Incentive Compatible Ranking Systems

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ABSTRACT

Ranking systems are a fundamental ingredient of multi-agent environments and Internet Technologies. These settings can be viewed as social choice settings with two distinguished properties: the set of agents and the set of alternatives coincide, and the agents' preferences are dichotomous, and therefore classical impossibility results do not apply. In this paper we initiate the study of incentives in ranking systems, where agents act in order to maximize their position in the ranking, rather than to obtain a correct outcome. We consider several basic properties of ranking systems, and fully characterize the conditions under which incentive compatible ranking systems exist, demonstrating that in general no such system satisfying all the properties exists.

Categories and Subject Descriptors

G.2.2 [**Discrete Mathematics**]: Graph Theory; J.4 [**Social and Behavioral Sciences**]: Economics; H.3.3 [**Information Storage and Retrieval**]: Information Search and Retrieval

General Terms

Algorithms, Economics, Human Factors, Theory

Keywords

Ranking systems, multi-agent systems, incentives, social choice

1. INTRODUCTION

The ranking of agents based on other agents' input is fundamental to multi-agent systems (see e.g. [14]). Moreover, it has become a central ingredient of a variety of Internet sites, where perhaps the most famous examples are Google's PageRank algorithm[11] and eBay's reputation system[13].

The ranking systems setting can be viewed as a variation of the classical theory of social choice[4], where the set of agents and the set of alternatives *coincide*. Specifically, we consider dichotomous ranking systems, in which the agents vote for a subset of the rest of the agents. This is a natural representation of the web page ranking setting[17], where the Internet pages are represented by the agents/alternatives, and the links are represented by votes.

Some basic work targeted at the foundations of ranking systems has been recently initiated. In particular, basic properties of ranking systems have been shown to be impossible to simultaneously accommodate[1], various known ranking systems have been recently compared with regard to certain criteria by [5], and several ranking rules have been axiomatized [2, 12, 16].

Although the above mentioned work consists of a significant body of rigorous research on ranking systems, the study did not consider the effects of the agents' incentives on ranking systems[1]. The issue of incentives has been extensively studied in the classical social choice literature. The Gibbard–Satterthwaite theorem [9, 15] shows that in the classical social welfare setting, it is impossible to aggregate the rankings in a strategy-proof fashion under some basic conditions. The incentives of the candidates themselves were considered in the context of elections[8], where a related impossibility result is presented. Another notion of incentives was considered in the case where a single agent may create duplicates of itself[7]. Furthermore, the computation of equilibira in the more abstract context of ranking games was also discussed[6].

In this paper we initiate research on the issue of incentives in ranking systems. We define two notions of incentive compatibility, where the agent is concerned with its expected position in the ranking under *affine* or *general* utility functions.

We then consider some very basic properties of ranking systems, which are satisfied by almost all known ranking systems, and prove that these properties cannot be all satisfied by an incentive compatible ranking system. This finding is far from trivial, as different ranking systems may require different manipulations by an agent in order to increase its rank in different situations. Furthermore, we show that when we assume only a subset of the basic properties, some artificial incentive compatible ranking systems can be constructed. Together, these results form a complete characterization of incentive compatible ranking systems under these

[1]A recent work on quantifying incentive compatbility of ranking systems[3] was based on a preliminary version of this paper.

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basic properties.

Our results expose some surprising and illuminating effects of some basic properties one may require a ranking system to satisfy on the existence of incentive compatible ranking systems.

This paper is structured as follows: In Section 2 we formally introduce the notion of ranking systems and in Section 3 we define some basic properties of ranking systems. In Section 4 we introduce our two notions of incentive compatibility. We then show a strong possibility result in Section 5, when we do not assume the *minimal fairness* property. In Section 6 we provide a full classification of the existence of incentive compatible ranking systems when we do assume minimal fairness. Section 7 provides some illuminating lessons learned from this classification. Finally, in Section 8 we introduce the isomorphism property and recommend further research with regard to the classification of incentive compatibility under isomorphism.

2. RANKING SYSTEMS

Before describing our results regarding ranking systems, we must first formally define what we mean by the words "ranking system" in terms of graphs and linear orderings:

DEFINITION 1. *Let A be some set. A relation $R \subseteq A \times A$ is called an* ordering *on A if it is reflexive, transitive, and complete. Let $L(A)$ denote the set of orderings on A.*

NOTATION 1. *Let $\preceq$ be an ordering, then $\simeq$ is the equality predicate of $\preceq$, and $\prec$ is the strict order induced by $\preceq$. Formally, $a \simeq b$ if and only if $a \preceq b$ and $b \preceq a$; and $a \prec b$ if and only if $a \preceq b$ but not $b \preceq a$.*

Given the above we can define what a ranking system is:

DEFINITION 2. *Let $\mathbb{G}_V$ be the set of all directed graphs on a vertex set V that do not include self edges*[2]*. A* ranking system *F is a functional that for every finite vertex set V maps graphs $G \in \mathbb{G}_V$ to an ordering $\preceq^F_G \in L(V)$.*

One can view this setting as a variation/extension of the classical theory of social choice as modeled by [4]. The ranking systems setting differs in two main properties. First, in this setting we assume that the set of voters and the set of alternatives coincide, and second, we allow agents only two levels of preference over the alternatives, as opposed to Arrow's setting where agents could rank alternatives arbitrarily.

3. BASIC PROPERTIES OF RANKING SYSTEMS

In order to classify the incentive compatibility features of ranking systems, we must first define the criteria for the classification. We define some very basic properties that are satisfied by almost all known ranking systems. Most properties have two versions – one weak and one strong, both satisfied by almost all known ranking systems.

First of all, we define the notion of a trivial ranking system, which ranks any two vertices the same way in all graphs.

[2]Our results are still correct when allowing self-edges, but for the simplicity of the exposition we assume none exist.

DEFINITION 3. *A ranking system F is called* trivial *if for all vertices v_1, v_2 and for all graphs G, G' which include these vertices: $v_1 \preceq^F_G v_2 \Leftrightarrow v_1 \preceq^F_{G'} v_2$. A ranking system F is called* nontrivial *if it is not trivial.*

A ranking system F is called infinitely nontrivial *if there exist vertices v_1, v_2 such that for all $N \in \mathbb{N}$ there exists $n > N$ and graphs $G = (V, E)$ and $G' = (V', E')$ s.t. $|V| = |V'| = n$, $v_1 \preceq^F_G v_2$, but $v_2 \prec^F_{G'} v_1$.*

A basic requirement from a ranking system is that when there are no votes in the system, all agents must be ranked equally. We call this requirement minimal fairness[3].

DEFINITION 4. *A ranking system F is* minimally fair *if for every graph $G = (V, \emptyset)$ with no edges, and for every $v_1, v_2 \in V$: $v_1 \simeq^F_G v_2$.*

Another basic requirement from a ranking system is that as agents gain additional votes, their rank must improve, or at least not worsen. Surprisingly, this vague notion can be formalized in (at least) two distinct ways: the monotonicity property considers the situation where one agent has a superset of the votes another has *in the same graph*, while the positive response property considers the addition of a vote for an agent *between graphs*. This distinction is important because, as we will see, the two properties are neither equivalent, nor imply each other.

NOTATION 2. *Let $G = (V, E)$ be a graph, and let $v \in V$ be a vertex. The predecessor set of v is $P_G(v) = \{v' | (v', v) \in E\}$. The successor set of v is $S_G(v) = \{v' | (v, v') \in E\}$. We may omit the subscript G when it is understood from context.*

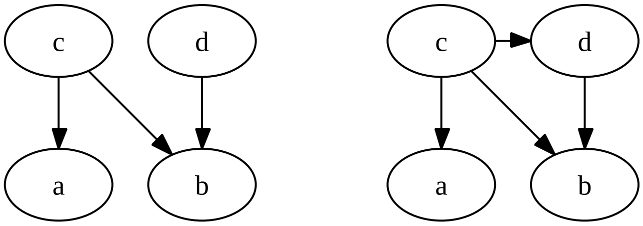
DEFINITION 5. *Let F be a ranking system. F satisfies* weak positive response *if for all graphs $G = (V, E)$ and for all $(v_1, v_2) \in (V \times V) \setminus E$, and for all $v_3 \in V \setminus \{v_2\}$: Let $G' = (V, E \cup (v_1, v_2))$. Then, $v_3 \preceq^F_G v_2$ implies $v_3 \preceq^F_{G'} v_2$ and $v_3 \prec^F_G v_2$ implies $v_3 \prec^F_{G'} v_2$. F furthermore satisfies* strong positive response *if $v_3 \preceq^F_G v_2$ implies $v_3 \prec^F_{G'} v_2$.*

DEFINITION 6. *A ranking system F satisfies* weak monotonicity *if for all $G = (V, E)$ and for all $v_1, v_2 \in V$: If $P(v_1) \subseteq P(v_2)$ then $v_1 \preceq^F_G v_2$. F furthermore satisfies* strong monotonicity *if $P(v_1) \subsetneq P(v_2)$ additionally implies $v_1 \prec^F_G v_2$.*

EXAMPLE 1. *Consider the graphs G_1 and G_2 in Figure 1. Further assume a ranking system F ranks $a \simeq^F_{G_1} d$ in graph G_1. Then, if F satisfies weak positive response, it must also rank $a \preceq^F_{G_2} d$ in G_2. If F satisfies the strong positive response, then it must strictly rank $a \prec^F_{G_2} d$ in G_2. However, if we do not assume $a \preceq^F_{G_1} d$, F may rank a and d arbitrarily in G_2.*

Now consider the graph G_1, and note that $P(a) = \{c\} \subsetneq \{c, d\} = P(b)$. This is the requirement of the weak (and strong) monotonicity property, and thus any ranking system F that satisfies weak monotonicity must rank $a \preceq^F_{G_1} b$, and it is satisfies strong monotonicity, it must strictly rank $a \prec^F_{G_1} b$.

[3]A stronger notion of fairness, the isomorphism property, will be considered in Section 8.



(a) G_1 (b) G_2

Figure 1: Example graphs for the basic properties of ranking systems

Note that the weak monotonicity property implies minimal fairness. This is due to the fact that when no votes are cast, all vertices have exactly the same predecessor sets and thus must be ranked equally.

Yet another simple requirement from a ranking system is that it does not behave arbitrarily differently when two sets of agents with their respective votes are considered one set.

Definition 7. *Let F be a ranking system and let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be graphs s.t. $V_1 \cap V_2 = \emptyset$ and let $v_1, v_2 \in V_1$ be two vertices. Let $G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$. F satisfies the* weak union condition *if $v_1 \preceq^F_{G_1} v_2 \Leftrightarrow v_1 \preceq^F_{G_1 \cup G_2} v_2$. Let $G' = (V_1 \cup V_2, E_1 \cup E_2 \cup E)$, where $E \subseteq V_1 \times V_2$ is in an arbitrary set of edges from V_1 to V_2. F satisfies the* strong union condition *if $v_1 \preceq^F_{G_1} v_2 \Leftrightarrow v_1 \preceq^F_{G'} v_2$.*

Surprisingly, we will see that even the weak union condition has great significance towards the existence of a ranking system or lack thereof. One reason for this effect, is that a ranking system satisfying this condition cannot behave differently depending on the size of the graph.

3.1 Satisfiability

As we have mentioned above, these properties are very basic and, with the exception of the strong union condition, all the properties above are satisfied by almost all known ranking systems such as the PageRank[11] ranking system (with a damping factor) and the authority ranking by the Hubs&Authorities algorithm[10]. These ranking systems *do not* satisfy the strong union condition, as in both systems outgoing links outside an agent's strongly connected component may affect ranks inside the strongly connected component, either by dividing the importance (in PageRank) or by affecting the hubbiness score in Hubs&Authorities.

Furthermore, the simple *approval voting* ranking system satisfies all the strong properties mentioned above including the strong union condition. The approval voting ranking system can be defined as follows:

Definition 8. *The* approval voting ranking system *AV is the ranking system defined by:*

$$v_1 \preceq^{AV}_G v_2 \Leftrightarrow |P(v_1)| \leq |P(v_2)|.$$

Fact 1. *The approval voting ranking system AV satisfies minimal fairness, strong monotonicity, strong positive response, the strong union condition, and infinite nontriviality.*

These facts lead us to believe that the properties defined above (perhaps with the exception of the strong union condition), should all be satisfied by any reasonable ranking system, at least in their weak form. We will soon show that this is not possible when requiring incentive compatibility.

4. INCENTIVE COMPATIBILITY

Ranking systems do not exist in empty space. The results given by ranking systems frequently have implications for the agents being ranked, which are the same agents that are involved in the ranking. Therefore, the incentives of these agents should in many cases be taken into consideration.

In our approach, we require that our ranking system will not rank agents better for stating untrue preferences, but we assume that the agents are interested only in their own ranking (and not, say, in the ranking of those they prefer).

We assume that for strict rankings (with no ties), for every agent count n, there exists a utility function $u_n : \mathbb{N} \mapsto \mathbb{R}$ that maps an agent's *rank* (i.e. the number of agents ranked below it) to a utility value for being ranked that way. We assume u_n is nondecreasing, that is every agent weakly prefers to be ranked higher.

This utility function can be extended to the case of ties, by treating these as a uniform randomization over the matching stict orders. Thus the utility of an agent with k agents strictly below it and m agents tied is

$$E[u_n] = u^*_n(k, m) = \frac{1}{m} \sum_{i=k}^{k+m-1} u_n(i).$$

We can now define the utility of a ranking for an agent as follows:

Definition 9. *The* utility *$u^F_G(v)$ of a vertex v in graph $G = (V, E)$ under the ranking system F and utility function u is defined as*

$$\begin{aligned} u^F_G(v) &= u^*_{|V|}\left(\left|\{v' : v' \prec v\}\right|, \left|\{v' : v' \simeq v\}\right|\right) = \\ &= \frac{1}{|\{v' : v' \simeq v\}|} \sum_{i=|\{v':v' \prec v\}|}^{\left|\{v':v' \preceq v\}\right|-1} u_n(i). \end{aligned}$$

This definition allows us to define a preference relation over rankings for each agent. Using this preference relation, we can now define the general notion of incentive compatiblity as immunity of utility to manipulation of outgoing edges:

Definition 10. *Let F be a ranking system. F is called* incentive compatible *under utility function u if for all graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ s.t. for some $v \in V$, and for all $v' \in V \setminus \{v\}, v'' \in V : (v', v'') \in E_1 \Leftrightarrow (v', v'') \in E_2$: $u^F_{G_1}(v) = u^F_{G_2}(v)$.*

A strong notion of incentive compatibility is compatibility under *any* utility function:

Definition 11. *Let F be a ranking system. F satisfies* strong incentive compatibility *if for any nondecreasing utility function $u : \mathbb{N} \times \mathbb{N} \mapsto \mathbb{R}$, F is incentive compatible under u.*

A simple utility function one may consider is the identity function $u_n(k) \equiv k$. This basic utility function means that any change in rank has the same significance. The utility of

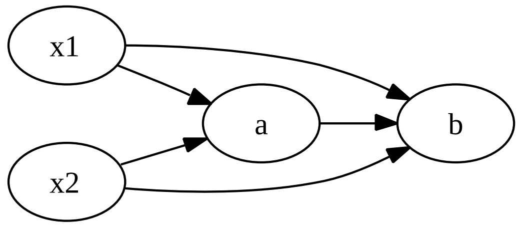


Figure 2: Example graph for ranking system F

a ranking with k weaker agents and m equal agents under this function is:

$$u_n^*(k,m) = \frac{1}{m} \sum_{i=k}^{k+m-1} u_n(i) = k + \frac{m-1}{2}.$$

It turns out that the preference relation over rankings produced by the identity utility function is the same as the one produced by any affine utility function $u(k) = a \cdot k + b$, as $u_n^*(k,m)$ in this case is simply $a \cdot (k + \frac{m-1}{2}) + b$. Therefore, it is interesting to look at incentive compatibility under an affine utility function u:

Definition 12. *Let F be a ranking system and let. F is called* weakly incentive compatible *if for every utility function $u : \mathbb{N} \times \mathbb{N} \mapsto \mathbb{R}$ such that $u_n(k) = a \cdot k + b$ for some constants $a, b \in \mathbb{R}$: F is incentive comaptible under u.*

Notation 3. *In order to prevent ambiguity, in the remainder of this paper we will use $r_G^F(v)$ ("rank") to denote $u_G^F(v)$ under the utility function $u_n(k) = k + \frac{1}{2}$. So that*

$$r_G^F(v) = \left|\{v' : v' \prec v\}\right| + \frac{1}{2}\left|\{v' : v' \simeq v\}\right|.$$

Note that due to the fact that all affine ranking functions give the same ordering over $u^*(k,m)$, we can, wlog, consider only $u_n(k) = k + \frac{1}{2}$ when proving weak incentive compatiblity or lack thereof.

Interestingly, we will see in the remainder of this paper that these incentive compatibility properties are very hard to satisfy, and no common nontrivial ranking system satisfies them. In particular, the PageRank, Hubs&Authorities and Approval Voting ranking systems mentioned above are not weakly incentive compatible.

Example 2. *One may think that under positive response, impossibility of weak incentive compatibility is a direct result of an alleged dominant strategy not to vote for any agent.*

However, this is not true, as sometimes the best response does involve voting for some agent. Consider the ranking system F defined by:

$$v_1 \preceq_G^F v_2 \Leftrightarrow |P(v_1)| + \tfrac{1}{3}|S(v_1)| \leq |P(v_2)| + \tfrac{1}{3}|S(v_2)|.$$

This ranking system satisfies strong positive response, but is not weakly incentive compatible. For example, in the graph depicted in Figure 2, the agent a can improve its rank either by not voting for b, or by voting for both x_1 and x_2. The maximal increase in a's rank is achieved by doing both.

Note that under this ranking system, agents do not have a dominant strategy that maximizes their rank, and thus there is no general dominant deviation that demonstrates lack of incentive compatibility.

5. POSSIBILITY WITHOUT MINIMAL FAIRNESS

To begin our classification of the existence of incentive compatible ranking systems, we first consider ranking systems which do not satisfy minimal fairness. We have already seen that minimal fairness is implied by weak monotonicity, so we cannot hope to satisfy weak monotonicity without minimal fairness. As it turns out, the strong versions of all the remaining properties considered above can, in fact, be satisfied simultaneously.

Proposition 1. *There exists a ranking system F_1 that satisfies strong incentive compatibility, strong positive response, infinite nontriviality, and the strong union condition.*

Proof. *Assume a lexicographic order $<$ over vertex names, and assume three consecutive vertices $v_1 < v_2 < v_3$. Then, F_1 is defined as follows (let $G = (V,E)$ be some graph):*

$$\begin{aligned} v \preceq_G^{F_1} u \quad \Leftrightarrow \quad & [v \leq u \wedge (v \neq v_2 \vee u \neq v_3)] \vee \\ & [v = v_2 \wedge u = v_3 \wedge (v_1, v_2) \notin E] \vee \\ & [v = v_3 \wedge u = v_2 \wedge (v_1, v_2) \in E]. \end{aligned}$$

That is, vertices are ranked strictly according to their lexicographic order, except when $(v_1, v_2) \in E$, whereas the ranking of v_2 and v_3 is reversed.

F_1 is infinitely nontrivial because graphs with the vertices v_1, v_2, v_3 are ranked differently depending on the existence of the edge (v_1, v_2), and these exist for any $|V| \geq 3$.

F_1 satisfies strong incentive compatibility because the only vertex that can make any change in the ranking is v_1 and it cannot ever change its own position in the ranking at all.

F_1 satisfies strong positive response because the ordering of the vertices remains unchanged by anything but the (v_1, v_2) edge, and is always strict. The addition of the (v_1, v_2) edge only increases the relative rank of v_2 as required.

Assume for contradiction that F_1 does not satisfy the strong union condition. Then, there exist two disjoint graphs $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$ and an edge set $E \subseteq V_1 \times V_2$ such that the ranking $\preceq_G^{F_1}$ of graph $G = (V_1 \cup V_2, E_1 \cup E_2 \cup E)$ is inconsistent with $\preceq_{G_1}^{F_1}$. First note that the only inconsistency that may arise is with the ranking of v_2 compared to v_3. Therefore, $\{v_2, v_3\} \subseteq V_1$. Furthermore, for the ranking to be inconsistent $(v_1, v_2) \notin E_1$ and $(v_1, v_2) \in E_1 \cup E_2 \cup E$ (the opposite is impossible due to inclusion). Furthermore, $v_2 \in V_1 \Rightarrow v_2 \notin V_2 \Rightarrow (v_1, v_2) \notin V_1 \times V_2 \Rightarrow (v_1, v_2) \notin E$. Thus we conclude that $(v_1, v_2) \in E_2$, and thus $v_2 \in V_2$, in contradiction to the fact that $v_2 \in V_1$. □

6. FULL CLASSIFICATION UNDER MINIMAL FAIRNESS

We are now ready to state our main results:

Theorem 1. *There exist weakly incentive compatible, infinitely nontrivial, minimally fair ranking systems F_2, F_3, F_4 that satisfy weak monotonicity; weak positive response; and the weak union condition repsectively. However, there is no weakly incentive compatible, nontrivial, minimally fair ranking system that satisfies any two of those three properties.*

Theorem 2. *There is no weakly incentive compatible, nontrivial, minimally fair ranking system that satisfies either* *one the four properties: strong monotonicity, strong positive response, the strong union condition and strong incentive compatibility.*

The proof of these two theorems is split into ten different cases that must be considered – three possibility proofs for F_2, F_3. and F_4, three impossibility results with pairs of weak properties, and four impossibility results with each of the strong properties. We will now prove each of these cases.

6.1 Possibility Proofs

PROPOSITION 2. *There exists a weakly incentive compatible ranking system F_2 that satisfies minimal fairness, weak positive response, and infinite nontriviality.*

PROOF. *Let v_1, v_2, v_3 be some vertices and let $G = (V, E)$ be some graph, then F_2 is defined as follows:*

$$v \preceq u \quad \Leftrightarrow \quad [v \neq v_3 \wedge u \neq v_2] \vee v = u \vee (v_1, v_3) \notin E \vee v_2 \notin V.$$

That is, F_2 ranks all vertices equally, except when the edge (v_1, v_3) exists. Then, F_2 ranks $v_2 \prec v \simeq u \prec v_3$ for all $v, u \in V \setminus \{v_2, v_3\}$.

F_2 satisfies minimal fairness because when no edges exist, the clause $(v_1, v_3) \notin E$ always matches, and thus all vertices are ranked equally, as required. F_2 satisfies infinite nontriviality, because for all $|V| \geq 3$ there exists a graph which includes the vertices v_1, v_2, v_3 and the edge (v_1, v_3), which is ranked nontrivially.

F_2 satisfies weak positive response because the only edge addition that changes the ranks of the vertices in the graph (the addition of (v_1, v_3)) indeed doesn't weaken the target vertex v_3.

F_2 is weakly incentive compatible because only v_1 can affect the ranking of the vertices in the graph (by voting for v_3 or not), but $r(v_1)$ is always $\frac{|V|}{2}$. □

PROPOSITION 3. *There exists a weakly incentive compatible ranking system F_3 that satisfies minimal fairness, the weak union condition, and infinite nontriviality.*

PROOF. *Let v_1, v_2, v_3 be some vertices and let $G = (V, E)$ be some graph, then F_3 is defined as follows:*

$$v \preceq u \quad \Leftrightarrow \quad [v \neq v_3 \wedge u \neq v_2] \vee v = u \vee \{(v_1, v_2), (v_1, v_3)\} \not\subseteq E.$$

That is, F_3 ranks all vertices equally, except when the edges $(v_1, v_2), (v_1, v_3)$ exist. Then, F_3 ranks $v_2 \prec v \simeq u \prec v_3$ for all $v, u \in V \setminus \{v_2, v_3\}$.

F_3 satisfies minimal fairness because when no edges exist, the clause $\{(v_1, v_2), (v_1, v_3)\} \not\subseteq E$ always matches, as required. F_3 satisfies infinite nontriviality, because for all $|V| \geq 3$ there exists a graph which includes the vertices v_1, v_2, v_3 and the edges $\{(v_1, v_2), (v_1, v_3)\}$, which is ranked nontrivially.

To prove F_3 satisfies the weak union condition, let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be some graphs such that $V_1 \cap V_2 = \emptyset$, and let $G = G_1 \cup G_2$. If $\{(v_1, v_2), (v_1, v_3)\} \not\subseteq E_1 \cup E_2$ then by the definition of F_3, it must rank all vertices in all graphs G_1, G_2, G equally, as required. Otherwise, for all $v, u \in (V_1 \cup V_2) \setminus \{v_2, v_3\}$: $v_2 \prec_G^{F_3} v \simeq_G^{F_3} u \prec_G^{F_3} v_3$. Assume wlog that $(v_1, v_2) \in E_1$ and thus $v_1, v_2 \in V_1$. But then also $(v_1, v_3) \in E_1$ and thus also $v_3 \in V_1$. By the definition of F_3, for all $v, u \in V_1 \setminus \{v_2, v_3\}$: $v_2 \prec_{G_1}^{F_3} v \simeq_{G_1}^{F_3} u \prec_{G_1}^{F_3} v_3$. As $v_1, v_2, v_3 \notin G_2$, trivially for all $v, u \in V_2$: $v \simeq_{G_2}^{F_3} u$, as required.

F_3 is weakly incentive compatible because only v_1 (if at all) can affect the ranking of the vertices in the graph (by voting for v_2 and v_3 or not), but $r(v_1)$ is always $\frac{|V|}{2}$. □

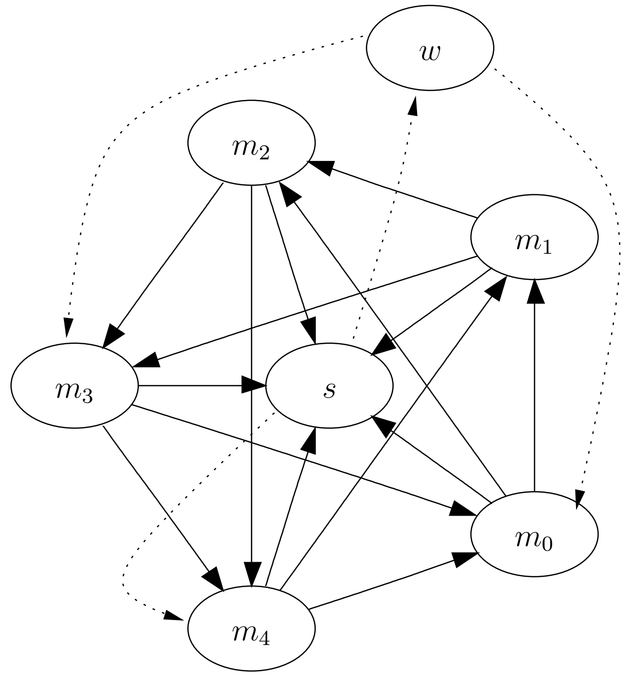


Figure 3: Nontrivially ranked graph for F_4

PROPOSITION 4. *There exists a weakly incentive compatible ranking system F_4 that satisfies minimal fairness, weak monotonicity, and infinite nontriviality.*

PROOF. *The ranking system F_4 ranks all vertices equally, except for graphs $G = (V, E)$ for which $|V| \geq 7$, where $V = \{w, s, m_0, \ldots, m_{n-1}\}$, and for all $i \in \{0, \ldots, n-1\}$: $(m_i, s) \in E$, $(m_i, w) \notin E$, and for all $j \in \{0, \ldots, n-1\}$: $(m_i, m_j) \in E$ if and only if $j = (i+1) \bmod n$ or $j = (i+2) \bmod n$. Figure 3 includes an example graph that satisfies these conditions. In such graphs, F_4 ranks $w \prec_G^{F_4} m_1 \simeq_G^{F_4} \cdots \simeq_G^{F_4} m_n \prec_G^{F_4} s$.*

F_4 is minimally fair by definition, as when there are no edges, all vertices are ranked equally. F_4 satisfies infinite nontriviality because such nontrivially ranked graphs G exist for all $|V| \geq 7$.

F_4 satisfies weak monotonicity because in the graphs that it doesn't rank all vertices equally we see that $P(w) \not\supseteq P(m_i) \not\supseteq P(s)$ for all $i \in \{0, \ldots n-1\}$, which is consistent with the ordering F_4 specifies.

To prove F_4 is weakly incentive compatible, we let G_1, G_2 be two graphs that differ only in the outgoing edges of a single vertex v, and show that $r_{G_1}^{F_4}(v) = r_{G_2}^{F_4}(v)$. Because all graphs in which not all vertices are ranked equally are of the form defined above, at least one of the graphs G_1, G_2 must have this form. Let us assume wlog that this graph is G_1, and mark the vertices of this graph as defined above.

Now consider two cases:

1. *If $v = w$ or $v = s$, then by the definition of F_4, $\preceq^{F_4}_{G_1} \equiv \preceq^{F_4}_{G_2}$, thus trivially, $r^{F_4}_{G_1}(v) = r^{F_4}_{G_2}(v)$, as required.*
2. *If $v = m_i$ for some $i \in \{0, \ldots, n-1\}$, then first note that $r^{F_4}_{G_1}(v) = \frac{|V|}{2}$. If G_2 is not of the form defined above then all its vertices are ranked equally and specifically $r^{F_4}_{G_2} = \frac{|V|}{2}$, as required. Otherwise, G_2 is of the form defined above. Let w' and s' be the w and s vertices for G_2 in the form defined above. By the definition, $2 \le |P_{G_1}(v)| \le 4$, while $|P_{G_2}(w')| \le 1$ and $|P_{G_2}(s')| \ge 5$. Therefore, $v \notin \{w', s'\}$. By the definition of F_4, $r^{F_4}_{G_2}(v) = \frac{|V|}{2}$, as required.*

□

6.2 Impossibility proofs with pairs of weak properties

We prove the impossibility results with pairs of weak properties, by assuming existence of a ranking system and analyzing the minimal graph in which the ranking system does not rank all agents equally. This is done in the following lemma:

Lemma 1. *Let F be a weakly incentive compatible minimally fair nontrivial ranking system. Then, there exists a graph $G = (V, E)$ and vertices $v_\perp, v_\top, v \in V$ such that:*

1. *For all graphs $G' = (V', E')$ where $|E'| < |E|$ or $|E'| = |E|$ and $|V'| < |V|$, $v_1 \simeq^F_{G'} v_2$ for all $v_1, v_2 \in V'$.*
2. *$r^F_G(v) = \frac{|V|}{2}$*
3. *$v_\perp \prec^F_G v \prec^F_G v_\top$*
4. *For all $v' \in V$: $v_\perp \preceq^F_G v' \preceq^F_G v_\top$.*
5. *$S(v) \neq \emptyset$ and for all $v' \in V$ such that $S(v') \neq \emptyset$: $v' \simeq^F_G v$.*

Proof. *Let $G = (V, E)$ be a minimal (in edges, then vertices) graph such that there exist v_1, v_2 where $v_1 \prec^F_G v_2$. Such a graph exists because F is nontrivial. This graph immediately satisfies condition 1. Let $v_\perp, v_\top$ be vertices such that for all $v' \in V$: $v_\perp \preceq^F_G v' \preceq^F_G v_\top$ (such vertices exist because $\preceq$ is an ordering). Note that these vertices satisfy condition 4.*

$E \neq \emptyset$ because minimal fairness will force $v_1 \simeq v_2$. Let $(v, v') \in E$ be some edge. From minimallity, $r^F_{(V, E \setminus \{(v,v')\})}(v) = \frac{|V|}{2}$. From weak incentive compatibility, $r^F_G(v) = \frac{|V|}{2}$, satisfying condition 2. Therefore,

$$\begin{aligned} \tfrac{1}{2}\left|\{v'|v' \prec v\}\right| + \tfrac{1}{2}\left|\{v'|v' \preceq v\}\right| &= \tfrac{1}{2}|V| \\ \left|\{v'|v' \prec v\}\right| + \left|\{v'|v' \preceq v\}\right| &= \left|\{v'|v' \preceq v\}\right| + \\ &\quad + \left|\{v'|v' \succ v\}\right| \\ \left|\{v'|v' \prec v\}\right| &= \left|\{v'|v' \succ v\}\right|. \end{aligned}$$

From the assumption that $v_1 \prec^F_G v_2$: $v_\perp \preceq^F_G v_1 \prec^F_G v_2 \preceq^F_G v_\top$. Therefore, $v_\perp \prec v$ or $v \prec v_\top$. But as $|\{v'|v' \prec v\}| = |\{v'|v' \succ v\}|$, and at least one is nonempty, both $v_\perp \prec v \prec v_\top$, satisfying condition 3.

Condition 5 is satisfied by noting that for all v' such that $S(v) \neq \emptyset$, $r^F_G(v') = \frac{|V|}{2} = r^F_G(v)$, and thus $v' \simeq^F_G v$. □

Now we can prove the impossibility results for any pair of weak properties:

Proposition 5. *There exists no weakly incentive compatible nontrivial ranking system that satisfies the weak monotonicity and weak positive response conditions.*

Proof. *Assume for contradiction a ranking system F that satisfies the conditions. First note that F is minimally fair, because in a graph with no edges, all vertices have exactly the same predecessor set. Thus, the conditions of Lemma 1 are satisfied, so we can let $G = (V, E)$ and $v, v_\perp, v_\top \in V$ be the graph and the vertices from the lemma.*

Now, let $(v_1, v_2) \in E$ be some edge. Let $G' = (V, E \setminus \{(v_1, v_2)\})$. By condition 1, $v_2 \simeq^F_{G'} v_\top$. By weak positive response, $v_\top \preceq^F_G v_2$. Since this is true for all $v_2 \in V$ with $P(v_2) = \emptyset$, and $v_\perp \prec^F_G v \prec^F_G v_\top$, we conclude that $P_G(v_\perp) = P_G(v) = \emptyset$. Now, by weak monotonicity $v_\perp \simeq^F_G v$, in contradiction to the fact that $v_\perp \prec^F_G v$. □

Proposition 6. *There exists no weakly incentive compatible nontrivial ranking system that satisfies the weak monotonicity and weak union conditions.*

Proof. *Assume for contradiction a ranking system F that satisfies the conditions. First note that F is minimally fair, because in a graph with no edges, all vertices have exactly the same predecessor set. Thus, the conditions of Lemma 1 are satisfied, so we can let $G = (V, E)$ and $v, v_\perp, v_\top \in V$ be the graph and the vertices from the lemma.*

Now let $G' = (V \cup \{x\}, E)$ be a graph with an additional vertex $x \notin V$. By the weak union condition, $v_\perp \prec^F_{G'} v$. By weak monotonicity, $x \preceq^F_{G'} v_\perp$. Therefore, by the weak union condition, $r^F_{G'}(v) = r^F_G(v) + 1 = \frac{|V|}{2} + 1$. Let $G'' = (V \cup \{x\}, E \setminus \{(v', v)|v' \in V\})$. By condition 1 and the fact that $S_{G'}(v) \neq \emptyset$, $r^F_{G''}(v) = \frac{|V|+1}{2}$. From weak incentive compatibility, $r^F_{G''}(v) = r^F_{G'}(v)$, which is a contradiction. □

Proposition 7. *There exists no weakly incentive compatible nontrivial minimally fair ranking system that satisfies the weak union and weak positive response conditions.*

Proof. *Assume for contradiction a ranking system F that satisfies the conditions. As the conditions of Lemma 1 are satisfied, let $G = (V, E)$ and $v, v_\perp, v_\top \in V$ be the graph and the vertices from the lemma. Now let $G_1 = (V \setminus \{v_\perp\}, E)$ and let $G_2 = (\{v_\perp\}, \emptyset)$. From conditions 3 and 5, $S(v_\perp) = \emptyset$. If $P_G(v_\perp) \neq \emptyset$, then by condition 1 in the graph $G' = (V, E \setminus \{(x, v_\perp)\})$ where $x \in P_G(v_\perp)$, $v_\top \preceq^F_{G'} v_\perp$. But then by weak positive response $v_\top \preceq^F_G v_\perp$ in contradiction to condition 3.*

Therefore, $P_G(v_\perp) = S_G(v_\perp) = \emptyset$. Thus, G_1 and G_2 satisfy the conditions of the weak union condition with regard to G. Therefore, $v \prec^F_G v_\top \Rightarrow v \prec^F_{G_1} v_\top$, in contradiction to condition 1, because the edge set is the same and $|V_1| < |V|$. □

6.3 Impossibility proofs with the strong properties

Proposition 8. *There exists no weakly incentive compatible minimally fair ranking system that satisfies strong positive response.*

Proof. *Assume for contradiction a ranking system F that satisfies the conditions. Assume a graph G with two vertices $V = \{v_1, v_2\}$ and no edges. By minimal fairness,*

$v_1 \simeq_G^F v_2$. *Now assume a graph* $G' = (V, \{(v_1, v_2)\})$ *with an added edge between* v_1 *and* v_2. *By strong positive response,* $v_1 \prec_G^F v_2$. *However, by weak incentive compatibility,* $1 = r_G^F(v_1) = r_{G'}^F(v_1) = \frac{1}{2}$, *which is a contradiction.* □

PROPOSITION 9. *There exists no weakly incentive compatible ranking system that satisfies strong monotonicity.*

PROOF. *Assume for contradiction a ranking system* F *that satisfies the conditions. Assume a graph* G *with two vertices* $V = \{v_1, v_2\}$ *and no edges. As* $P_G(v_1) = P_G(v_2)$, *by strong monotonicity,* $v_1 \simeq_G^F v_2$. *Now assume a graph* $G' = (V, \{(v_1, v_2)\})$ *with an added edge between* v_1 *and* v_2. *As* $P_{G'}(v_1) \subsetneq P_{G'}(v_2)$, $v_1 \prec_G^F v_2$. *However, by weak incentive compatibility,* $1 = r_G^F(v_1) = r_{G'}^F(v_1) = \frac{1}{2}$, *which is a contradiction.* □

PROPOSITION 10. *There exists no nontrivial strongly incentive compatible minimally fair ranking system..*

PROOF. *We will prove that for any* $G = (V, E)$, *and for any* $v_1, v_2 \in V$: $v_1 \preceq_G^F v_2$. *We will use the incentive function* $u_n(k) = n^k$, *which gives a different value for each* $u_n^*(k, m)$. *The proof is by induction on* $|E|$.

Induction Base: *Assume* $E = \emptyset$, *and let* $v_1, v_2 \in V$ *be vertices. By minimal fairness,* $v_1 \preceq v_2$.

Inductive Step: *Assume correctness for* $|E| \leq n$ *and prove for* $|E| = n + 1$. *Assume for contradiction that for some* $v_1, v_2 \in V$: $v_2 \prec v_1$. *Let* $v \in V$ *be a vertex such that* $S(v) \neq \emptyset$ *(such a vertex exists because* $|E| > 0$*). Note that* $|\{x \in V | v \simeq_G^F x\}| < |V|$, *because otherwise* $v_1 \preceq_G^F x \preceq_G^F v_2$. *Let* $E' = E \setminus \{(v, x) | x \in V\}$ *and* $G' = (V, E')$. *By the assumption of induction,* $|\{x \in V | v \simeq_{G'}^F x\}| = |V|$. *Thus,* $|\{x \in V | v \prec_{G'}^F x\}| = 0$. *By strong incentive compatibility,* $0 \leq |\{x \in V | v \prec_G^F x\}| \leq |\{x \in V | v \prec_{G'}^F x\}| = 0$, *thus* $|V| = |\{x \in V | v \simeq_{G'}^F x\}| \leq |\{x \in V | v \simeq_G^F x\}| < |V|$ *which yields a contradiction.* □

PROPOSITION 11. *There exists no weakly incentive compatible nontrivial minimally fair ranking system that satisfies the strong union condition.*

PROOF. *Assume for contradiction a ranking system* F *that satisfies the conditions. As the conditions of Lemma 1 are satisfied, let* $G = (V, E)$ *and* $v, v_\perp, v_\top \in V$ *be the graph and the vertices from the lemma. Now let* $G_1 = (V \setminus \{v_\top\}, E \setminus \{(v', v_\top) \in E | v' \in V\})$ *and let* $G_2 = (\{v_\top\}, \emptyset)$. *From conditions 3 and 5,* $S(v_\top) = \emptyset$ *and thus* G_1 *and* G_2 *satisfy the conditions of the strong union condition with regard to* G. *Therefore,* $v_\perp \prec_G^F v \Rightarrow v_\perp \prec_{G_1}^F v$, *in contradiction to condition 1, because* $|E_1| \leq |E|$ *and* $|V_1| < |V|$. □

7. SOME ILLUMINATING LESSONS

Theorems 1 and 2 teach us some surprising lessons about the implications of various versions of the basic properties.

7.1 Strong incentive compatibility is different than weak incentive compatibility

We have seen in Proposition 10 that, as one would expect, strong incentive compatibility is impossible when assuming minimal fairness. However, it turns out that when we slightly weaken the requirement of incentive compatibility to cover only the *expected* rank of the agent, Proposition 4 shows us this is possible. This means that the level of incentive compatibility has an effect on the existence of ranking systems.

7.2 Positive Response is not the same as Monotonicity

The Positive response and Monotonicity properties seem, at a glance, to be very similar, as they both informally require that the more votes an agent has, the higher it is ranked. However, looking more deeply, we see that the Positive Response properties require this behavior to be manifested across graphs, while the Monotonicity properties require that the effect be seen within a single graph.

This leads to interesting facts, such as not being able to nontrivially satisfy both Weak Monotonicity and Weak Positive response with incentive compatibility (Proposition 5), while each of the properties could be satisfied separately (Propositions 4 and 1) . Furthermore, Strong Monotonicity cannot be satisfied at all (Proposition 9) with weak incentive compatibility, while Strong Positive Response *can* be satisfied even with strong incentive compatibility (Proposition 1).

7.3 The Weak Union property matters

Recall that the weak union property requires that when two disjoint graphs are put together, the subgraphs must still be ranked as before.

This property might seem trivial, but the impossibility results in Theorem1 imply that this property has a part in inducing impossibility. The reason for this is twofold:

- The combination of two graphs adds more options for the agents in both subgraphs to vote for, which in order to preserve incentive compatibility, must all preserve the agent's relative rank in the combined graph.
- The weak union property further implies that the ranking system must not rely on the number of vertices in the graph, and moreover, that the minimal nontrivially ranked graph for a given ranking system must be connected.

8. THE ISOMORPHISM PROPERTY AND FURTHER RESEARCH

Most of the ranking systems we have seen up to now in the possibility proofs take advantage of the names of the vertices to determine the ranking. A natural requirement from a ranking system is that the names assigned to the vertices will not take part in determining the ranking. This is formalized by the isomorphism property.

DEFINITION 13. *A ranking system* F *satisfies* isomorphism *if for every isomorphism function* $\varphi : V_1 \mapsto V_2$, *and two isomorphic graphs* $G \in \mathbb{G}_{V_1}, \varphi(G) \in \mathbb{G}_{V_2}$: $\preceq_{\varphi(G)}^F = \varphi(\preceq_G^F)$.

It turns out that the ranking system F_4 from the possibility proof for weak incentive compatibility and weak monotonicity (Proposition 4) satisfies isomorphism as well, and thus there exists an weakly incentive compatible ranking system satisfying isomorphism and weak monotonicity. The existence of weakly incentive compatible ranking systems satisfying isomorphism in conjunction with either the weak union property or the weak positive response is an open question.

Another natural extension of this work, is to consider weaker notions of incentive compatibility, where agents may have beneficial deviations, but the amount or magnitude of such deviations is bounded. In a pending paper, we address the quantification of such weaker notions of incentive compatibility.

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