LONG RANGE ORDER AND GIANT COMPONENTS OF QUANTUM RANDOM GRAPHS

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Abstract. Mean field quantum random graphs give a natural generalization of classical Erdős-Rényi percolation model on complete graph $G_N$ with $p = \beta/N$. Quantum case incorporates an additional parameter $\lambda \geq 0$, and the short-long range order transition should be studied in the $(\beta, \lambda)$-quarter plane. In this work we explicitly compute the corresponding critical curve $\gamma_c$, and derive results on two-point functions and sizes of connected components in both short and long range order regions. In this way the classical case corresponds to the limiting point $(\beta_c, 0) = (1, 0)$ on $\gamma_c$.

1. Introduction and Results

1.1. Classical Erdős-Rényi random graphs. In the classical Erdős - Rényi model of random graphs each two vertices $i \neq j$ of the complete graph $G_N = \{1, \ldots, N\}$ are connected with probability $p = \beta/N$ independently from all other edges. The phase transition [JLR] occurs at the critical value

$$\beta_c = 1.$$ (1.1)

Namely, for $\beta > \beta_c$ with probabilities of order $1 - o(1)$ there is a unique giant connected component of size $O(N)$, whereas for $\beta < \beta_c$ all the connected components of $G_N$ have sizes of the order $O(\log N)$ or less.

1.2. Quantum version of Erdős-Rényi random graphs. Let us formulate now a quantum version of Erdős-Rényi random graphs. As we shall briefly explain in the end of the section, both the motivation and the choice of terminology comes from the stochastic geometric (Fortuin-Kasteleyn type) representation of quantum Curie-Weiss model in transverse magnetic field, which was originally developed in the general ferromagnetic context in [CKP, AKN].

There are two parameters $\beta \in (0, \infty]$ - the inverse temperature and $\lambda \in [0, \infty)$ - the strength of the transversal field. The case $\beta = \infty$ corresponds to the ground state, and the case $\lambda = 0$ brings us back to the context of classical random graphs discussed above.

As it has been pointed to us by Geoffrey Grimmett a nearest neighbour ground state ($\beta = \infty$) version of the model we consider here has been studied in [BG] under the name of continuous time percolation.

Given $\beta \in [0, \infty]$ let us use $S_\beta$ to denote the circle of length $\beta$ under the convention $S_\infty = \mathbb{R}$. The model is built on the space $\mathfrak{G}_N^\beta = G_N \times S_\beta$, that is to each site $i \in G_N$ we attach a copy $S_\beta^i$ of $S_\beta$. With a slight abuse of notation we shall also write $S_\beta^i = i \times S_\beta$. 

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Our next step is to make finite ($\beta < \infty$) or countable ($\beta = \infty$) random number of holes in each $S_\beta^i$ and draw finite ($\beta < \infty$) or countable ($\beta = \infty$) random number of links between various points of $\Theta_N$ with the same time coordinates, that is between points of the type $(i,t)$ and $(j,t)$ where $i \neq j$ and $t \in S_\beta$. Both operations are going to be performed with the help of independent Poisson point processes over the time space $S_\beta$ and, eventually, will lead to a splitting of $\Theta_N^\beta$ into a finite ($\beta < \infty$) or countable ($\beta = \infty$) number of disjoint maximal connected components,

$$\Theta_N^\beta \setminus \mathcal{H} = \mathcal{C}_1 \lor \cdots \lor \mathcal{C}_n,$$

(1.2)

where $\mathcal{H}$ is the set of the holes. An example for $N = 3$ is given on Figure 1.2. For each fixed $x \in \Theta_N^\beta$ the probability $P_{N}^{\beta,\lambda}(x \in \mathcal{H}) = 0$. Thus, for given $x \in \Theta_N^\beta$ the notion $\mathcal{C}(x)$ of the connected component containing $x$ in the decomposition (1.2) is well defined.

 Processes of holes $\mathcal{H}_i$. For each $i \in G_N$ the process of holes $\mathcal{H}_i$ is the Poisson point process on $S_\beta^i$ with intensity $\lambda$. $\mathcal{H}_i$-s are assumed to be independent for different $i$-s. For $\beta < \infty$ the punched circle $S_\beta^i \setminus \mathcal{H}_i$ consists of $n$ disjoint maximal connected intervals,

$$S_\beta^i \setminus \mathcal{H}_i = I_1^i \cup I_2^i \cup \cdots \cup I_n^i.$$  

(1.3)

Of course $n = 1$ whenever the cardinality $\# \mathcal{H}_i = 0, 1$.
In the case $\beta = \infty$ the punched real line $S^i_{\beta} \setminus \mathcal{H}_i$ is split into a countable disjoint union of connected intervals,

$$S^i_{\infty} \setminus \mathcal{H}_i = \bigcup_{r=-\infty}^{\infty} I^i_r,$$

(1.4)

where we label $I^i_0$ the interval which contains $(i, 0)$.

In the sequel we shall use $|I^i_k|$ to denote the length of $I^i_k$, and we shall write $\mathcal{H}$ for the total collection of all the holes,

$$\mathcal{H} = \cup_i \mathcal{H}_i \subset \mathfrak{S}_{N}^\beta.$$

Processes of links $L_{ij}$ and decomposition (1.2). With each (unordered) pair of vertices $i, j \in G_N$ we associate a copy $S^i_{\beta}$ of $S_{\beta}$ and a Poisson point process $L_{ij}$ on $S^i_{\beta}$ with intensity $1/N$. Processes $L_{ij} = L_{ji}$ are assumed to be independent for different $(i, j)$ and also independent of the processes of holes $\mathcal{H}_i$.

Two intervals $I^i_k$ and $I^j_l$ either in the decomposition (1.3) or accordingly, in the case $\beta = \infty$, in the decomposition (1.4) are said to be connected if there exists $t \in S^i_{\beta}$ such that $t \in L_{ij}$, whereas $(i, t) \in I^i_k$ and $(j, t) \in I^j_l$. The decomposition (1.2) of $\mathfrak{S}_{N}^\beta \setminus \mathcal{H}$ into maximal connected components is, thereby, well defined.

Relation to the classical Erdős-Rényi random graph. If $\lambda = 0$ then there are no holes and $S^i_{\beta} \setminus \mathcal{H}_i$ always contains only one connected component, which of course equals to $S^i_{\beta}$ itself. In the latter case, the probability ($\beta < \infty$) that $S^i_{\beta}$ and $S^j_{\beta}$ are connected equals to $1 - e^{-\beta/N}$ and we are back to the original Erdős-Rényi setup.

1.3. Phase transition in the $(\beta, \lambda)$-plane. The critical curve $\gamma_c$ in the $(\beta, \lambda)$-coordinate quarter plane is implicitly given by (see Figure 1.3)

$$F(\beta, \lambda) = \frac{2}{\lambda} \left(1 - e^{-\lambda \beta}\right) - \beta e^{-\lambda \beta} = 1.$$ (1.5)

The branching process picture behind (1.5) is explained in Subsection 2.3. It is easy to check that $\gamma_c$ is in fact a graph of a function $\lambda_c = \lambda_c(\beta)$ defined on $\beta \in [1, \infty)$. Consider the decomposition (Figure 1.3) of the off-critical region

$$\mathbb{R}^2_+ \setminus \gamma_c = A_{LRO} \cup A_{SRO},$$

where

$$A_{LRO} = \{(\beta, \lambda) \in \mathbb{R}^2_+ : F(\beta, \lambda) > 1\}.$$ (1.6)

LRO and SRO above stand for the long (respectively short) range order.

Our main result states that for $(\beta, \lambda) \in A_{LRO}$ there is a long range order in the sense that the probability of two points $(i, j)$ and $(j, s)$ being connected does not vanish when the size of the system tends to infinity. Contrary to this such probability vanishes in the $N \to \infty$ limit whenever $(\beta, \lambda) \in A_{SRO}$. The survival of probabilities of connections is related to an emergence of an $O(N)$-giant connected component in the disjoint decomposition (1.2) in the LRO regime; in particular for $(\beta, \lambda) \in A_{SRO}$ typical connected component of any point $(i, t) \in \mathfrak{S}_{N}^\beta$ is of order $O(\log N)$. This is a quantum version of Erdős-Rényi phase transition phenomenon and, since

$$\lim_{\lambda \to 0} F(\beta, \lambda) = \beta,$$

the classical case is recovered in the limiting $\lambda = 0$ case.
Figure 2. Decomposition of the $$(\beta, \lambda)$$ quarter plane into the short range and long range regions.

We proceed with several exact alternative statements of this result.

1.4. Long and short range order. In the sequel we shall use $\mathbb{P}_N^{\beta, \lambda}(\cdot)$ for the joint product measure of all the processes $H_i$ of holes and all the processes $L_{ij}$ of links as defined above.

Let us say that two points $(i, t), (j, s) \in S_{N, \beta}$ are connected if they belong to the same connected component in the decomposition (1.2). We shall denote the latter event as \{$(i, t) \leftrightarrow (j, s)$\}.

Theorem A. If $(\beta, \lambda) \in A_{\text{SRO}}$, then

$$\mathbb{P}_N^{\beta, \lambda}((i, t) \leftrightarrow (j, s)) = O\left(\frac{\log N}{N}\right)$$

uniformly in $t, s \in S_{\beta}$ and $i \neq j$.

On the other hand, if $\beta < \infty$ and $(\beta, \lambda) \in A_{\text{LRO}}$, then there exists $p = p(\beta, \lambda) \in (0, 1)$, such that

$$\mathbb{P}_N^{\beta, \lambda}((i, t) \leftrightarrow (j, s)) = p(\beta, \lambda)^2 (1 + o(1)),$$

also uniformly in $t, s \in S_{\beta}$ and $i \neq j$.

1.5. Emergence of the giant component. Each connected cluster $C_k$ in the decomposition (1.2) consists of disjoint union of intervals

$$C_k = \bigcup_l J_k^l,$$

where each $J_k^l$ coincides with some $I_r^i$ in one of the decompositions (1.3) ((1.4) in the $\beta = \infty$ case). Define,

$$|C_k| = \sum_l |J_k^l|.$$

For any fixed $(i, t) \in S_{N, \beta}$ the probability

$$\mathbb{P}_N^{\beta, \lambda}((i, t) \in \mathcal{H}) = 0.$$
Thus, in general position, \((i, t) \in \Theta^β \setminus \mathcal{H}\) and there exists \(\mathcal{C}_k(i, t)\) (which from now on we shall denote as \(\mathcal{E}(i, t)\)) such that \((i, t) \in \mathcal{C}_k\). Evidently the distribution of \(|\mathcal{E}(x)|\) does not depend on a particular \(x = (i, t)\).

If \(\beta < \infty\) we also define the maximal cluster size

\[
\mathcal{M} = \max_k |\mathcal{C}_k|,
\]

(1.9)

and the next to maximal cluster size,

\[
\mathcal{M}^{\text{next}} = \max \{|\mathcal{C}_k| : |\mathcal{C}_k| \neq \mathcal{M}\}.
\]

(1.10)

These definitions would clearly make little sense if \(\beta = \infty\).

**Theorem B.** If \((\beta, \lambda) \in A_{\text{SRO}}\), then for every \(\kappa > 0\) there exists \(c = c(\beta, \lambda, \kappa) < \infty\), such that

\[
P^β_\lambda(\mathcal{E}(x)| > c \log N) = o\left(\frac{1}{N^\kappa}\right).
\]

(1.11)

Furthermore, if \(\beta < \infty\), then

\[
P^β_\lambda(\mathcal{M} > c \log N) = o\left(\frac{1}{N^{\kappa - 1}}\right)
\]

(1.12)

If, however, \(\beta < \infty\) and \((\beta, \lambda) \in A_{\text{LRO}}\) then there exists a sequence of positive numbers \(\epsilon_N(\beta, \lambda) \to 0\) such that,

\[
P^β_\lambda\left(\left|\frac{|\mathcal{E}(x)|}{N} - \rho\right| < \epsilon_N\right) = p(\beta, \lambda)(1 - o(1)),
\]

(1.13)

where \(\rho = \rho(\beta, \lambda) = \beta p(\beta, \lambda) > 0\), and \(p(\beta, \lambda)\) is the same probability as in (1.8). Furthermore, in the \(\beta < \infty\) case, there exists a constant \(c = c(\beta, \lambda) < \infty\) such that

\[
P^β_\lambda(\mathcal{E}(\rho, \epsilon_N, c)) = 1 - o(1),
\]

(1.14)

where the event \(\mathcal{E}(\rho, \epsilon_N, c)\) is defined via

\[
\mathcal{E}(\rho, \epsilon_N, c) = \left\{ \left(\frac{\mathcal{M}}{N} - \rho\right) < \epsilon_N \right\} \cap \{\mathcal{M}^{\text{next}} < c \log N\}.
\]

(1.15)

In this paper we shall prove the short range order parts of Theorem A and Theorem B for all \(\beta \leq \infty\), whereas in the long range order case we shall concentrate only on the case of positive temperatures \(\beta < \infty\). A treatment of the LRO properties of the ground state case \(\beta = \infty\) requires an additional coupling with branching random walks in the sense of [Bi]. Specifically, in the \(\beta = \infty\) case in order to show that two large connected clusters intersect one should also control the time spread-off of each of these clusters. The corresponding results may be interesting in their own right and we relegate them to the forthcoming [IL1].

After the first draft of this paper became available online, part of our results (for \(\beta < \infty\)) was reproven using different methods in [Ja].

1.6. **Relation to Curie-Weiss model in transverse field.** The Hamiltonian of the Curie-Weiss model in transversal field is given by

\[
-\frac{1}{2N} \sum_{i \neq j} \sigma_i^z \sigma_j^z - \lambda \sum_i \sigma_i^x,
\]

(1.16)

where \(\sigma_i^z, \sigma_i^x\) are Pauli spin 1/2 matrices. As it has been discovered in [CKP, AKN] (in a general ferromagnetic context) path-integral type representation of the Curie-Weiss leads
to the following modification $\tilde{P}_{N}^{\beta,\lambda}$ of our basic product measure of “holes” and “links”, which is in fact the Fortuin-Kasteleyn representation of (1.16)

$$\tilde{P}_{N}^{\beta,\lambda}(dH, dL) = \frac{1}{Z_N(\beta, \lambda)} 2^{\#c(H, L)} P_{N,2}^{\beta,\lambda}(dH, dL),$$  \hspace{1cm} (1.17)

where $\#c(H, L)$ is the number of maximal connected components in the decomposition (1.2), and the measure $P_{N,2}^{\beta,\lambda}$ is constructed in the same way as $P_{N}^{\beta,\lambda}$ except that the process of links $L_{ij}$ has a double intensity $2/N$ for each pair $i \neq j$. In particular, the two point function could be expressed in terms of this FK-measure as

$$\langle \sigma_i^z \sigma_j^z \rangle_{\beta,\lambda} = \tilde{P}_{N}^{\beta,\lambda}((i, 0) \leftrightarrow (j, 0)).$$

A sample path large deviation analysis [IL2] of (1.16) indicates that the long/short range order Curie-Weiss critical curve in the $(\beta, \lambda)$-plane is given by

$$F_{CW}(\beta, \lambda) = \frac{1}{\lambda} \tanh(\lambda\beta) = 1.$$  \hspace{1cm} (1.18)

Similar results were independently obtained in [BCC].

1.7. FKG properties of $P_{N}^{\beta,\lambda}(\cdot)$. Many of our arguments rely on the following FKG (Fortuin-Kasteleyn-Ginibre) property of $P_{N}^{\beta,\lambda}(\cdot)$:

Let us define the partial order of the probability space, $\Omega$, in which $(H, L)$ takes values in the following way:

$$(H', L') \gg (H, L) \iff H' \subseteq H \text{ and } L' \supseteq L.$$  

In other words, in the decomposition of $\Theta_{N}^{\beta}$ generated by $(H', L')$ there are less holes and more links than in the decomposition corresponding to $(H, L)$.

In the sequel we shall say that $A$ is an increasing (decreasing) event if for all $(H, L) \in A$,

$$\text{if } (H', L') \gg (H, L) \text{ then } (H', L') \in A.$$  

As it has been proven in [AKN, CKP] a probability measure $P_{N}^{\beta,\lambda}(\cdot)$ has the positive association property: if both $A$ and $B$ are increasing (decreasing) events, then

$$P_{N}^{\beta,\lambda}(A \cap B) \geq P_{N}^{\beta,\lambda}(A) P_{N}^{\beta,\lambda}(B).$$

1.8. Structure of the paper. Our proof is built upon the classical treatment (see e.g. [BGJ]). An essential additional complication to be encountered is that in the quantum case two different clusters may share a spatial component without intersecting. In other words it can happen that there exists an index $i \in G_N$ and two disjoint clusters $C_1$ and $C_2$ such that $C_1 \cap \tilde{S}_i^j \neq \emptyset$ for $l = 1, 2$.

In Section 2 we set up most of the relevant notation and develop our basic inductive construction of percolation clusters. In the $\beta < \infty$ case the genealogical structure of percolation clusters could be ignored and, accordingly, our exposition could be slightly simplified. However, the multi-index notation we employ will become indispensable in the $\beta = \infty$ case [IL1] and, besides, we feel that it gives a rather natural way to describe connected clusters.

Section 3 is devoted to the proofs of all our main results.
2. Construction of connected clusters

2.1. Underlying probability space. Let $M_+$ be the countable set of all finite multi-indices $\alpha$.

$$M_+ = \{ \alpha = (\alpha_1, \ldots, \alpha_n) ; \alpha_i \in \mathbb{N} , \ n = 1, 2, \ldots \} .$$

There is a total order on $M_+$: we shall say that $\alpha < \gamma$ if either $\alpha$ has less entries (belongs to an older generation) than $\gamma$ or else if $\alpha$ is less than $\gamma$ in the lexicographical order. The underlying probability measure which we proceed to call $P_{N}^{\beta, \lambda} (\cdot)$ is a product measure

$$P_{N}^{\beta, \lambda} (\cdot) = \bigotimes_{\alpha \in M_+} Q_{N, \alpha}^{\beta, \lambda} .$$

Each measure $Q_{N, \alpha}^{\beta, \lambda} (\cdot)$ generates an interval $I[\alpha] \subseteq S_{\beta}$ and, subsequently, a point process $X_+ [\alpha]$ on $S_{\beta}^N = S_{\beta} \times S_{\beta}$ according to the following procedure, whose relation to the background $(\mathcal{H}, \mathcal{L})$ process of holes and links should be obvious:

Construction of $I[\alpha]$. Let $U, V$ be two independent Exp($\lambda$) random variables. If $U + V \geq \beta$, then $I = S_{\beta}$. Otherwise, $I = (-V, U) \subseteq S_{\beta}$.

In the sequel we shall refer to the distribution of the interval $I[\alpha]$, or more precisely to the distribution of its length $|I[\alpha]|$, just constructed as to $\Gamma_{\beta}(2, \lambda)$ distribution. Obviously, in the $\beta = \infty$ case $|I|$ of $I$ is distributed as $\Gamma (2, \lambda)$ variable.

Construction of the number of offsprings and of the point process $X_+ [\alpha]$. Given $I[\alpha]$ and, in particular the length $|I[\alpha]|$, sample the number of off-springs $\xi_+[\alpha]$ from the Poisson distribution

$$\xi_+[\alpha] \sim \text{Poisson} (\langle I[\alpha] \rangle) .$$

We shall denote the (unconditional) distribution of $\xi_+[\alpha]$ constructed above as $\Xi_{\beta}(2, \lambda)$.

With $I[\alpha]$ and $\xi_+[\alpha]$ fixed, sample

$$X_+ [\alpha] = \{ x_1 [\alpha], \ldots, x_{\xi_+} [\alpha] \} = \{ (d_1 [\alpha], \tau_1 [\alpha]), \ldots, (d_{\xi_+} [\alpha], \tau_{\xi_+} [\alpha]) \} ,$$

where the departure times $\tau_1, \ldots, \tau_{\xi_+}$ are i.i.d. random variables with the uniform distribution on $I[\alpha]$, whereas the departure destinations $d_1, \ldots, d_{\xi_+}$ are i.i.d. uniform Uni($G_N$) random variables. For the latter use we define the set of all departure destinations from $\alpha$.

$$D_+ [\alpha] = \text{proj}_{G_N} X_+ [\alpha] .$$

2.2. Construction of $\mathcal{C}(x)$. Let $x \in S_{\beta}^N$. The connected cluster $\mathcal{C}(x)$ (see Subsection 1.2) is a disjoint union,

$$\mathcal{C}(x) = \bigcup_k i_k \times J_k$$

(2.19)

of intervals $J_k \subseteq S_{\beta}$ with spatial coordinates $i_k$. These disjoint intervals will be labeled by a subset $M^x \subset M_+$ of multi-indices. We shall always record the multi-indices from $M^x$ in their increasing order, $M^x = \{ \underline{\alpha}_1, \underline{\alpha}_2, \ldots \}$. In this way we denote the spatial coordinate $i_k = i[\underline{\alpha}_k]$ and the associated interval $J_k = J[\underline{\alpha}_k]$ and rewrite (2.19) as

$$\mathcal{C}(x) = \bigcup_{\alpha \in M^x} i[\alpha] \times J[\alpha] .$$

(2.20)

Our construction of $M^x$ and, accordingly, of $\{ i[\underline{\alpha}], J[\underline{\alpha}] \}$ is an inductive one: At each stage we screen a certain multi-index $\underline{\alpha}$ and keep track of

- $M^x(\underline{\alpha})$ - set of multi-indices which were already saturated into $M^x$ before $\underline{\alpha}$.
• $R^x(\alpha)$ - set of multi-indices (including $\alpha$ itself) which are potential candidates for the membership in $M^x$ and which are yet to be screened.

Both $M^x$ and $R^x$ are updated once $\alpha$ is screened. The construction is complete whenever we finish an update with $R^x = \emptyset$.

Notice that at each stage we also keep track both of space coordinates $i[\alpha] \in G_N$ and of time coordinates $t[\alpha] \in S_\beta$ for all multi-indices $\alpha$ from $M^x \cup R^x$. On the other hand, we sample intervals $J[\alpha]$ and the associated point processes $X[\alpha] \subseteq X^x[\alpha]$ only at the moment when $\alpha$ is screened.

The fact the construction below indeed reproduces the correct distribution of $C(x)$ is straightforward once we try to think about all the Poisson processes involved in terms of the usual approximation by Bernoulli trials.

**Initial stage.** For $x = (j, t)$ set

$$\underline{\omega}_1 = (1) \quad i[\underline{\omega}_1] = j \quad t[\underline{\omega}_1] = t \quad M^x = \emptyset \quad R^x = \{\underline{\omega}_1\}.$$  

**Screening stage.** If $R^x$ is empty, then stop. Otherwise, choose $\alpha$ to be the minimal element of $R^x$ (and set $R^x(\alpha) = R^x$ and $M^x(\alpha) = M^x$). There are two steps to be performed and several cases to be considered:

**STEP 1** Deciding whether $\alpha$ is to be included into $M^x$ (Cases 2 and 3 below) and, if yes, sampling of $J[\alpha]$.

**CASE 1** If there exists $\gamma \in M^x(\alpha)$ (and then necessarily satisfying $\gamma < \alpha$) such that

$$i[\alpha] = i[\gamma] \quad \text{and} \quad t[\alpha] \in J[\gamma],$$

then remove $\alpha$ from $R^x$ and proceed to screen the next multi-index of $R^x$.

**CASE 2** If $i[\alpha] \neq i[\gamma]$ for every $\gamma \in M^x(\alpha)$, which means that

$$C_\alpha(x) \triangleq \bigvee_{\gamma \in M^x(\alpha)} i[\gamma] \times J[\gamma]$$

(2.21)  

does not hit $S^{|[\alpha|}_\beta$, then set $J[\alpha] = t[\alpha] + I[\alpha]$, where $I[\alpha]$ is sampled from $P_{N}^{\beta \lambda} (\cdot)$ as described in Subsection 2.1.

**CASE 3** In the remaining case,

$$C_\alpha(x) \cap S^{|[\alpha|}_\beta \neq \emptyset \quad \text{but} \quad (i[\alpha], t[\alpha]) \not\in C_\alpha(x).$$

In such a situation define $J[\alpha]$ as the connected component of $t[\alpha]$ of

$$(t[\alpha] + I[\alpha]) \setminus (C_\alpha(x) \cap S^{|[\alpha|}_\beta).$$

If either CASE 2 or CASE 3 took place then add $\alpha$ to $M^x$, remove $\alpha$ from $R^x$ and proceed with the second step.

**STEP 2** Generating descendants of $\alpha$. Sample $\xi_{+}[\alpha]$ and, accordingly, the point process $X_{+}[\alpha]$ from the underlying distribution $P_{N}^{\beta \lambda} (\cdot)$. Screen all $k = 1, \ldots, \xi_{+}$ departures of $X_{+}$ as follows:

**CASE 1** If $t[\alpha] + \tau_k \not\in J[\alpha]$, then ignore this $k$-th departure.

**CASE 2** Otherwise register $k$-th departure as follows: Add $(\alpha, k)$ to $R^x$ and set,

$$i[(\alpha, k)] = d_k \quad \text{and} \quad t[(\alpha, k)] = t[\alpha] + \tau_k.$$

Return to the beginning of the screening stage. \qed
In the sequel we shall use the following notation: For each $\alpha \in \mathbf{M}^x$ we define $N^x(\alpha)$ as the set of all registered descendants of $\alpha$. The corresponding point process is $\mathbf{X}[\alpha] = \{(i[\gamma], \tau[\gamma]): \gamma \in N^x(\alpha)\}$. Finally, we shall denote the set of all spatial coordinates of registered descendants of $\alpha$ as $\mathbf{D}[\alpha] = \text{proj}_{\mathbf{G}} N^x[\alpha]$. 

2.3. The critical curve. As it becomes clear from the above construction of $C(x)$, the size of $\mathbf{M}^x$ is stochastically dominated by the total population size of Galton-Watson process with offspring distribution $\Xi_{\beta}[2, \lambda]$. Let $\xi \sim \Xi_{\beta}[2, \lambda]$. Evidently, $\xi$ has finite exponential moments. Furthermore, 

$$E_{N}^{\beta,\lambda}(\xi) = N E_{N}^{\beta,\lambda}(1 - e^{-|I|/N}) = O\left(\frac{1}{N}\right) + E_{N}^{\beta,\lambda}(|I|).$$

Now (with the usual convention $0 \cdot \infty = 0$ in the $\beta = \infty$ case), 

$$E_{N}^{\beta,\lambda}(|I|) = E(U + V; U + V < \beta) + \beta P(U + V \geq \beta),$$

where, as before, $U$ and $V$ are two independent exponential $\text{Exp}(\lambda)$ random variables. Since $U + V \sim \Gamma(2, \lambda)$, 

$$P(U + V \geq \beta) = \int_{\beta}^{\infty} \lambda^2 te^{-\lambda t} dt = (\lambda \beta + 1)e^{-\lambda \beta}.$$

In the same fashion, 

$$E(U + V; U + V \leq \beta) = \int_{0}^{\beta} \lambda^2 t e^{-\lambda t} dt$$

$$= \frac{2}{\lambda} \left(1 - e^{-\lambda \beta}\right) - \beta e^{-\lambda \beta} = F(\beta, \lambda),$$

Consequently, 

$$E_{N}^{\beta,\lambda}(|I|) = \frac{2}{\lambda} \left(1 - e^{-\lambda \beta}\right) - \beta e^{-\lambda \beta} = F(\beta, \lambda), \quad (2.22)$$

which is, of course, precisely the right hand side of (1.5).

2.4. LRO: size of $\mathbf{S}^x$. In the LRO-case of $F(\beta, \lambda) > 1$ we shall confine the discussion to the case of finite $\beta < \infty$. Define 

$$\mathbf{S}^x = \text{proj}_{\mathbf{G}} \mathbf{C}(x).$$

In other words $\mathbf{S}^x$ is the set of all different spatial coordinates of $\mathbf{C}(x)$. Obviously, 

$$|\mathbf{C}(x)| \leq \beta \#\mathbf{S}^x.$$ 

In fact, as we shall see in Section 3 a converse is also true in the sense that large size of $\#\mathbf{S}^x$ necessarily implies that $|\mathbf{C}(x)|$ is also large. Meanwhile: 

Lemma 2.1. In the LRO-case let $\delta > 0$ be such that $(1 - \delta)F(\beta, \lambda) > 1$. Then, for every $\kappa > 0$ there exists $c_1 = c_1(\beta, \lambda, \kappa)$, such that 

$$P_{N}^{\beta,\lambda}(\#\mathbf{S}^x \in [c_1 \log N, \delta N]) = o\left(\frac{1}{N^\kappa}\right).$$

(2.23)
In the sequel we shall say that $C(x)$ is small if $\#S^x < c_1 \log N$ and, accordingly, that it is large if $\#S^x > \delta N$.

Proof of Lemma 2.1. Let us go back to STEP 2 of the screening stage of Subsection 2.2. A multi-index descendant $\gamma \in N^x(\alpha)$ of $\alpha \in M^x$ is called free if its colour $i[\gamma]$ is encountered for the first time in the course of our construction of $C(x)$. Formally, $\gamma$ is free if

$$i[\gamma] \notin \{i[\mu] : \mu < \gamma \text{ and } \mu \in M^x(\alpha) \cup R^x(\alpha) \cup N^x(\alpha)\}.$$ 

Notice that any free descendant of $\alpha$ will be duly included into $M^x$ at later screening stages. Also notice that the set $F^x \subseteq M^x$ of all free multi-indices is in one-to-one correspondence with $S^x$. Let us define

$$S^x(\alpha) = \{i[\gamma] : \gamma \in M^x(\alpha) \cup R^x(\alpha)\}.$$

In other words $S^x(\alpha)$ is the set of all different spatial coordinates of $C(x)$ which were generated before screening of $\alpha$. Then the number $\eta_f(\alpha)$ of free descendants of $\alpha$ is given by

$$\eta_f(\alpha) = \sum_{i \in G_N \setminus S^x(\alpha)} I_{i \in D(\alpha)}.$$ 

(2.24)

If $\alpha$ is itself free and, moreover, $\#S^x(\alpha) < \delta N$, then $\eta_f(\alpha)$ stochastically dominates and, in fact, could be coupled in a straightforward way with a random variable $\eta(\alpha)$ whose conditional on $I[\alpha]$ distribution is binomial $\text{Bin} \left( (1 - \delta)N, 1 - e^{-|I[\alpha]|/N} \right)$. Of course, $E^{\beta,\lambda}_N (\eta) \to (1 - \delta)F(\beta, \lambda) > 1$ as $N \to \infty$.

Let us, thereby, consider an i.i.d. family $\{\eta_k\}$ of random variables distributed as above, which is coupled with the collection $\{\eta_f(\alpha)\}$ in the following way: Write all free multi-indices $F^x = \{\alpha^1, \alpha^2, \ldots\}$ in their increasing order. Then $\eta_k \leq \eta_f(\alpha^k)$ on the event $\{\#S^x(\alpha^k) < \delta N\}$. Since $\#S^x = \#F^x$, and

$$\#S^x \geq \sum_{\alpha \in F^x} \eta_f(\alpha),$$ 

(2.25)

we infer that

$$P^{\beta,\lambda}_N \left( \#S^x \in [c_1 \log N, \delta N] \right) \leq \sum_{n = c_1 \log N}^{\delta N} P^{\beta,\lambda}_N \left( \sum_{1}^{n} \eta_k \leq n \right),$$

and (2.23) follows from a usual LD upper bound, as $E^{\beta,\lambda}_N (\eta) > 1$. \hfill $\Box$

2.5. Coupling between two clusters. Recall that we are recording the multi-indices $M^x = \{\alpha_1, \alpha_2, \ldots\}$ according to the order in which they were saturated or, equivalently, in their increasing order. With each $k = 1, 2, \ldots$ we associate a $\sigma$-algebra

$$\Sigma^x_k = \sigma \left( X_+[\alpha], I[\alpha] ; \lambda \leq k \right).$$ 

(2.26)

In order to avoid ambiguities we set $\alpha_k \equiv \alpha_m^x$, for $k \geq m^x$, where $m^x \Delta \equiv \#M^x$. Clearly, $\Sigma^x_k$ contains all the information on the growth of $C(x)$ up to the $k$th (or, in the case $k > m^x$, up to the last) saturation. In particular, the eventual number of intervals $m^x$ in the decomposition (2.19) is a stopping time with respect to the filtration $\{\Sigma^x_k\}$.

Our proof of various claims in the LRO case is based on the following modification of the growth algorithm described in Subsection 2.2: Let $x, y$ be two points of $\Theta_N$. 


First of all we shall grow $\mathcal{C}(x)$ according to the rules of Subsection 2.2 until a certain stopping time $S$ (with respect to the filtration $\{\Sigma^x_t\}$). We shall define $S$ below. In any case, the relevant notation is (see (2.21)),

$$\mathcal{C}_k(x) = \mathcal{C}_{\Delta k+1}(x), \ M^x_k = M^x(\Delta k+1) \quad \text{and} \quad R^x_k = R^x(\Delta k+1).$$

Accordingly, $\mathcal{C}_S(x)$ is the shape of the cluster whose growth was suspended at $S$, and $R^x_S$ is the set of all multi-indices which were registered but not yet saturated (or dropped) by time $S$.

Secondly, we shall grow $\mathcal{C}(y)$ according to the following modified set of rules, which should necessarily take into account already existing information as recorded in $\Sigma^x_S$. Growth of $\mathcal{C}(y)$ will be halted at some random stage and certainly once it will become clear that $\mathcal{C}(x)$ and $\mathcal{C}(y)$ are going to merge. In the sequel we shall use $T$ to denote the stage at which we halt the construction of $\mathcal{C}(y)$. We use $\tilde{M}^y$, $\tilde{R}^y$ etc notation to stress that we construct $\mathcal{C}(y)$ after all the data for $\Sigma^x_S$ has been already sampled. If $\tilde{M}^y = \{\tilde{\gamma}_1, \tilde{\gamma}_2, \ldots\}$ is the (ordered) set of all saturated multi-indices of $\mathcal{C}(y)$, then $T$ will be a the stopping time with respect to the filtration

$$\tilde{\Sigma}^y_0 = \Sigma^x_S \quad \text{and} \quad \tilde{\Sigma}^y_k = \sigma (\Sigma^x_S, \Sigma^y_k),$$

where $\Sigma^y_k$ is defined as in (2.26).

**The initial stage.** If $y \in \mathcal{C}_S(x)$, then set $T = 0$ and halt immediately. Otherwise proceed as in the initial stage of one-cluster construction of Subsection 2.2 (except for setting $\tilde{\gamma}_1 = (2)$, so that $x$ and $y$ related multi-indices are distinguished).

**An update stage.** There are several corrections which are due to the already existing information contained in $\Sigma^x_S$. Let $\tilde{\gamma} \in \tilde{R}^y$ be a multi-index under screening.

CORRECTION 1 In STEP 1 if $(i[\gamma], t[\gamma]) \in \mathcal{C}_S(x)$, then remove $\tilde{\gamma}$ from $\tilde{R}^y$ and proceed to screen the next multi-index of $\tilde{R}^y$.

CORRECTION 2 In CASE 2 and CASE 3 of STEP 1 define the interval $J[\gamma]$ as the connected component of $t[\gamma]$ of

$$\left((t[\gamma] + I[\gamma]) \setminus \left(\mathcal{C}_\downarrow(y) \cap \mathcal{C}_S(x) \cap S^x_\downarrow[\gamma] \right)\right).$$

CORRECTION 3 If, after $J[\gamma]$ is sampled, there exists $\alpha \in \tilde{R}^x_S$ such that $i[\alpha] = i[\gamma]$ and $t[\alpha] \in J[\gamma]$, then the clusters of $x$ and $y$ are going to merge. If $\tilde{\gamma} = \tilde{\gamma}_k$, then set $T = k$ and halt the construction of $\mathcal{C}(y)$.

2.6. **Merging of two large clusters.** Recall that a cluster $\mathcal{C}(x)$ is called large if $\#(S^x) > \delta N$, where $\delta > 0$ is some fixed number satisfying $(1 - \delta)F(\beta, \lambda) > 1$.

**Lemma 2.2.** Let $\beta < \infty$. Then for every $\kappa > 0$,

$$\mathbb{P}^x_\kappa (\mathcal{C}(x) \cap \mathcal{C}(y) = \emptyset \ | \ both \ \mathcal{C}(x) \ and \ \mathcal{C}(y) \ are \ large) = o \left(\frac{1}{N^\kappa}\right), \quad (2.27)$$

uniformly in $x, y \in \mathcal{S}_N^\beta$.
Proof of Lemma 2.2 The proof relies on the construction of clusters and, accordingly, on the coupling between $\mathcal{C}(x)$ and $\mathcal{C}(y)$ as introduced in Subsection 2.2 and Subsection 2.5.

STEP 1 We start by growing the cluster $\mathcal{C}(x)$ until the stopping time $S$ which is described as follows: Let $\epsilon > 0$ be small, e.g. $3\epsilon < \delta$ would suffice. Then,

$$S = \max\{k \leq m^x : \#S^x(\alpha_k) < \epsilon N\}. \quad (2.28)$$

Evidently, $S$ is a stopping time with respect to the filtration $(\Sigma_k^x)$.

Let $F_S^x$ be the set of all free multi-indices of $\mathcal{C}(x)$ which were generated by time $S$. In this notation,

$$S^x(\underline{\alpha_s}) \cup D(\underline{\alpha}) = \bigcup_{\underline{\alpha} \in \mathcal{F}^x_S} i[\underline{\alpha}],$$

We shall distinguish between the free multi-indices $F_{S-}^x = F_S^x \cap M_S^x$ which were already saturated by time $S$ and the remaining free multi-indices $F_{S+}^x = F_S^x \setminus F_{S-}^x$. Let $\nu > 0$ be such that

$$(1 - 3\nu)(1 - \delta)F(\beta, \lambda) > 1. \quad (2.29)$$

We claim that once $\mathcal{C}(x)$ is large, then up to $o\left(\frac{1}{N^2}\right)$-probabilities the cardinality of unsaturated free multi-indices $\#F_{S+}^x$ exceeds $3\nu\epsilon N$, or, more precisely, the cardinality of saturated multi-indices $\#F_{S-}^x$ does not exceed $(1 - 3\nu)\epsilon N$. Indeed, by construction $\#F_S^x > \epsilon N$, whereas $\#S^x(\underline{\alpha_s}) < \epsilon N$. On the other hand, as in (2.25),

$$\#S^x(\underline{\alpha_s}) \geq \sum_{\underline{\alpha} \in \mathcal{F}^x_{S-}} \eta_k.$$

Consequently,

$$P_N^{\beta, \lambda}\left(\#F_{S+}^x < 3\nu\epsilon N : \mathcal{C}(x) \text{ is large}\right) \leq P_N^{\beta, \lambda}\left(\#F_{S-}^x > (1 - 3\nu)\epsilon N : \#(S^x(\underline{\alpha_s})) < \epsilon N\right)$$

$$\leq P_N^{\beta, \lambda}\left(\sum_{1}^{(1-3\nu)\epsilon N} \eta_k < \epsilon N\right) = O\left(\frac{1}{N^\kappa}\right), \quad (2.30)$$

since $(1 - 3\nu)P_N^{\beta, \lambda}(\eta_k)$ is still larger than 1 whenever $\nu$ complies with (2.29).

STEP 2 We shall now switch to the coupled construction of $\mathcal{C}(y)$. First of all we shall adjust the notion of the set $F^y$ of free multi-indices of $\mathcal{C}(y)$: Recall that free multi-indices are generated at STEP 2 of the screening stage when we consider family of multi-index descendants $N^y(\alpha)$ of freshly saturated multi-indices $\alpha \in M^y$. We shall say that $\gamma \in N^y(\alpha)$ is free if, as before, its colour $i[\gamma]$ is encountered for the first time in the course of construction of $\mathcal{C}(y)$, but, in addition, $i[\gamma] \notin \{i[\mu] : \mu \in M_S^x\}$.

The latter set is in one-to-one correspondence with $F_{S-}^x$. As a result, on the event $\{\mathcal{C}(y) \text{ is large} \text{ and } \mathcal{C}(y) \cap \mathcal{C}(x) = \emptyset\}$,

$$\#F^y \geq \#S^y - \#F_{S-}^x \geq \delta N - (1 - 3\nu)\epsilon N \geq 2\epsilon N,$$

where the first inequality (up to $o\left(\frac{1}{N^2}\right)$-probability) follows from (2.30) and the second inequality follows from our choice of $3\epsilon < \delta$. Define now a $\{\Sigma_k^y\}$-stopping time

$$T = \min\left\{k : \#\tilde{F}^y(\Sigma_k) \geq \nu\epsilon N\right\},$$
where, $\tilde{F}^y(\gamma_y)$ is the set of all (modified) free multi-indices which were generated during the first $k$ screening sessions. Since by construction (end of Subsection 2.1) $\#D_+[\alpha]$ is a family of i.i.d random variables with exponentially decaying tails, we can safely assume that, up to $o\left(\frac{1}{N^\alpha}\right)$-probability
\[
\#\tilde{F}^y(\gamma_y) \leq 2\nu \epsilon N. \tag{2.31}
\]
Similarly to (2.30), set $\tilde{F}^y_{T,-} = \tilde{M}^y_T \cap \tilde{F}^y_T$. Then,
\[
\mathbb{P}_{N}^{\beta,\lambda} \left( \#\tilde{F}^y_{T,-} > (1-3\epsilon)\#\tilde{F}^y_T ; \ #\tilde{F}^y_T \in [\nu \epsilon N, 2 \nu \epsilon N] \right) = o\left(\frac{1}{N^\alpha}\right).
\]
Consequently, up to $o\left(\frac{1}{N^\alpha}\right)$-probability, at least $3\nu \epsilon^2 N$ modified free multi-indices from $\tilde{F}^y_T$ were not saturated by time $T$. Let us denote the latter set of multi-indices as $\tilde{F}^y_{S,T}$,
\[
\#\tilde{F}^y_{S,T} \geq 3\nu \epsilon^2 N. \tag{2.32}
\]
On the other hand, the number of all saturated multi-indices from $\tilde{F}^y_T$ is, in view of (2.31), bounded above by $2\nu \epsilon N$, which falls short of at least $3\nu \epsilon$ free unsaturated multi-indices of $\tilde{F}^y_{S,+}$. Thus the cardinality of the set
\[
\tilde{F}^x_{S,T} \triangleq \left\{ \alpha \in F^x_{S,+} : i[\alpha] \notin \text{proj}_\gamma C_T(y) \right\}
\]
satisfies,
\[
\#\tilde{F}^x_{S,T} \geq \nu \epsilon N. \tag{2.33}
\]
STEP 3 In view of (2.32) and (2.33) let us summarize the above results as follows: Consider the event $\{C(x) \cap C(y) = \emptyset \}$ both $C(x)$ and $C(y)$ are large and let $S$ and $T$ be stopping times defined as above. Then, up to $o\left(\frac{1}{N^\alpha}\right)$-probability, there exists a set of unsaturated free multi-indices $\tilde{F}^y_{S,T}$ and, accordingly, a set of unsaturated free multi-indices $\tilde{F}^y_{S,T}$ such that,
\begin{enumerate}
  \item $\#\tilde{F}^x_{S,T} \geq \nu \epsilon N$ and $\#\tilde{F}^y_{S,T} \geq 3\nu \epsilon^2 N$.
  \item For every $\alpha \in \tilde{F}^x_{S,T} \cup \tilde{F}^y_{S,T}$ and every $\gamma \in M^x_S \cup \tilde{M}^y_T$ the colours $i[\alpha] \neq i[\gamma]$.
  \item Any two different multi-indices of $\tilde{F}^x_{S,T}$, respectively of $\tilde{F}^y_{S,T}$, have different colours.
\end{enumerate}
Set
\[
S^{x,y}_{S,T} = \left\{ i[\alpha] : \alpha \in \tilde{F}^x_{S,T} \cup \tilde{F}^y_{S,T} \right\}.
\]
Property (2) above means that none of the processes of holes $\mathcal{H}_i$ was ever sampled for $i \in S^{x,y}_{S,T}$. Similarly none of the processes $\mathcal{L}_{ij}$ of links was ever sampled for $i, j \in S^{x,y}_{S,T}$. Since the family of processes
\[
\{\mathcal{H}_i, \mathcal{L}_{ij}\}_{i,j \in S^{x,y}_{S,T}}
\]
is, conditionally on $S^{x,y}_{S,T}$, independent of $\Sigma^y_{S,T}$, we infer that up to a $o\left(\frac{1}{N^\alpha}\right)$-probability,
\[
\mathbb{P}_{N}^{\beta,\lambda} \left( C(x) \not\subset C(y) \ | \ \Sigma^y_{S,T} \right) \leq \max_{v_1,\ldots,v_L} \mathbb{P}_{N}^{\beta,\lambda} \left( \bigcap_{i,j} \{ I(v_i) \not\subset I(u_j) \} \right), \tag{2.34}
\]
where $L = \nu \epsilon N$, $M = 3\nu \epsilon^2 N$ and the maximum above is over all colour disjoint collections (see property (3) above) of points $\{v_1, \ldots, v_L\} \subset \Theta^x_N$ and $\{u_1, \ldots, u_M\} \subset \Theta^y_N$. We have used the obvious notation in (2.34): For $z = (i, t) \in \Theta^x_N$, $I(z)$ denotes the interval which contains $t$ in the decomposition (1.3) of $\Theta^x_N$. 


STEP 4 It remains to derive an upper bound on the right-hand side of (2.34). Set $K = L - M = \nu(1 - 3\nu)eN$. By our choice of $\nu$ in (2.29) the number $K$ is positive and, moreover, proportional to $N$. At least $K$ of colour disjoint $u_j$-s have spatial coordinates different from any of $u_i$ spatial coordinates. There is no loss to assume that this property is enjoyed by the first $K$ points $\{v_1, \ldots, v_K\}$. Thus, the right hand side of (2.34) is bounded above by

$$\max_{v_1, \ldots, v_K} \max_{u_1, \ldots, u_M} \mathbb{P}_N^{\beta, \lambda} \left( \bigcap_{i,j} \{ I(v_i) \not\leftrightarrow I(u_j) \} \right), \quad (2.35)$$

where the maximum is now over all possible $K + L$ colour disjoint points of $\mathcal{G}^\beta_N$. By usual large deviation estimates there is a constant $c = c(\beta, \nu) > 0$ such that up to $e^{-cN}$-probability there are at least $e^{-\lambda \beta K}/2$ of $v_j$-s lie on $S_\beta$ circles without holes, and at least $e^{-\lambda \beta M}/2$ of $u_j$-s which also lie on $S_\beta$ circles without holes. This is a reduction to the computation employed in the classical case: Indeed, for any two $v_i$ and $u_j$ with spatial coordinates $i \neq j$,

$$\mathbb{P}_N^{\beta, \lambda} (I(v_i) \not\leftrightarrow I(u_j) \mid \mathcal{H}_i = \emptyset, \mathcal{H}_j = \emptyset) = e^{-\beta/N}. \quad (3.37)$$

Consequently, the expression in (2.35) is bounded above by

$$e^{-cN} + \exp \left\{ -\beta e^{-2\lambda \beta KM} \right\}. \quad (3.36)$$

Since $KM/N = 3\nu^2(1 - 3\nu)e^2N$, the proof of Lemma 2.2 is complete. \hfill \Box

3. Proofs of main results

3.1. Short range order. We employ the construction and the notation of Subsection 2.1 and Subsection 2.2.

Proof of (1.7) and of (1.11) for $\beta < \infty$. Let $\mathcal{Z}_+$ be the sub-critical Galton-Watson process with offspring distribution $\Xi_\beta(2, \lambda)$ (Subsection 2.1). The members of $\mathcal{Z}_+$ are naturally labeled by multi-indices from $\mathbb{M}_+$ and, furthermore, it is straightforward to couple the construction of $\mathcal{Z}_+$ with that of $\mathcal{C}(x)$ in such a way that

$$M^x \subseteq \mathcal{Z}_+ \quad \text{and} \quad |\mathcal{C}(x)| \leq \sum_{\alpha \subseteq \mathcal{Z}_+} |I[\alpha]| \overset{\Delta}{=} |\mathcal{Z}_+|. \quad (3.37)$$

As a result for every $\kappa > 0$ there exists $c > 0$ such that

$$\mathbb{P}^{\beta, \lambda}_N (\#(S^x) > c \log N) \leq \mathbb{P}^{\beta, \lambda}_N (\#\mathcal{Z}_+ > c \log N) = o \left( \frac{1}{N^\kappa} \right). \quad (3.36)$$

Since, $|\mathcal{C}(x)| \leq \beta \#S^x$, the bound (1.11) readily follows in the $\beta < \infty$ case.

Furthermore, if $x = (i, t)$ and $j \neq i$, then by the $G_N$-permutation invariance,

$$\mathbb{P}^{\beta, \lambda}_N (j \in S^x \mid \#S^x = n) = \frac{n - 1}{N - 1}, \quad (3.37)$$

which, in view of (3.36), implies the SRO bound (1.7) at any $\beta \leq \infty$. 


Proof of (1.11) in the $\beta = \infty$ case. First of all notice that in the $\beta = \infty$ case, the underlying distribution $\Xi_\infty(2, \lambda)$ of offsprings is just negative binomial (for the number of failures),

$$P(\xi_+ = k) = (k + 1) \frac{\lambda^2}{(1 + \lambda)^{k+2}} \quad k = 0, 1, 2, \ldots,$$

(3.38)

whereas the conditional distribution of the interval length $|I|$ is

$$|I| \sim \Gamma (k + 2, 1 + \lambda) \quad \text{given } \xi_+ = k \text{ for } k = 0, 1, 2, \ldots$$

(3.39)

In the $\beta = \infty$ case formula (3.39) implies that the conditional distribution of $|Z_+|$ given the value of the total size of the population $\#Z_+$ is precisely that of the sum of $3\#Z_+ - 1$ independent exponential $\text{Exp}(1 + \lambda)$ random variables. Therefore,

$$P^\beta_N (|Z_+| > c \log N) \leq P^\beta_N \left( \#Z_+ > \frac{1 + \lambda}{6} c \log N \right)$$

$$+ \max_n \left\{ \frac{3n}{c(1 + \lambda) \log N} \right\} \sum_{i=1}^n \left( \eta_i - \mathbb{E}^\beta_N (\eta_i) \right) > \frac{c}{2} \log N \right),$$

(3.40)

where $\eta_1, \eta_2, \ldots$ are i.i.d. $\text{Exp}(1 + \lambda)$ random variables. Clearly, for every $\kappa > 0$ one can choose $c$ sufficiently large so that the right-hand side of (3.40) is bounded above by $1/N^\kappa$.

3.2. LRO: Proof of Theorem B. For the rest of the paper we shall assume that $\beta < \infty$ and that $F(\beta, \lambda) > 1$.
**Probability of large cluster.** Recall that we say that \( \mathcal{C}(x) \) is large if the number of spatial coordinates satisfies \#\( S^x \geq \delta N \), where \( \delta \) is a fixed small number, \( (1 - \delta) F(\beta, \lambda) > 1 \).

Both upper and lower bounds on \( \mathbb{P}_N^{\beta, \lambda}(\mathcal{C}(x) \text{ is large}) \) follow, as in the classical case, from comparison with appropriate Galton-Watson processes.

The upper bound is straightforward: As in Subsection 3.1 let \( Z_+ \) be the Galton-Watson process with offspring distribution \( \Xi_\beta(2, \lambda) \). Define \( p(\beta, \lambda) \) as the survival probability of \( Z_+ \), i.e.,

\[
1 - p = E_N^{\beta, \lambda} \left( (1 - p)^{\xi_+} \right),
\]

(3.41)

where \( \xi_+ \sim \Xi_\beta(2, \lambda) \). By the very construction of \( \mathcal{C}(x) \) in Subsection 2.2,

\[
\mathbb{P}_N^{\beta, \lambda}(\mathcal{C}(x) \text{ is large}) \leq p(\beta, \lambda) \left( 1 + o(1) \right).
\]

(3.42)

The lower bound is slightly more delicate. It relies on a coupling with yet another Galton-Watson process \( Z_f^f \), which lives on free multi-indices \( F^x \) of \( \mathcal{C}(x) \). Specifically, as before set \( F^x = \{ \alpha^*_1, \alpha^*_2, \ldots \} \). Recall the definition of random variables \( \eta_f(\alpha^*_k) \) in (2.24). On the event \( \{ \#S^x < c \log N \} = \{ \#F^x < c \log N \} \), the sequence \( \{ \eta_f(\alpha^*_k) \} \) can be coupled with an i.i.d sequence \( \{ \eta^*_k \} \), such that

\[
\forall \alpha^*_k \in F^x \quad \eta_f(\alpha^*_k) \geq \eta^*_k,
\]

whereas the distribution of \( \eta^*_k \) is given by: First sample \( |I| \) from \( \Gamma_\beta(2, \lambda) \), and then sample \( \eta^*_k \) from Bin \((N - c \log N, 1 - e^{-|I|/N})\).

Let \( p_N(\beta, \lambda) \) be the survival probability of the Galton-Watson process with the offspring distribution specified by \( \eta \) above. Clearly,

\[
\lim_{N \to \infty} p_N(\beta, \lambda) = p(\beta, \lambda).
\]

(3.42)

On the other hand, by Lemma 2.1,

\[
\mathbb{P}_N^{\beta, \lambda}(\mathcal{C}(x) \text{ is large}) = 1 - \mathbb{P}_N^{\beta, \lambda}(\#S^x \leq c \log N) \geq 1 - \mathbb{P}_N^{\beta, \lambda} \left( Z_f^f \text{ dies out} \right),
\]

up to \( o(1) \) probabilities. In view of (3.42) this gives a complementary lower bound and, consequently,

\[
\mathbb{P}_N^{\beta, \lambda}(\mathcal{C}(x) \text{ is large}) = p(\beta, \lambda) \left( 1 + o(1) \right).
\]

(3.43)

**Size of \( \mathcal{M} \).** It is a straightforward exercise to deduce from (3.43), Lemma 2.1 and from the FKG properties of \( \mathbb{P}_N^{\beta, \lambda} (\cdot) \) that in the LRO regime

\[
\mathbb{P}_N^{\beta, \lambda} \left( \exists x \in \mathcal{G}_N^\beta \text{ s.t. } \mathcal{C}(x) \text{ is large} \right) = 1 - O \left( \frac{1}{N^\kappa} \right),
\]

for every \( \kappa > 0 \). Furthermore, it is equally straightforward to deduce from Lemma 2.2 that

\[
\mathbb{P}_N^{\beta, \lambda} \left( \exists \text{ unique large cluster} \mid \exists \text{ large cluster} \right) = 1 - O \left( \frac{1}{N^\kappa} \right).
\]

Let us denote such unique large cluster as \( \mathcal{C}^* \). Of course,

\[
\mathbb{P}_N^{\beta, \lambda} (x \in \mathcal{C}^*) = \mathbb{P}_N^{\beta, \lambda} (\mathcal{C}(x) = \mathcal{C}^*) = \mathbb{P}_N^{\beta, \lambda} (\mathcal{C}(x) \text{ is large}) \left( 1 - O \left( \frac{1}{N^\kappa} \right) \right),
\]

\[
\mathbb{P}_N^{\beta, \lambda} (\mathcal{C}(x) \text{ is large}) \left( 1 - O \left( \frac{1}{N^\kappa} \right) \right).
\]
does not depend on \(x \in \mathcal{G}_N^\beta\). In particular, up to \(O\left(\frac{1}{N}\right)\)-quantities,

\[
\mathbb{E}_N^{\beta,\lambda}(M^2) = \mathbb{E}_N^{\beta,\lambda} \left( \sum_{i=1}^{N} \sum_{j=1}^{N} \int_0^\beta \mathbb{1}_{(i,t) \in \mathcal{C}} \ dt \right) = \mathbb{E}_N^{\beta,\lambda} \left( \sum_{i=1}^{N} \sum_{j=1}^{N} \int_0^\beta \mathbb{1}_{\{i\in\mathcal{C}^*\ \text{and both} \ (i,t) \ \text{is large}\}} \ dt \right)
\]

(3.44)

where we have used (3.43) in the last step. Accordingly, define \(\rho(\beta, \lambda) = N\beta p(\beta, \lambda)\). Let us compute,

\[
\mathbb{E}_N^{\beta,\lambda}(M^2) = \sum_{i=1}^{N} \sum_{j=1}^{N} \int_0^\beta \mathbb{P}_N^{\beta,\lambda}(\mathcal{C}(i,t) \ \text{is large}) \ dt = N\beta^2 p(\beta, \lambda)^2 (1 + o(1)),
\]

However, for \(i \neq j\) and any \(t, s \in [0, \beta),\)

\[
\mathbb{P}_N^{\beta,\lambda}((i, t) \in \mathcal{C}^*: (j, s) \in \mathcal{C}) = \mathbb{P}_N^{\beta,\lambda}((i, t) \rightarrow (j, s)) (1 + o(1)) = p(\beta, \lambda)^2 (1 + o(1)),
\]

as it follows from the proof of Theorem A in Subsection 3.3 below. Therefore,

\[
\mathbb{E}_N^{\beta,\lambda}(M^2) \leq \beta^2 N + N^2 \beta^2 p(\beta, \lambda)^2 (1 + o(1)).
\]

As a result,

\[
\text{Var}_N^{\beta,\lambda} \left( \frac{M}{N} \right) = o(1),
\]

and both (1.13) and (1.15) follow.

3.3. LRO: Proof of Theorem A. Define \(x = (i, t)\) and \(y = (j, s)\), then for \(i \neq j\)

\[
\mathbb{P}_N^{\beta,\lambda}((i, t) \rightarrow (j, s)) = \mathbb{P}_N^{\beta,\lambda}(\mathcal{C}(x) \cap \mathcal{C}(y) \neq \emptyset)
\]

By (3.37),

\[
\mathbb{P}_N^{\beta,\lambda}(\mathcal{C}(x) \cap \mathcal{C}(y) \neq \emptyset | \mathcal{C}(y) \text{ is small}) \leq \mathbb{P}_N^{\beta,\lambda}(j \in S^x | \#S^x \leq c \log N) = O \left(\frac{\log N}{N}\right)
\]

Hence,

\[
\mathbb{P}_N^{\beta,\lambda}(\mathcal{C}(x) \cap \mathcal{C}(y) \neq \emptyset; \text{both} \ \mathcal{C}(x) \text{ and} \ \mathcal{C}(y) \text{ are small}) \leq \mathbb{P}_N^{\beta,\lambda}(\mathcal{C}(x) \cap \mathcal{C}(y) \neq \emptyset | \mathcal{C}(x) \text{ is small}) = O \left(\frac{\log N}{N}\right)
\]

(3.45)

Note that if \(\mathcal{C}(x) \cap \mathcal{C}(y) \neq \emptyset\), then, up to a \(O\left(\frac{1}{N^x}\right)\)-probability, either both \(\mathcal{C}(x)\) and \(\mathcal{C}(y)\) are large or both \(\mathcal{C}(x)\) and \(\mathcal{C}(y)\) are small. Thus, in view of (3.45),

\[
\mathbb{P}_N^{\beta,\lambda}(\mathcal{C}(x) \cap \mathcal{C}(y) \neq \emptyset)
\]

\[
= \mathbb{P}_N^{\beta,\lambda}(\mathcal{C}(x) \cap \mathcal{C}(y) \neq \emptyset; \text{both} \ \mathcal{C}(x) \text{ and} \ \mathcal{C}(y) \text{ are large}) + O \left(\frac{\log N}{N}\right).
\]

On the other hand we already know from Lemma 2.2 that two large clusters merge with high \((1 - o\left(\frac{1}{N^x}\right))\) probability. Therefore,

\[
\mathbb{P}_N^{\beta,\lambda}(\mathcal{C}(x) \cap \mathcal{C}(y) \neq \emptyset) = \mathbb{P}_N^{\beta,\lambda}(\text{both} \ \mathcal{C}(x) \text{ and} \ \mathcal{C}(y) \text{ are large}) + O \left(\frac{\log N}{N}\right).
\]

The asymptotic formula (3.43) implies that up to \(o(1)\)-terms,

\[
\mathbb{P}_N^{\beta,\lambda}(\text{both} \ \mathcal{C}(x) \text{ and} \ \mathcal{C}(y) \text{ are large}) = 2p(\beta, \lambda) - 1 + \mathbb{P}_N^{\beta,\lambda}(\text{both} \ \mathcal{C}(x) \text{ and} \ \mathcal{C}(y) \text{ are small}).
\]
Since both \( \{ C(x) \text{ is small} \} \) and \( \{ C(y) \text{ is small} \} \) are decreasing events, by the FKG properties of \( P^\beta_N(\cdot) \),
\[
P^\beta_N(\text{both } C(x) \text{ and } C(y) \text{ are small}) \geq (1 - p(\beta, \lambda))^2 (1 + o(1)).
\]
To get a complimentary upper bound, notice that
\[
P^\beta_N( S^x \cap S^y \neq \emptyset | \text{both } C(x) \text{ and } C(y) \text{ are small}) \leq 2c \log N \mathbb{P}^\beta_N(i \in S^y | \#S^y \leq c \log N) = O \left( \frac{(\log N)^2}{N} \right),
\]
as it follows from (3.37) and \( G_N \)-permutation invariance. Thus,
\[
\mathbb{P}^\beta_N(\text{both } C(x) \text{ and } C(y) \text{ are small}) = \mathbb{P}^\beta_N(\text{both } C(x) \text{ and } C(y) \text{ are small}; S^x \cap S^y = \emptyset) + O \left( \frac{(\log N)^2}{N} \right)
\]
However,
\[
\mathbb{P}^\beta_N(\text{C(x) is small} | S^x \cap S^y = \emptyset; \text{C(y) is small}) \leq \mathbb{P}_{\tilde{\beta}}^\lambda_N(\text{C(x) is small})
\]
where \( \tilde{\beta} = \beta \frac{N}{N - c \log N} \).
As \( N \) tends to \( \infty \), the right hand side above converges to \( 1 - p(\beta, \lambda) \). Thereby, we are able to conclude,
\[
\mathbb{P}^\beta_N(\text{both } C(x) \text{ and } C(y) \text{ are small}) = (1 - p(\beta, \lambda))^2 (1 + o(1))
\]
and (1.8) follows.

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