S| + 1$ coalition members from M_2.*

Lemma 3.6. *Let G be an RSG with 2 strictly increasing resources, and let s be a NE of G. Suppose there is a unidirectional deviation by coalition $\Gamma \subset T_i$ that has a weakly-profitable deviation to a profile $s = (s'_\Gamma, s_{-\Gamma})$. Then, $s_J = s_{J'}$ $\forall J, J' \in T_i$.*

Definition 3.7. A vector $(\sigma_1, \sigma_2, \ldots \sigma_m)$ is *lexicographically smaller than $(\hat{\sigma}_1, \hat{\sigma}_2, \ldots \hat{\sigma}_m)$ if for some i, $\sigma_i < \hat{\sigma}_i$ and $\sigma_k = \hat{\sigma}_k$ for all $k < i$.*

An action profile s is cost-wise lexicographically smaller than s' if the cost vector $c(s) = (c_1(l_1(s)), \ldots, c_m(l_m(s)))$, sorted in non-increasing order, is smaller lexicographically than $c(s')$, sorted in non-increasing order. We denote this relationship by $s \prec s'$.

Lemma 3.8. *Let s be a cost-wise lexicographic minimum of an RSG game G with non-decreasing resources. Then, s is a NE of G.*

Proof. Suppose by way of contradiction that there exists a job J that can improve its cost by migrating to a resource s'_J. i.e., $c_J(s) > c_J(s')$. But since $c_{s_J}(s') \leq c_{s_J}(s)$ and $c_{s'_J}(s') < c_{s_J}(s)$, and in addition $c_{s_J}(s) > c_{s'_J}(s)$, we reach a contradiction to the lexicographic minimality of s. □

Theorem 3.9. *Any RSG with 2 strictly increasing resources admits a T-SSE for any T.*

3.3 The case of m non-decreasing non-identical resources

We next consider the more general case of non-decreasing cost functions. We show that if $|T_i| \leq 2\ \forall i$ a T-SSE always exists. While this theorem follows as a special case of [10], we choose to present it due to the simplicity of the proof. Before formulating our theorem, we present the following lemma.

Lemma 3.10. *Let G be an RSG with m non-decreasing resources, and let s be a NE of G. Given a coalition T_j, if it holds that $|\{J|J \in T_j, s_J = M_i\}| \leq 1\ \forall i$, then T_j has no weakly profitable deviation.*

With this we are ready to state the theorem.

Theorem 3.11. *Every RSG G with non-decreasing resources in which $|T_i| \leq 2\ \forall i$ admits a T-SSE.*

Proof. Let s be a NE of G and consider a T-spread-out-s assignment (we abuse notation and use s to denote both the Ne and the T-spread-out NE). In s, there might be only a single resource that contains more than a single member of each coalition, denote it M_i. Since s is a NE, no singleton can deviate. By Lemma 3.10, no coalition (of size 2) that is assigned to different resources can deviate either. Thus, we should only consider deviations of pairs (recall $|T_i| \leq 2\ \forall i$) that are assigned to the same resource. Suppose jobs J, J' are assigned to M_i and let s' be a weakly profitable deviation. If both players deviate, it contradicts s being a NE. Thus, we should only consider a deviation in which $s'_{J'} = M_i$ and $s'_J = M_k$, $k \neq i$.

It must hold that $c_J(s') \leq c_J(s)$, thus $b_k(l_k(s')) = b_k(l_k(s)+1) \leq b_i(l_i(s))$, and by s being a NE it must hold that $b_k(l_k(s)+1) \geq b_i(l_i(s))$; thus $b_k(l_k(s)+1) = b_i(l_i(s))$. Therefore, the cost of player J' must strictly reduce; i.e., $b_i(l_i(s)-1) < b_i(l_i(s))$.

We next claim that s' is also a NE. To show this we show that s' is also a NE. By $b_k(l_k(s)+1) = b_i(l_i(s))$ a unilateral deviation from M_i to M_k or in the other direction are not profitable. A unilateral deviation from $M_l, l \neq i, k$ to M_i is not profitable either since by s being a NE, a job from M_l cannot improve by deviating to M_k, thus by $b_k(l_k(s)+1) = b_i(l_i(s))$ it cannot improve by deviating to M_i either. By s being a NE, it follows that all other unilateral deviations are not profitable either. A similar argument shows that after each deviation of a pair that is assigned to M_i, we should again consider only such deviations. But this process is limited by the number of pairs that are assigned to M_i, which is finite. We conclude that this process must converge to a T-SSE. □

3.4 The case of 2 non-decreasing resources

We now turn to another extension. In this subsection we consider the existence of T-SSE for two identical resources, but allow the resources to have general non-decreasing cost functions.

The following lemma provides a condition that must be satisfied in order for a unilateral deviation to occur. It holds for non-identical resources too, and has been used in section 3.2 as well.

Lemma 3.12. *Let G be an RSG with 2 non-decreasing resources with cost functions $b_1(\cdot)$ and $b_2(\cdot)$, and let s be a cost-wise lexicographic minimum, T-spread-out NE of G with loads $l_1(s)$ and $l_2(s)$ respectively. A unidirectional deviation of $\Gamma \subseteq T_i$ is possible only from a resource that contains all the members of T_i.*

Lemma 3.13. *Let G be an RSG with 2 non-decreasing identical resources, and let s be a NE of G. If there exists a weakly profitable bi-directional deviation, there also exists a weakly profitable uni-directional deviation of a smaller coalition.*

With this, we are ready to state the theorem.

Theorem 3.14. *Every RSG with 2 non-decreasing identical resources admits a T-SSE for any T.*

4 T-Strong Equilibrium (T-SE) Existence in Repeated RSG

Consider a *one-shot game* G and an integer R. We define the repeated game $\hat{G} =< G, R >$ as R plays of G, where in period t players choose strategies $(s_1, \ldots, s_n) \in (S_1, \ldots, S_n)$ after observing the actions taken by all the users in all previous periods. A strategy of player J is a function p_J specifying the action of player J at time t, given the history up to time $t-1$. Player J's cost in $\hat{G}$ is $c_J(p) = \sum_{t=1}^{R} c_J(s_1(t), \ldots, s_n(t))$, where $s_J(t)$ is player J's one-shot game action in period t according to p_J.

There is a crucial difference between SE and NE in repeated games. Suppose s is a NE of the game G. Then, playing s in every round of the repeated game must be a NE of the repeated game. In contrast, if s is a SE of the game G, it is not necessarily the case that playing s in every round of the repeated game is a SE of the repeated game.

For example, while every RSG admits a SE, even on the very simple RSG that is composed of two identical resources with non-decreasing cost functions and 3 players, its repeated version might not admit a SE.

Observation 4.1. *[15]. There exists a repeated RSG with 2 identical non-decreasing resources and 3 players that does not admit a SE.*

Similarly, if s is a T-SE of the game G, it is not necessarily the case that playing s in every round of the repeated game is a T-SE of the repeated game (as exemplified by Theorem 4.1). Thus, characterizing the set of repeated games that admit a T-SE is a challenging goal.

4.1 The general case

The following theorem shows that every repeated RSG on m non-decreasing resources admits a T-SE if T contains no singletons.

We first define a Γ-minimal player and present several lemmas that will be used in the proof of the theorem.

Definition 4.2. Let G be a one-shot game and let s be an action profile in G. Player i is said to be Γ^s-minimal if for any action profile $s' = (s'_\Gamma, s_{-\Gamma})$ it holds that $c_i(s) \leq c_i(s')$.

Lemma 4.3. *Let $\hat{G} =< G, R >$, and let s be a strategy profile s.t. $\forall T_i\ \exists J \in T_i$ s.t. J is T_i^s-minimal $\forall r \in R$. Then, playing s in every round of $\hat{G}$ is a T-SE of $\hat{G}$.*

In our characterization we use the following definition.

Definition 4.4. Let $\mathfrak{G}$ denote a family of games, and $\mathfrak{T}$ denote a family of T-structures. The family $\mathfrak{T}$ is said to be a condition for the existence of T-SE in $\mathfrak{G}$ if (i) every game $G \in \mathfrak{G}$ in which $T \in \mathfrak{T}$ admits a T-SE; and (ii) $\exists T \notin \mathfrak{T}$ s.t. there exists $G \in \mathfrak{G}$ with T that does not admit a T-SE.

In what follows we provide a full characterization of the family $\mathfrak{T}$ that serves as a condition for the existence of T-SE in repeated RSGs with m non-decreasing resources. We first introduce a family $\mathfrak{T}$ that guarantees the existence of T-SE.

Theorem 4.5. *Every repeated RSG with m non-decreasing resources admits a T-SE if $|T_i| \geq 2\ \forall i$.*

Proof. Let G be a one-shot RSG with m non-decreasing resources. Let C be a set containing an arbitrary single agent from every T_i, and let M_j be a resource s.t. $b_j(|C|) \geq b_i(|C|)\ \forall i \neq j$. Since $|T_i| \geq 2\ \forall i$, it holds that $|N \setminus C| \geq |C|$. Let $G' =< M, N, b'(\cdot), c(\cdot) >$ be the game induced by G where all players $J \in N \setminus C$ are assigned to M_j; i.e., $b'_j(l) = b_j(l + |N \setminus C|)$ and $b'_i(l) = b_i(l)\ \forall i \neq j$.

We shall prove that there exists a NE s_C of players $J \in C$ in G' in which $s_J \neq M_j\ \forall J \in C$. To see this, consider best-response dynamics starting from an arbitrary assignment of the players $J \in C$ on resources $M \setminus M_j$. Since this is a potential game, best-response dynamics must converge to a NE. For any profile s of the players $J \in C$, it holds that $c_k(l_k(s)) \leq c_k(|C|) \leq c_j(|C|) \leq c_j(|N \setminus C|)\ \forall k \in \{1, \ldots, m\}$. Therefore, we can assume WLOG that in the best-response dynamics players do not migrate to resource M_j. This concludes the statement.

Let $s = (s_C, s_{N\setminus C})$, where $s_J = M_j\ \forall J \in N \setminus C$ and s_C is a NE of players $J \in C$ in G' in which $s_J \neq M_j\ \forall J \in C$. We claim that playing s in every stage is a T-SE of the repeated game.

For every i, let J_i be the player J s.t. $J \in T_i \cap C$. We claim that s is T_i-minimal for $J_i\ \forall i$. First, since s is a NE, J_i cannot reduce its cost by a unilateral deviation. In addition, $s_J = M_j\ \forall J \in T_i \setminus \{J_i\}$. Thus, to show that s is T_i-minimal for J_i, it is sufficient to show that J_i's cost will not be reduced in the profile s' where all the players in $T_i \setminus \{J_i\}$ migrate from M_j and J_i migrates to M_j. Denote $s_{J_i} = M_i$. It holds that $l_i(s') \leq |C|$ and $l_j(s') \geq |C|$. It follows that:

$$c_{J_i}(s) = b_i(l_i(s)) \leq b_i(|C|) \leq b_j(|C|) \leq b_j(l_j(s')) = c_{J_i}(s')$$

Thus, s is T_i-minimal for J_i in the one-shot game, and by Lemma 4.3, playing s in every round of the game constitutes a T-SE. □

In addition, every repeated RSG on m non-decreasing resources admits a T-SE if $|T_i| \leq 2\ \forall i$. We first introduce a lemma that will be used here and in the sequel.

Lemma 4.6. *Let G be an RSG with m non-decreasing resources, and let s be a NE of G s.t. $\forall i\ \forall j\ |\{J | J \in T_j, s_J = M_i\}| \leq 1$ (i.e., no resource contains more than a single representative of each coalition). Then, playing s in every round constitutes a T-SE of the repeated game.*

With this we are ready to state the theorem.

Theorem 4.7. *Every repeated RSG with m non-decreasing resources admits a T-SE if $|T_i| \leq 2\ \forall i$.*

In addition, there exists a repeated RSG that does not adhere to the structure described above that does not admit a T-SE, as the following theorem shows.

Theorem 4.8. *There exists a repeated RSG with 2 identical non-decreasing resources s.t. $|T_1| = 1$ and $|T_2| = 3$ that does not admit a T-SE.*

Proof. Let G be an RSG with two identical resources with cost function $b(\cdot)$ and four players, where $T = \{T_1, T_2\}$, s.t. $|T_1| = 3$ and $|T_2| = 1$, $b(1) + 2b(3) < 3b(2)$, and $b(2) < b(3)$.

Consider the game $\hat{G} =< G, 3 >$. Suppose by way of contradiction that the repeated game above admits a T-SE s. In the third (and last) stage of the game, the singleton can never share a resource with more than one additional player, since if it does, it incurs a cost of at least $b(3)$ and by deviating it can incur a cost of at most $b(2)$. Second, the three players in T_1 cannot all share a resource, since if one of them deviates, all three players reduce their cost from $b(3)$ to $b(2)$. Therefore, in the third stage, every resource should be assigned exactly 2 players. Using a backward induction argument, under the profile s, in every stage of the game two players should be assigned to every resource, i.e., $l_i(s) = 2 \ \forall i \in M$. Consider the following deviation s' of T_1: each player in T_1 is left alone in one of the stages and has a load of 3 in the other two stages. For every player $J \in T_1$, it holds that $c_J(s') = 2c(3) + c(1) < 3c(2) = c_J(s)$. Therefore, s' is a strongly-profitable deviation of T_1 and the game admits no T-SE. □

The following characterization follows.

Corollary 4.9. *Let $\mathfrak{G}$ be the family of repeated games with non-decreasing resources. $\mathfrak{T}$ is a condition for the existence of T-SE in the family $\mathfrak{G}$ if $|T_i| \leq 2 \ \forall i$, or $|T_i| \geq 2 \ \forall i$.*

Note that the construction given in Theorem implies that the existence of T-SE in one-shot RSGs does not apply to general congestion games. This can easily be verified by constructing a congestion game that consists of three networks that are composed serially, where each network is composed of 2 parallel edges with the cost functions and T-structure given in the example above.

4.2 The case of majority of singletons

For special cases, we have a more refined characterization.

Definition 4.10. Let $\mathfrak{G}$ denote a family of games, and $\mathfrak{T}$ denote a family of T-structures. The family $\mathfrak{T}$ is said to be a strong condition for the existence of T-SE in $\mathfrak{G}$ if (i) every game $G \in \mathfrak{G}$ in which $T \in \mathfrak{T}$ admits a T-SE; and (ii) $\forall T \notin \mathfrak{T}$ there exists $G \in \mathfrak{G}$ with T that does not admit a T-SE.

In particular, if the majority of the players are singletons and the resources are identical, we fully characterize the T-structures that serve as the strong condition for the existence of T-SE in repeated games. The characterization is slightly different for odd and even number of players.

4.2.1 Odd number of players

Theorem 4.11. *Let $\mathfrak{G}$ be the family of repeated games with identical non-decreasing resources, an odd number of players and a majority of singletons. $\mathfrak{T}$ is a strong condition for the existence of T-SE in the family $\mathfrak{G}$ if and only if $|T_i| \leq 3 \ \forall i$.*

The theorem above follows directly from the following two theorems. The first one identifies the T-structures under which there always exists a game that does not admit any T-SE.

Theorem 4.12. *For every T s.t. $\exists i$ s.t. $|T_i| \geq 4$, there exists a repeated RSG on identical resources with an odd number of players and a majority of singletons that does not admit a T-SE.*

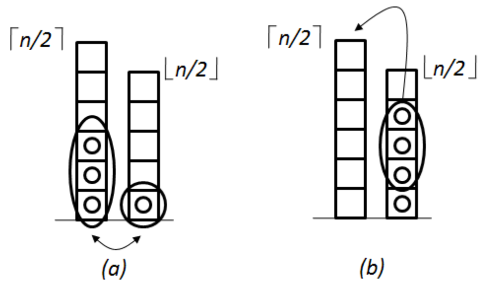

Figure 1: (a) a bidirectional deviation leaving a single member alone; (b) a unidirectional deviation leaving a single member in the less-loaded machine.

Proof. Let $max_i|T_i| = x$, and consider an RSG on two identical strictly increasing resources M_1, M_2, repeated x rounds, such that

$$(x-1) \cdot b(\left\lceil \frac{n}{2} \right\rceil + x - 1) + b(\left\lceil \frac{n}{2} \right\rceil - x + 1) < x \cdot b(\left\lfloor \frac{n}{2} \right\rfloor) \tag{1}$$

(this is a valid cost function since $x \geq 4$). Suppose towards contradiction that there exists a T-SE in the repeated game and denote it s. In the last stage of the game, it must hold that $|l_1(s) - l_2(s)| \leq 1$. To see this, suppose towards contradiction that $l_1(s) > l_2(s) + 1$ (WLOG), then there must be a singleton on M_1. By monotonicity, $b(l_1(s) + 1) < b(l_1(s))$, thus the singleton can improve its cost by migrating to M_2. Following a backward induction argument, it must hold that in every round of the game $l_1(s) - l_2(s) \leq 1$, thus in every round $l_1(s) = \lfloor \frac{n}{2} \rfloor$ and $l_2(s) = \lceil \frac{n}{2} \rceil$ (or vice versa).

In every round, $\forall J \in T_i$, J is assigned to a machine with load at least $\lfloor \frac{n}{2} \rfloor$, thus its total cost in the repeated game is at least $x \cdot b(\lfloor \frac{n}{2} \rfloor)$.

The coalition can deviate such that in every round one of the members of the coalition (a different member in every round) will be left alone. One can easily verify that a coalition member who is left alone will incur a cost of at most $b(\lceil \frac{n}{2} \rceil - x + 1)$ (the most extreme case is the one in which one of the coalition members is assigned to the less-loaded resource in every round of the game, see Figure 1(a) for an illustration of a coalition of size 4). In addition, each of the coalition members other than the one left alone will incur a cost of at most $b(\lceil \frac{n}{2} \rceil + x - 1)$ (the most extreme case is the one in which all the coalition members were assigned to the less-loaded resource and $x - 1$ of them migrate to the more-loaded one, see Figure 1(b) for an illustration of a coalition of size 4).

After the deviation, the cost of each coalition member is at most $(x-1) \cdot b(\lceil \frac{n}{2} \rceil + x - 1) + b(\lceil \frac{n}{2} \rceil - x + 1)$, which is smaller than $x \cdot b(\lfloor \frac{n}{2} \rfloor)$ (its cost in s) by Equation 1. □

The following theorem shows that under a majority of singletons and an odd number of agents, if all the coalitions are of size at most 3, a T-SE always exists.

Theorem 4.13. *For every repeated RSG $\hat{G}$ on m identical non-decreasing resources with a majority of singletons and an odd number of players, if $|T_i| \leq 3$ $\forall i$, $\hat{G}$ admits a T-SE.*

4.2.2 Even number of players

Theorem 4.14. *Let $\mathfrak{G}$ be the family of repeated games with identical non-decreasing resources, an even number of players and a majority of singletons. $\mathfrak{T}$ is a strong condition for the existence of*

T-SE in the family $\mathfrak{G}$ *if and only if* $|T_i| \leq 2\ \forall i$.

The proof is based on the following two theorems.

Theorem 4.15. *For every* T *s.t.* $\exists i$ *s.t.* $|T_i| \geq 3$*, there exists a repeated RSG on identical resources with an even umber of players and a majority of singletons that does not admit a* T*-SE.*

The following theorem shows that under a majority of singletons and an even number of players, if all the coalitions are of size at most 2, a T-SE always exists.

Theorem 4.16. *For every repeated RSG on* m *identical non-decreasing resources with a majority of singletons and an even number of players, if* $|T_i| \leq 2\ \forall i$*,* G *admits a* T*-SE.*

Proof. Let s be a T-spread-out NE of the one-shot game. Since $|M| \geq 2$ and $|T_i| \leq 2\ \forall i$, no two members of the same coalition are assigned to the same resource. By Lemma 4.6, playing s in every round of the game constitutes a T-SE of the repeated game. □

References

[1] N. Andelman, M. Feldman, and Y. Mansour. Strong Price of Anarchy. In *SODA'07*, 2007.

[2] Itai Ashlagi, Piotr Krysta, and Moshe Tennenholtz. Social context games. In *WINE-08*, 2008.

[3] Robert Aumann. Acceptable Points in General Cooperative n-Person Games. In *Contributions to the Theory of Games*, volume 4, 1959.

[4] Amir Epstein, Michal Feldman, and Yishay Mansour. Strong Equilibrium in Cost Sharing Connection Games. In *ACMEC*, 2007.

[5] Amos Fiat, Haim Kaplan, Meital Levi, and Svetlana Olonetsky. Strong Price of Anarchy for Machine Load Balancing. In *ICALP*, 2007.

[6] D. Fotakis, S. Kontogiannis, and P. Spiraklis. Atomic Congestion Games Among Coalitions. In *ICALP*, pages 573–584, 2006.

[7] Ara Hayrapetyan, Eva Tardos, and Tom Wexler. The Effect of Collusion in Congestion Games. In *38th ACM Symposium on Theory of Computing*, 2006.

[8] Ron Holzman and Nissan Law-Yone. Strong equilibrium in congestion games. *Games and Economic Behavior*, 21:85–101, 1997.

[9] Ron Holzman and Nissan Law-Yone (Lev-tov). Network structure and strong equilibrium in route selection games. *Mathematical Social Sciences*, 46:193–205, 2003.

[10] Sergey Kuniavsky and Rann Smorodinsky. Coalitional Congestion Games. In *Technion MSc Thesis*, 2007.

[11] Stefano Leonardi and Piotr Sankowski. Network Formation Games with Local Coalitions. In *PODC*, 2007.

[12] D. Monderer and L. S. Shapley. Potential games. *Games and Economic Behavior*, 14:124–143, 1996.

[13] P.Borm, Giovanni Facchini, F. van Megen, S. Tijs, and M. Voorneveld. Congestion games and potentials reconsidered. *International Game Theory Review*, 46:283–299, 1999.

[14] R. W. Rosenthal. A class of games possessing pure-strategy nash equilibria. *International Journal of Game Theory*, 2:65–67, 1973.

[15] Moshe Tennenholtz and Aviv Zohar. Learning equilibrium in congestion games. In *AAMAS-09*, 2009.

A Missing Proofs from Section 2

Proof. of Observation 2.5:

Let s be a SE. This means that no coalition Γ has a strongly-profitable deviation from s. In particular, none of the coalitions T_i has a strongly-profitable deviation; thus s is a T-SE. The same argument holds for T-SSE under weakly-profitable deviations. □

Proof. of Corollary **??** In a 2-player game, it is either the single-coalition case or the fully-distributed case. In the former case, a T-SSE exists by Claim 2.6. In the latter case, a T-SSE exists since a NE always exists. □

B Missing Proofs from Section 3

Proof. of Lemma 3.2:

Since s is a NE, for any two resources M_i, M_j, $b_i(s) \leq b_j(l_j(s)+1)$ (otherwise a player assigned to M_i can improve by deviating to M_j). Suppose by way of contradiction that there exists a resource M_k s.t. $l_k(s') \geq l_k(s)+2$. Then, there must exist a player J s.t. $s'_J = M_k$ and $s_J = M_j$ for some resource $M_j \neq M_k$. We show that this player's cost increases after the deviation. It holds that $c_J(s) = b_j(s) \leq b_k(l_k(s)+1) < b_k(l_k(s)+2) \leq b_k(l_k(s')) = c_J(s')$, where the first inequality follows by s being a NE, and the second inequality follows from strict monotonicity. □

Proof. of Theorem 3.4:

Let s be a NE of G. We claim that a T-spread-out-s assignment is a T-SSE. Suppose by way of contradiction there is a weakly profitable deviation of some coalition to a profile s'. Since the resources are identical and strictly increasing there must exist L s.t. $l_i(s) \in \{L, L+1\}\ \forall i$. Denote by k the number of resources of load L. Players assigned to resources of load L and $L+1$ are denoted "low" and "high" players, respectively.

We first claim that $l_i(s') \geq L\ \forall i$. Suppose by way of contradiction that $\exists i$ s.t. $l_i(s') \leq L-1$. Then, in order to assign all the low jobs, there must exist k additional resources of load at most L. But then it must hold that $\exists i$ s.t. $l_i(s') \geq L+2$, contradicting Lemma 3.2. We conclude that $l_i(s') \in \{L, L+1\}\ \forall i$. Since the total number of players remains the same, there must exist k resources of load L and $m-k$ resources of load $L+1$ in s'.

For every low job J, it must hold that $l_i(s') \leq L$, thus, for every high job it must hold that $l_i(s') = L+1$. Therefore, no job in the coalition strictly improves its load, and the statement follows. □

Proof. of Lemma 3.5:

In every coalitional deviation, at least one resource's load should increase, otherwise, if a coalition's member strictly improves, there must be another coalition's member that becomes worse-off. By Lemma 3.2, each resource can increase by at most a single job after the deviation. M_2 cannot increase, since if it does, there must be a coalition member who suffers a cost of $b_2(l_2(s)+1) > b_2(l_2(s)) \geq b_1(l_1(s))$ and thus strictly increases its cost. Thus, in any coalitional deviation, M_1 increases by a single job; that is $|S|$ members migrate from M_1 to M_2, and $|S|+1$ migrate from M_2 to M_1. Since $l_1(s') > l_1(s)$, $b_1(l_1(s')) > b_1(l_1(s))$, implying that no coalition member stays on M_1 in s'. □

Proof. of Lemma 3.6:

Suppose WLOG that $s_J = M_2 \ \forall J \in \Gamma$, and suppose by way of contradiction that $\exists J \in T_i$ s.t. $s_J = M_1$. Then, $c_J(s) < c_J(s') \ \forall J \in T_i \setminus \Gamma$; a contradiction. □

Proof. of Theorem 3.9:

Let s be a cost-wise lexicographic minimum T-spread-out profile, and suppose WLOG that $l_2(s) \geq l_1(s)$. By Lemma 3.8, s is a NE.

If $b_1(l_1(s)) = b_2(l_2(s))$, it is easy to verify that no coalition has a weakly-profitable deviation: consider a deviation s'. If $l_1(s) = l_1(s')$ and $l_2(s) = l_2(s')$, all players incur the same cost in s' as in s. Otherwise, if one of the resources grows by a single job (see Lemma 3.2), that job's cost increases. Thus, we need to consider only two cases as follows:

case a: $b_1(l_1(s)) < b_2(l_2(s))$.

Consider a deviation s'. By Lemma 3.2, the load of every resource can increase by at most a single job. Since $b_1(l_1(s)) < b_2(l_2(s))$, if M_2 increases, there must be a job in the coalition whose cost increases, thus it must be the case that $l_1(s') = l_1(s)+1$ and $l_2(s') = l_2(s)-1$. The deviation to s' can have one of two structures: (i) a unidirectional deviation from M_2 to M_1, where the jobs that stay on M_2 improve and the ones migrating to M_1 are indifferent. This case can occur only if $\exists i$ s.t. $s_J = M_2 \ \forall J \in T_i$ (by Lemma 3.12, to be proven in section 3.4). In this case $b_2(l_2(s')) = b_2(l_2(s)-1) < b_2(l_2(s))$ and $b_2(l_2(s)) = b_1(l_1(s)+1) = b(l_1(s'))$. Thus

$$b_2(l_2(s')) = b_2(l_2(s)-1) < b_1(l_1(s)+1) = b_1(l_1(s')); \tag{2}$$

or (ii) a bidirectional deviation with a structure as described in Lemma 3.5. In this case it must hold that $b_2(l_2(s)) \geq b_1(l_1(s)+1) = b(l_1(s'))$ (so that the jobs migrating to M_1 do not get worse off) and $b_2(l_2(s')) = b_2(l_2(s)-1) \leq b_1(l_1(s))$ (so that the jobs migrating to M_2 do not get worse off). It is easy to see that if one of the above two inequalities is strict, s' is cost-wise lexicographically smaller than s, contradicting the minimality of s. Therefore,

$$b_2(l_2(s)) = b_1(l_1(s)+1) \tag{3}$$

and

$$b_2(l_2(s)-1) = b_1(l_1(s)) \tag{4}$$

Integrating the strict monotonicity into the equations we get $b_2(l_2(s')) = b_2(l_2(s)-1) < b_2(l_2(s)) = b_1(l_1(s)+1) = b_1(l_1(s'))$. We conclude that in (i) as well as in (ii) Equation 2 must hold.

After this stage, for every i it holds that the number of players from T_i that are on M_1 can be greater than the number of players from T_i that are on M_1 by at most 1. This claim holds in s since

s is a T-spread-out assignment. In case(i) (unidirectional deviation), M_1 contains a single agent of the deviating coalition, and the other coalitions are like in s. In case (ii) (bidirectional deviation), all the members of the deviating coalition that were on M_1 deviated to M_2, and a number of players greater than this number by 1 moved to M_1, so the claim holds for the deviation coalition, and it obviously holds for the other coalitions (just like in s).

We next consider a weakly-profitable deviation s'' from s'. s'' cannot be a unidirectional deviation from M_2 to M_1 since $b_1(l_1(s')) > b_2(l_2(s'))$. s'' cannot be a unidirectional deviation from M_1 to M_2 either since M_1 does not contain whole coalitions (see Lemma 3.6). Thus, it can only be a bidirectional deviation. In any bidirectional deviation s'', M_2 should increase by a single job, thus $l_1(s'') = l_1(s') - 1 = l_1(s)$ and $l_2(s'') = l_2(s') + 1 = l_2(s)$. Since the cost of the job migrating to M_1 cannot increase, it must hold that $b_1(l_1(s)) = b_1(l_1(s'')) \leq b_2(l_2(s')) = b_2(l_2(s) - 1)$. If the deviation to s' was bidirectional (case (ii)), then $b_2(l_2(s) - 1) = b_1(l_1(s))$ (see Equation 4), and $b_2(l_2(s)) = b_1(l_1(s) + 1)$ (by Equation 3). Therefore, the only players who can strictly improve by this deviation are the jobs that stay on M_1. But since the number of players migrating from M_1 to M_2 is greater by 1 than the number of players moving in the opposite direction, it must hold that in s' the number of players from the deviating coalition assigned to M_1 is greater by at least 2 than those assigned to M_2. But this contradicts our claim above (showing that it can be greater by at least 1).

If, however, the deviation to s' was unidirectional (case (i)), then a bidirectional from s' to s'' may still be possible, and s'' is a cost-wise lexicographic minimum (since the resources' loads according to s'' are identical to s). Therefore, s'' resembles s in almost all aspects, except that the number of i s.t. T_i is fully assigned to M_2 is smaller by 1 than in s.

Since the number of coalitions is finite, we can repeat this process until there are no coalitions that are fully assigned to M_2. Then, the first stage deviation must be a bidirectional one, and the obtained assignment must be a T-SSE from the exact same reasoning as above (since the number of players from the deviating coalition assigned to M_1 cannot be greater by at least 2 than those assigned to M_2).

case b: $b_1(l_1(s)) > b_2(l_2(s))$. We show that in this case s is a T-SSE. In any deviation s', M_2 should increase by a single job. By Lemma 3.12 the deviation cannot be unidirectional since there does not exist any coalition that is assigned fully to M_1, and a singleton cannot unilaterally deviate since s is a NE. Therefore, it can only be a bidirectional deviation of a coalition T_i, where all $J \in T_i$ assigned to M_2 move to M_1 and a number of players greater by 1 move from M_1 to M_2 (by Lemma 3.5). In this case, it must hold that $b_2(l_2(s)) \geq b_1(l_1(s) - 1)$ (so that the cost of the players moving to M_1 does not increase) and $b_2(l_2(s)+1) \leq b_1(l_1(s))$ (so that the cost of the players moving to M_2 does not increase). In addition, since s is a NE, it must hold that $b_2(l_2(s) + 1) \geq b_1(l_1(s))$, thus $b_2(l_2(s) + 1) = b_1(l_1(s))$.

After such a deviation, there cannot remain any coalition members on M_2 (since the load on M_2 increases, thus its cost by strict monotonicity). Additionally, there cannot remain any coalition members on M_1, since s is a T-spread-out assignment in which the number of coalition members on M_1 can be greater than the number of coalition members on M_2 by at most 1 for any coalition). But since there must be a coalition member whose cost strictly decreases, it can only be members that migrated from M_2 to M_1. Thus, it must hold that $b_2(l_2(s)) > b_1(l_1(s) - 1)$. We get: $b_2(l_2(s)) > b_1(l_1(s) - 1) = b_1(l_1(s'))$ and $b_2(l_2(s')) = b_2(l_2(s) + 1) = b_1(l_1(s))$, implying that s' is cost-wise lexicographically smaller than s, contradicting the minimality of s. □

Proof. of Lemma 3.10:

Suppose by way of contradiction there is a weakly profitable deviation of T_j to $s' = (s_{-T_j}, s'_{T_j})$. Then, $\exists J \in T_j$ $s.t.$ $c_J(s') < c_J(s)$. Let $s_J = M_i$. It is easy to see that the only way for J to reduce its cost is by deviating to another resource, M_j, from which another job J' migrates (otherwise, it contradicts s being a NE) and $b_j(l_j(s')) = b_j(l_j(s)) < b_i(l_i(s))$.

J' cannot migrate to M_i since $b_j(l_j(s)) < b_i(l_i(s))$, neither can J' migrate to a resource $M_{j'}$ from which no job migrates, since if it does, $b_{j'}(l_{j'}(s)+1) \leq b_j(l_j(s)) < b_i(l_i(s))$, contradicting s being a NE. Thus, J' can only migrate to a resource from which another job migrates.

We prove by induction that each job that leaves a resource must migrate to a resource from which another job migrates. The base of the induction is the first job, as described above. Suppose that the claim holds up to the t^{th} resource $M_{j_1}, M_{j_2}, \ldots M_{j_t}$. For every $k \in \{1, \ldots, t\}$, it holds that $b_{j_k}(l_{j_k}(s)) \leq b_{j_{k-1}}(l_{j_{k-1}}(s)) \leq \ldots b_{j_2}(l_{j_2}(s)) < b_{j_1}(l_{j_1}(s))$, thus $b_{j_k}(l_{j_k}(s) < b_{j_1}(l_{j_1}(s))$ $\forall k$. Now consider the job that leaves resource M_t. First, it cannot migrate to M_{j_1} since if it does it holds that $b_{j_1}(l_{j_1}(s)) \leq b_{j_t}(l_{j_t}(s)) < b_{j_1}(l_{j_1}(s))$; in contradiction. Second, it cannot migrate to any resource M_k, $k \in \{2, \ldots, t-1\}$, since if it does it holds that $b_{j_k}(l_{j_k}(s)+1) \leq b_{j_t}(l_{j_t}(s)) < b_{j_1}(l_{j_1}(s))$, contradicting s being a NE. Third, it cannot migrate to another resource M_w from which no job migrates, since if it does it holds that $b_w(l_w(s)+1) \leq b_{j_t}(l_{j_t}(s)) < b_{j_1}(l_{j_1}(s))$, contradicting s being a NE. Thus, it must migrate to another resource from which another job migrates. Since the number of machines that contain jobs in T_j is finite, the last job leaving its own resource cannot migrate to any other resource. □

Proof. of Lemma 3.12:

Suppose by way of contradiction that T_i is spread out on the two resources and suppose WLOG that $b_1(l_1(s)) \leq b_2(l_2(s))$. The migrating jobs cannot reduce their costs, otherwise it contradicts s being a NE. Thus, the jobs that stay on their resources must be the ones reducing their costs, and those that migrate are indifferent.

case a: deviation of Δ jobs from M_2 to M_1. We obtain a vector $b_2(l_2(s)-\Delta), b_1(l_1(s)+\Delta)$ s.t. $b_2(l_2(s)-\Delta) < b_2(l_2(s))$. In addition, the jobs on M_1 cannot get worse off, thus it must hold that $b_1(l_1(s)) = b_1(l_1(s)+\Delta)$. But then s' is cost-wise lexicographically smaller than s; reaching a contradiction.

case b: deviation of Δ jobs from M_1 to M_2. We obtain a vector $b_2(l_2(s)+\Delta), b_1(l_1(s)-\Delta)$. The jobs from M_1 cannot get worse off thus it must hold that $b_2(l_2(s)+\Delta) = b_2(l_2(s)) = b_1(l_1(s))$. In addition, the jobs on M_1 must improve thus $b_1(l_1(s)-\Delta) < b_1(l_1(s))$. But then $s' \prec s$; reaching a contradiction. □

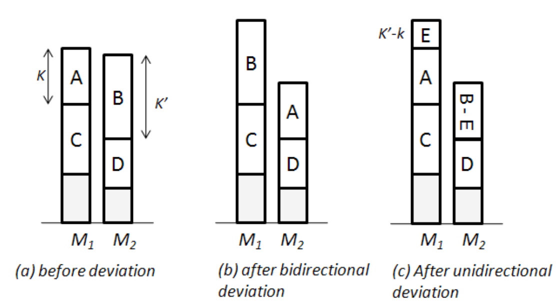

Figure 2: An illustration of a bidirectional deviation (b) and a smaller unidirectional deviation (c).

Proof. of Lemma 3.13:

Let $s' = (s'_\Gamma, s_{-\Gamma})$ be the profile obtained by the bidirectional deviation. Let $A = \{J | s_J = M_1, s'_J = M_2\}, B = \{J | s_J = M_2, s'_J = M_1\}, C = \{J | J \in \Gamma, s_J = M_1, s'_J = M_1\}, D = \{J | J \in \Gamma, s_J = M_2, s'_J = M_2\}$. Denote $k = |A|, k' = |B|$, and suppose WLOG that $k' > k$ (one can easily verify that if $k = k'$ a weakly profitable deviation does not exist). Denote $\Delta = k' - k$ (see Figure 2 for an illustration).

We first show that if $C \neq \emptyset$ there exists a smaller unidirectional deviation s'' from s in which Δ agents migrate from M_2 to M_1. Among the agents in B, we denote $E = \{J | s_J = M_2, s''_J = M_1\}$, $E \subseteq B$.

$\forall J \in C$, $c_J(s) = b(l_1(s))$, and $c_J(s') = b(l_1(s) + \Delta)$. Since J's cost cannot increase it holds that $b(l_1(s)) \geq b(l_1(s) + \Delta)$. On the other hand, by monotonicity $b(l_1(s)) \leq b(l_1(s) + \Delta)$; thus

$$b(l_1(s)) = b(l_1(s) + \Delta). \tag{5}$$

$\forall J \in A$, $c_J(s) = b(l_1(s))$, and $c_J(s') = b(l_2(s) - \Delta)$; thus $b(l_1(s)) \geq b(l_2(s) - \Delta)$. Also, $c_J(s'') = b(l_1(s) + \Delta) = b(l_1(s)) = c_J(s)$, by Equation 5.

$\forall J \in B$, $c_J(s) = b(l_2(s))$, and $c_J(s') = b(l_1(s) + \Delta)$; thus $b(l_2(s)) \geq b(l_1(s) + \Delta)$.

$\forall J \in E$, $c_J(s'') = b(l_1(s) + \Delta) = c_J(s')$, and for $J \in B \setminus E$, $c_J(s'') = b(l_2(s) - \Delta) \leq b(l_1(s)) = b(l_1(s) + \Delta) = c_J(s')$, where the inequality follows from the argument about agents in A above, and the last equality follows from Equation 5.

$\forall J \in D$, $c_J(s) = b(l_2(s))$, and $c_J(s') = b(l_2(s) - \Delta)$; thus $b(l_2(s)) \geq b(l_2(s) - \Delta)$. Also, $c_J(s'') = b(l_2(s) - \Delta) = c_J(s')$.

It follows that the only possible way for s' to be a weakly profitable deviation while s'' is not is if $\forall J \in A$, $c_J(s') < c_J(s)$; i.e., $b(l_2(s) - \Delta) < b(l_1(s))$, and for all agents $J \in \Gamma$, $c_J(s'') = c_J(s)$. Suppose by way of contradiction that this is the case. If $b(l_2(s) - \Delta) < b(l_2(s))$, agents in $B \setminus E$ strictly improve, a contradiction to $c_J(s'') = c_J(s)$ $\forall J \in \Gamma$; thus $b(l_2(s) - \Delta) = b(l_2(s))$. We get: $b(l_1(s) + \Delta) \leq b(l_2(s)) = b(l_2(s) - \Delta) < b(l_1(s))$, where the first inequality follows from the argument about $j \in B$ in s', but this is in contradiction to the monotonicity of $b(\cdot)$. Thus, if $C \neq \emptyset$ and s' is a weakly profitable bidirectional deviation, then s'' is a weakly profitable unidirectional deviation.

It is left to show that if $C = \emptyset$ the statement holds as well. In the deviation to s', it must hold that $\exists J \in \Gamma$ s.t. $c_J(s') < c_J(s)$.

Suppose that $\forall J \in B$ $c_J(s') < c_J(s)$; i.e., $b(l_1(s) + \Delta) < b(l_2(s))$. But since $b(l_2(s)) \leq b(l_1(s) + 1)$ (from s being a NE) we get $b(l_1(s) + \Delta) < b(l_1(s) + 1)$, in contradiction to the monotonicity of $b(\cdot)$ (since $\Delta \geq 1$). Thus $c_J(s') \geq c_J(s)$ $\forall J \in B$.

Next suppose that $\forall J \in A$ $c_J(s') < c_J(s)$; i.e., $b(l_2(s) - \Delta) < b(l_1(s))$. By monotonicity, $l_2(s) - \Delta < l_1(s)$, in contradiction to $C = \emptyset$ (since all the agents $J \in \Gamma$ assigned to M_1 by s must migrate). This concludes the proof.

Since there must exist a player that strictly reduces its cost, it must hold that $\forall J \in D$ $c_J(s') < c_J(s)$; i.e., $b(l_2(s) - \Delta) < b(l_2(s))$, but since s is a NE, $b(l_2(s)) \leq b(l_1(s) + 1)$. We get: $b(l_2(s) - \Delta) < b(l_1(s) + 1)$. From monotonicity, $l_2(s) - \Delta < l_1(s) + 1$, thus $l_2(s') \leq l_1(s)$, implying that $l_1(s') \geq l_2(s)$ (by $l_1(s) + l_2(s) = l_1(s') + l_2(s')$). But since $C = \emptyset$, all the agents $J \in \Gamma$ s.t. $s_J = M_1$ migrate to M_2, thus all the agents $J \in \Gamma$ s.t. $s_J = M_2$ migrate to M_1, concluding that $D = \emptyset$. Therefore, if $C = \emptyset$ the statement holds as well. □

Proof. of Theorem 3.14:

Let s be a cost-wise lexicographically minimal, T-spread-out profile with loads $l_1(s)$ and $l_2(s)$, s.t. $l_1(s) \geq l_2(s)$ WLOG; thus, by monotonicity, $b(l_1(s)) \geq b(l_2(s))$. It is easy to see that for every T_i s.t. $|T_i| \geq 2$, if $\exists J \in T_i$ s.t. $s_J = M_2$, then there must exist $J \in T_i$ s.t. $s_J = M_1$.

Consider a coalitional deviation of minimal size from s to s'. By Lemma 3.13, it must be a unidirectional coalition, and by Lemma 3.12, it can only be of a set $S \subseteq T_i$ from resource M_1 (since any coalition greater than 2 has at least one member on M_1 by the structure of s), s.t. $b(l_1(s) - |S|) < b(l_1(s))$, and $b(l_1(s)) = b(l_2(s) + |S|) \geq b(l_2(s))$, where the last inequality follows from monotonicity. Note that, as in s, also in s' only M_1 contains full coalitions (except for singletons). We next show that s' is also a NE.

By the minimality of the deviation, $b(l_1(s) - |S| + 1) = b(l_1(s))$, and as stated above $b(l_2(s) + |S|) = b(l_1(s))$; thus $b(l_1(s) - |S| + 1) = b(l_2(s) + |S|) = b(l_2(s'))$, so a unilateral deviation from M_2 to M_1 is not profitable. As for a unilateral deviation from M_1 to M_2, this is obviously not profitable since $b(l_1(s')) = b(l_1(s) - |S|) < b(l_1(s)) = b(l_2(s) + |S|) = b(l_2(s'))$. We conclude that s' is also a NE.

We claim that s' is a T-SSE. By Lemma 3.13 it is sufficient to show that there is no weakly profitable unidirectional deviation. Again, since $b(l_1(s')) = b(l_1(s) - |S|) < b(l_1(s)) = b(l_2(s) + |S|) = b(l_2(s'))$, there is no weakly profitable unidirectional deviation from M_1 to M_2, thus it is left to show that there is no weakly profitable unidirectional deviation from M_2 to M_1.

Consider a deviation of a set $S', |S'| \geq 1$ from M_2 to M_1. By the minimality of the deviation to s' it follows that $b(l_1(s) - |S| + 1) = b(l_1(s))$ and $b(l_1(s) - |S|) < b(l_1(s))$. We get: $b(l_1(s) - |S| + |S'|) \geq b(l_1(s) - |S| + 1) = b(l_1(s)) > b(l_1(s) - |S|)$. Since in s' only M_1 contains full coalitions, any unidirectional deviation from M_2 to M_1 will hurt the coalition members that are assigned to M_1 in s'. This concludes the proof. □

C Missing Proofs from Section 4

Proof. of Lemma 4.3:

For every coalition T_i there exists a player $J \in T_i$ such that no deviation of its coalition in any round can reduce its cost. Since this is true for every coalition T_i, no strongly profitable deviation exists; thus, s is a T-SE. □

Proof. of Lemma 4.6:

By Lemma 3.10, s is a T-SE of the one-shot game in every round of the repeated game. Thus, for every coalition T_i, there exists a player $J \in T_i$ for which s is T_i-minimal. Thus, by Lemma 4.3, no coalition has a strongly-profitable deviation in the repeated game, and the proof follows. □

Proof. of Theorem 4.7:

Let G be a one-shot RSG with m non-decreasing resources. Let C be a set containing one representative of every T_i s.t. $|T_i| = 2$, and let C' be the set of their partners. Let M_j be a resource s.t. $b_j(|C|) \geq b_i(|C|)\ \forall i \neq j$. Let $G' =< M, N, b'(\cdot), c(\cdot) >$ be the game induced by G where all players $J \in C$ are assigned to M_j; i.e., $b'_j(l) = b_j(l + |C|)$ and $b'_i(l) = b_i(l)\ \forall i \neq j$.

We shall prove that there exists a NE $s_{N \setminus C}$ of players $J \in N \setminus C$ in G' in which $s_J \neq M_j\ \forall J \in C'$. Since a NE is not sensitive to the identities of the players, it is sufficient to show that there exists a

NE in which $|\{J|s_J \in M \setminus M_j\}| \geq |C'|$. To see this, consider best-response dynamics starting from an arbitrary assignment of the players $J \in N \setminus C$. Since this is a potential game, best-response dynamics must converge to a NE. We show that for any profile s, if $l_j(s) \geq |N - C|$ we can assume WLOG that in the best-response dynamics players do not migrate to resource M_j. Since $|N-C| \geq C$, if $l_j(s) \geq |N-C|$, then $b_k(l_k(s)) \leq b_k(|C|) \leq b_j(|C|) \leq b_j(|N \setminus C|) \leq b_j(l_j(s))\ \forall k \neq j$. Thus, player assigned to M_k cannot improve by migrating to M_j. We conclude that there exists a NE s of G in which the members of each coalition of size 2 are spread out.

We claim that playing s in every stage is a T-SE of the repeated game. Obviously, none of the singletons can deviate since playing a NE in every stage of the repeated game is also a NE in the repeated game. In addition, since in s each coalition is spread out, Lemma 4.6 implies that no coalition of size 2 can migrate and this concludes the proof. □

Proof. of Theorem 4.13:

case a: $|M| \geq 3$. Let s be a T-spread-out NE of the one-shot game. Since $|M| \geq 3$, no two members of one coalition are assigned to the same resource. By Lemma 4.6, playing s in every round of the game is a T-SE of the repeated game.

case b: $|M| = 2$. First, we claim that there exists a NE s of the one-shot game s.t. $|l_2(s) - l_1(s)| \leq 1$ (i.e., one of the resources has load $\lceil \frac{n}{2} \rceil$ and the second resource has load $\lfloor \frac{n}{2} \rfloor$), WLOG suppose $l_2(s) - l_1(s) \leq 1$. Suppose by way of contradiction that for every profile s which is a NE of the one-shot game $l_2(s) - l_1(s) > 1$. Then $b(l_2(s)) \geq b(l_1(s)+1)$ and it is easy to verify that the profile in which a single player migrates from M_1 to M_2 is a NE as well; a contradiction.

Among all the NE s in which $l_2(s) - l_1(s) \leq 1$, we consider the following profile s^*: For every coalition T_i s.t. $|T_i| = 3$, we assign one of the jobs to M_1 together with one singleton and the other two to M_2 (so that the two resources have equal size). We repeat this process until there are no more coalitions of size 3 (we have a sufficient number of singletons since there is a majority of singletons). Then, for every coalition of size 2, we assign one of the players to M_1 and the second to M_2. Finally, we assign the left-out singletons on the two resources alternately s.t. the last singleton is assigned to M_2; thus $l_2(s) = l_1(s) + 1$ (recall that n is odd).

We claim that playing s^* in every round of the repeated game constitutes a T-SE of the repeated game. By Lemma 4.3 it is sufficient to show that for every coalition T_i, there exists a player $J \in T_i$ which is $T_i^{s^*}$-minimal in the one-shot game. The claim is trivial for coalitions of size 1. For coalitions of size 2, the claim follows form Lemma 3.10. It is left to show that the claim holds for coalitions of size 3. We claim that for every coalition T_i of size 3, the profile s^* is $T_i^{s^*}$-minimal for the player assigned to M_1. To see this, note that if it migrates to M_2, even if the two members of its coalition migrate from M_2, it cannot reduce its cost since $l_2(s) = l_1(s) + 1$, which implies that $l_1(s) = l_2(s) - 1$, which will be the load on M_2 after the deviation. □

Proof. of Theorem 4.15: Let $\max_i |T_i| = x$, and consider an RSG on two identical strictly increasing resources, repeated x rounds. If $x = 3$, consider the game in which:

$$(x-1) \cdot b(\frac{n}{2}+1) + b(\frac{n}{2}-1) < x \cdot b(\frac{n}{2}). \tag{6}$$

In every NE s of the last stage of the game, it must hold that $l_1(s) = l_2(s)$. To see this note that if $l_1(s) \geq l_2(s) + 2$ then there must be a singleton assigned to M_1 (recall that there is a majority of singletons), but this singleton can reduce its cost by deviating to M_2. Then the cost of every member of T_i in every stage of the repeated game is $b(\frac{n}{2})$. Regardless of the assignment of the coalition in every stage, T_i can deviate such that each coalition member is left alone in one stage and incurs a cost of at most $b(\frac{n}{2}-1)$, while incurring a cost of at most $b(\frac{n}{2}+1)$ in the other stages. Thus, each member of the coalition incurs a cost of at most $(x-1)\cdot b(\frac{n}{2}+1)+b(\frac{n}{2}-1)$, which is strictly smaller than $x \cdot b(\frac{n}{2})$ by Equation 6; thus it is a strongly-profitable deviation of T_i.

□