Rigorous probabilistic analysis of equilibrium crystal shapes

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The rigorous microscopic theory of equilibrium crystal shapes has made enormous progress during the last decade. We review here the main results that have been obtained, both in two and higher dimensions. In particular, we describe how the phenomenological Wulff and Winterbottom constructions can be derived from the microscopic description provided by the equilibrium statistical mechanics of lattice gases. We focus on the main conceptual issues and describe the central ideas of the existing approaches. © 2000 American Institute of Physics.

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A. Phenomenological Wulff construction

1. Equilibrium crystal shapes

The phenomenological theory of equilibrated crystals dates back at least to the beginning of the century.\(^1\) Suppose that two different thermodynamic phases (say crystal and its vapor) coexist at a certain temperature \(T\). Assuming that the whole system is in equilibrium, in particular, that the volume \(v\) of the crystalline phase is well defined, what could be said about the region this phase occupies? Of course, the issue cannot be settled in the language of bulk free energies—these do not depend neither on the shape, nor even on the prescribed volume \(v\) of the crystal. Instead, possible phase regions are quantified by the value of the free energy of the crystal–vapor interface, or by the total surface tension between the crystal and the vapor. (In this review, our point of view is that of mathematical physics; for an exposition of the problem from the viewpoint of theoretical physics, we refer to Ref. 2 and references therein.) Equilibrium shapes correspond, in this way, to the regions of minimal interfacial energy. This is an isoperimetric-type problem: The surface tension \(\tau_\beta\) (where, throughout the article, \(\beta\) denotes the inverse temperature, \(\beta = 1/T\)) is an anisotropic function of the local direction of the interface. Thus, assuming that the crystal occupies a region \(V \subset \mathbb{R}^d\), the corresponding contribution \(W_\beta(V)\) to the free energy is equal to the integral of \(\tau_\beta\) over the boundary \(\partial V\) of \(V\) (Fig. 1).

The Wulff variational problem could then be formulated as follows:

\[
(WP)_v \quad W_\beta(V) \rightarrow \min \quad \text{Given: } \text{vol}(V) = v.
\]

As in the usual isoperimetric case \((WP)_v\) is scale invariant,

\[
\forall s > 0, \quad W_\beta(s V) = s^{d-1} W_\beta(V).
\]

Consequently, any dilatation of an optimal solution is itself optimal, and one really talks here in terms of optimal shapes.

The canonical way to produce an optimal shape is given by the following Wulff construction (Fig. 2): Define

\[
\mathcal{K} = \bigcap_{n \in \mathbb{R}^{d-1}} \{ x \in \mathbb{R}^d : x \cdot n \equiv \tau_\beta(n) \}.
\]

(1.1.1)

\[
W_\beta(V) = \int_{\partial V} \tau_\beta(n) d\mathcal{H}^{d-1}_n
\]

FIG. 1. The free energy of the crystal–vapor interface is given by the integral of the anisotropic surface tension \(\tau_\beta\) over \(\partial V\). \(\mathcal{H}^{d-1}_n\) is the \((d-1)\)-dimensional Hausdorff measure.
It would be convenient to normalize $K$ as
$$K_1 \triangleq \frac{d}{\text{vol}(K)} K.$$ We refer to $K_1$ as to the normalized, or unit volume, Wulff shape. The variational theory of \((\text{WP})_v\), which we briefly address in the subsequent section, states that any solution to \((\text{WP})_v\) can be obtained by a shift of the corresponding dilatation $K_v \triangleq \frac{d}{\text{vol}(K)} K_1$ of $K_1$.

2. Variational methods

The corresponding literature is rather rich and diverse, here we merely attempt to facilitate the orientation of the reader and to introduce some notations that will be useful in the sequel. Since the half-spaces $H(\mathbf{n})$ in (1.1.1) are convex, so is the Wulff shape $K$. Furthermore, in all the problems we consider here, the surface tension $\tau_0$ is bounded above and below,

$$0 < \min_{\mathbf{n} \in S_{d-1}} \tau_0(\mathbf{n}) \leq \max_{\mathbf{n} \in S_{d-1}} \tau_0(\mathbf{n}) < \infty. \quad (1.1.2)$$

Accordingly, equilibrium crystal shapes are bounded and have nonempty interiors, $0 \in \text{int}(K_v)$. The fact that $K$ is optimal follows from the general Brunn–Minkowski theory: Let $\tau_0^\ast(\mathbf{x})$ be the support function of $K$, $\tau_0^\ast(\mathbf{x}) = \sup\{\mathbf{y} \cdot \mathbf{x} | \mathbf{y} \in K\}$. Of course, if the homogeneous extension of $\tau_0$, $\tau_0(\mathbf{x}) \triangleq \|\mathbf{x}\|_2 \tau_0\left(\frac{\mathbf{x}}{\|\mathbf{x}\|_2}\right)$, is convex, then $\tau_0$ and $\tau_0^\ast$ coincide. In general, $\tau_0^\ast(\mathbf{x})$ is the convex lower-semicontinuous regularization of $\tau_0$, in particular, $\tau_0^\ast(\mathbf{x}) = \tau_0(\mathbf{x})$. Nevertheless, for the Wulff shape $K$,

$$\mathcal{W}_0^\ast(K) \triangleq \int_{\partial K} \tau_0^\ast(\mathbf{n}) d\mathcal{H}^{d-1}_x = \int_{\partial K} \tau_0(\mathbf{n}) d\mathcal{H}^{d-1}_x,$$

where, as before, $\mathbf{n}$ is the outward normal to $\partial V$ in $x$ and $\mathcal{H}^{d-1}$ is the $(d-1)$-dimensional Hausdorff measure in $\mathbb{R}^d$.

On the other hand, the action of the regularized functional $\mathcal{W}_0^\ast$ could be extended to any compact set $V \subset \mathbb{R}^d$ in terms of the mixed volume,

$$\mathcal{W}_0^\ast(V) = \liminf_{\epsilon \to 0} \frac{1}{\epsilon} (\text{vol}(V + \epsilon K) - \text{vol}(V));$$
the latter definition coincides with the integral definition of $\mathcal{W}_1^{**}$ for regular $V$. The Brunn–Minkowski inequality, 3

$$\text{vol}(A + B) \geq (\text{vol}(A))^{1/d} + (\text{vol}(B))^{1/d},$$

implies that for any regular $V$ with $\text{vol}(V) = \text{vol}(K)$,

$$\mathcal{W}_1^*(V) \geq \mathcal{W}_1^{**}(V) \geq d \cdot \text{vol}(K) = \mathcal{W}_1(K).$$

Of course, we have been rather sloppy above, and we refer the reader to the work of Refs. 4–6 for a comprehensive discussion and results, including the history of the variational Wulff problem. The language employed in the latter works is that of the geometric measure theory, and we proceed with setting up some of the corresponding notation, which will also turn out to be useful for the 1d approach to the microscopic justification of the Wulff construction, as described in Sec. II of this review. In the latter case, the macroscopic state of the system will be determined by the value of an order parameter that specifies the phase of the system. In the systems that we will consider, the pure phases are characterized by their averaged density, which are encoded by two values, $\rho_1(\beta)$ and $\rho_0(\beta)$, for example, $\rho_0$ for the crystal and $\rho_1$ for the vapor. [In fact, we shall derive all the results in the symmetrized spin language, in which case the two values will be $\pm m^*(\beta)$, where $m^*(\beta)$ is the spontaneous magnetization (see Sec. II) at the inverse subcritical temperature $\beta > 0$.] For a given temperature, it is convenient to replace this order parameter by a parameter with values $\pm 1$. We suppose that the macroscopic region of $\mathbb{R}^d$ where the system is confined is the unit torus $\mathbb{T}^d = (\mathbb{R}/\mathbb{Z})^d$. The macroscopic system is described by a function $v$ taking values $\pm 1$ and the fact that $v_r = 1$ for some $r$ in $\mathbb{T}^d$ means that locally at $r$ the system is in equilibrium in the phase $m^* = m^*(\beta)$.

For any measurable set $V$ in $\mathbb{T}^d$, the perimeter of $V$ is defined by

$$P(V) = \sup \left\{ \int_V \text{div} \phi(x) dx \bigg| \phi \in C^1(\mathbb{T}^d, \mathbb{R}^d), \ |\phi| \leq 1 \right\}. \quad (1.1.4)$$

A function $v$ with values $\pm 1$ is said to be of bounded variation in $\mathbb{T}^d$ if the perimeter of the set $\{v = 1\}$ is finite. We denote by $\text{BV}(\mathbb{T}^d, \{± 1\})$ the set of functions of bounded variation on $\mathbb{T}^d$ with values $\pm 1$ (see Ref. 7 for a review). For any $v$ in $\text{BV}(\mathbb{T}^d, \{± 1\})$, there exists a generalized notion of the boundary of $\{v = 1\}$ called reduced boundary and denoted by $\partial^* v$. If $\{v = 1\}$ is a regular set, $\partial^* v$ coincides with the usual boundary $\partial v$. Furthermore, a blow-up Theorem (see Ref. 7, p. 199) ensures that for all $x$ in $\partial^* v$ an approximate tangent plane can be defined locally. This will imply the existence of a unit vector $n_x$ called the measure theoretic unit normal to $\{v = 1\}$ at $x$. For any $x$ in $\mathbb{R}^d$ and any vector $n$, we define the half-spaces,

$$H^+(x, n) = \{ y \in \mathbb{R}^d \mid (y - x) \cdot n \geq 0 \},$$

$$H^-(x, n) = \{ y \in \mathbb{R}^d \mid (y - x) \cdot n \leq 0 \}.$$

Then for all $x$ in $\partial^* v$, there is a unit vector $n_x$ such that

$$\lim_{r \to 0} \frac{1}{r^d} \text{vol}(B(x, r) \cap \{v = 1\} \cap H^+(x, n)) = 0,$$

$$\lim_{r \to 0} \frac{1}{r^d} \text{vol}(B(x, r) \cap \{v = -1\} \cap H^-(x, n)) = 0,$$

$$\lim_{r \to 0} \frac{1}{r^d} \text{vol}(B(x, r) \cap \{v = 1\} \cap H^+(x, n)) = 0,$$

$$\lim_{r \to 0} \frac{1}{r^d} \text{vol}(B(x, r) \cap \{v = -1\} \cap H^-(x, n)) = 0,$$
where \( B(x,r) \) is the ball of radius \( r \) centered in \( x \). The previous property shows that the reduced boundary is not too wild (see Fig. 3). In fact, it is possible to prove that a set of finite perimeter has “measure theoretically a \( C^1 \) boundary.”

The functional \( W_\beta \) can be extended on \( L_1(\mathbb{T}^d) \) as follows:

\[
W_\beta(v) = \begin{cases} 
\int_{\partial^* v} \tau_\beta(n_x)d\mathcal{H}^{d-1}_x, & \text{if } v \in BV(\mathbb{T}^d, \{1\}), \\
\infty, & \text{otherwise.}
\end{cases}
\] (1.1.5)

Under the assumption that the homogeneous extension (1.1.3) of \( \tau_\beta \) is convex, a result by Ambrosio and Braides (see Ref. 8, Theorem 2.1) ensures that \( W_\beta \) is lower semicontinuous with respect to \( L_1 \) convergence. In certain cases (attractive interactions) the convexity of \( \tau_\beta \) can be derived from the properties of the corresponding microscopic system, as will be explained later.

To any measurable subset \( A \) of \( \mathbb{T}^d \), we associate the function \( 1_A \) and simply write \( W_\beta(A) = W_\beta(1_A) \). In this new setting, the isoperimetric problem is to find the minimizers of

\[
\min \left\{ W_\beta(v) \mid v \in BV(\mathbb{T}^d, \{1\}), \int_{\mathbb{T}^d} m^* v_x \, dr \leq m \right\},
\] (1.1.6)

where \( m \) belongs to \( [\bar{m}(\beta), m^*(\beta)] \). The parameter \( \bar{m} \) is chosen such that the minima of the variational problem above are translates of the set \( K_m \) deduced from the Wulff shape \( K \) by dilatation in order to satisfy the volume constraint. This restriction enables us to exclude pathological minimizers that occur from the periodicity. Nevertheless, notice that the precise shape or the uniqueness of the minimizers of the variational problem will be irrelevant for the microscopic derivation of the Wulff construction.

### 3. Stability properties

In two dimensions Wulff solutions to (WP)\(_{\epsilon} \) are stable in the metric of Hausdorff distance: let \( V \) be a connected and simply connected subset of \( \mathbb{R}^2 \) with a rectifiable boundary \( \partial V \). Assume that \( \text{Area}(V) \geq 1 \). Then,

\[
\min_x d_{11}(V, x + K_1) \leq c_{\epsilon} \sqrt{W_\beta(V) - W_\beta(K_1)}.
\] (1.1.7)

This result has been established in Ref. 9 as a generalization of the classical Bonnesen inequality.

If \( V \) consists of several connected and simply connected components, \( V = \bigcup_{i=1}^n V_i \), and the total surface tension of \( V \) is close to the optimal,

\[
W_\beta(V) = \sum_{i=1}^n W_\beta(V_i) \approx W_\beta(K_1) + \epsilon,
\]
then, again assuming that Area($V$) = $\sum_{i=1}^n$ Area($V_i$) = 1, an easy consequence of (1.1.7) implies [see (2.9.7) and (2.9.8) in Ref. 9] that actually all but one components of $V$ are small, and that the only large component, say $V_1$, is close to a shift of $K_i$. Namely

$$\sum_{i=2}^n \text{Area}(V_i) \leq c_3 \epsilon^2 \quad \text{and} \quad \sum_{i=2}^n \mathcal{W}_\beta(V_i) \leq c_3 \epsilon,$$

and $V_1$ satisfies (1.1.7).

These stability properties are indispensable for a sharp justification of the phenomenological Wulff construction directly from the microscopic assumptions on the local interparticle interactions (see Sec. III E).

As far as we understand, stability properties of higher-dimensional isoperimetric problems are much less studied. Already in three dimensions the Hausdorff distance is, of course, not an adequate measure of stability. Trivial rate-free stability properties in $L_1$ simply follow from the uniqueness of Wulff solutions and the compactness of BV balls in $L_1$. On a more qualitative side, there are well-studied stability properties in the class of convex sets and, also, for sets with a smooth boundary.\textsuperscript{10} We feel, however, that the statistical stability under the microscopic approximations in the problems we consider here might be better than the impartial stability of the corresponding variational problems. A result of this sort is supposed to appear in Ref. 11.

4. Winterbottom problem

The Wulff variational problem provides a description of an equilibrium crystal shape deep inside a region filled with gas phase. If, however, the spatial extent of the system is finite, it may occur that the boundary of the surrounding vessel exhibits a preference toward the crystal phase. In such a situation, the equilibrium state may not be given by the Wulff shape anymore, but may have the crystal attached to the boundary. We discuss briefly the simplest model of such an interaction between an equilibrium crystal and an attractive substrate. Suppose, for simplicity, that our system is contained in the half-space $H = \{ x \in \mathbb{R}^d : x(d) \geq 0 \}$; the boundary of this half-space, the hyperplane $W = \{ x \in \mathbb{R}^d : x(d) = 0 \}$ represents the boundary of the vessel and is called the wall.

We also suppose to simplify the analysis, and because these assumptions will always be satisfied, that $\tau_W(n) = \tau_{\beta}(n)$, and that the homogeneous extension of $\tau_{\beta}$ is convex. (In the models we consider in this paper, this is a consequence of FKG inequality.)

To model the degree of attractiveness of the wall, we introduce a new thermodynamical quantity, the wall free energy $\tau_{\text{bd}}(\beta, \eta)$, which depends on both the inverse temperature $\beta$ and the "chemical structure" of the wall $\eta$, and modify the free energy functional accordingly,

$$W_{\beta, \eta}(V) = W_{\beta}(V) + (\tau_{\text{bd}}(\beta, \eta) - \tau_{\beta}^w) \mathcal{H}^{d-1}(\partial V \cap W),$$

where $\tau_{\beta}^w = \tau_{\beta}(e_k)$, $e_k \in \mathbb{R}^d$ with $e_k(k) = \delta_{kd}$. The wall free energy replaces therefore the surface tension $\tau_{\beta}$ along the wall. At equilibrium, a thermodynamical stability argument shows that $\tau_{\text{bd}}(\beta, \eta) \leq \tau_{\beta}^w$ (this can also be proved in some microscopic models; see Sec. IV), so that this last term is always nonpositive. The new variational problem is

$$(\text{WBP})_{\beta, \eta} \quad W_{\beta, \eta}(V) \rightarrow \min \quad \text{given:} \ V \subset H, \ \text{vol}(V) = \nu.$$

It has first been studied in Ref. 12 and is called the Winterbottom variational problem. Let us now discuss what its solution looks like. It turns out that there are three cases to consider.

1. $\tau_{\text{bd}}(\beta, \eta) = \tau_{\beta}^w$. In this case, $W_{\beta, \eta}(V) = W_{\beta}(V)$ and therefore the solution is the Wulff shape associated to $\tau_{\beta}$. The equilibrium crystal is not attached to the wall. This can happen even if $a \text{ priori}$ the chemical structure of the wall is such that it is energetically favorable for the crystal to lay on the wall; see Sec. IV for a discussion from a microscopic point of view.

2. $|\tau_{\text{bd}}(\beta, \eta)| < \tau_{\beta}^w$. Now the wall is really attractive for the crystal shape. The solution of the variational problem is given by a suitably rescaled version of the following set (see Fig. 4):
\[ K^n = K \cap \{ x \in \mathbb{R}^d : x(d) \approx -\tau_{\text{bd}}(\beta, \eta) \} , \]

so that the volume constraint is satisfied (notice that this variational problem is still scale invariant); see Ref. 13 for a simple proof.

(3) \[ \tau_{\text{bd}}(\beta, \eta) = -\tau_0^\phi . \] This is a somewhat pathological case. Indeed, the solution of the variational problem is completely degenerate, the solution being unbounded. A minimizing sequence is, for example,

\[ R_n = \{ x \in H : |x(k)| \leq n, k = 1, \ldots, d - 1, 0 \leq x(d) \leq n^{1-d} \phi \} . \]

As \( n \to \infty \), \( R_n \) covers the whole wall with a film of vanishingly small width; the limiting value of the surface free energy functional is 0. This describes the regime of so-called complete wetting, where the wall so strongly prefers the crystal that it wants to prevent any contact with the gas phase.

5. Microscopic justification

Microscopic models we consider here are simple lattice gas-type models (in the magnetic interpretation), which are going to be defined precisely in the next section. The prototype situation when the Wulff construction is thought to be recovered as a law of large numbers as the size of the microscopic system tends to infinity could be loosely described as follows: Suppose that the particles of a certain substance live on the vertices of the integer lattice \( \mathbb{Z}^d \), so that each vertex of \( \mathbb{Z}^d \) could be either occupied by a particle or remain vacant. Thus, various particle configurations \( n \) could be labeled by points of \( \{0,1\}^{\mathbb{Z}^d} \) where one puts \( n_i = 1 \) if there is a particle at site \( i \in \mathbb{Z}^d \), and \( n_i = 0 \), otherwise. These random configurations are sampled from a Gibbs distribution \( \Gamma \), which takes into account the assumptions on the microscopic interactions between the particles. The strength of the interaction is quantified by the value \( \beta = 1/T \) of the inverse temperature; the larger the \( \beta \) (respectively, the smaller the temperature \( T \)), the stronger the interaction. In many instances sufficiently low temperatures give rise to two stable phases—the low-density phase (which we call vapor) with an average particle density per site \( \rho_l \) and the high-density phase (crystal) with a corresponding average density \( \rho_h \), \( 0 < \rho_l < \rho_h < 1 \).

Suppose now that all the particles are confined to a large finite volume vessel \( \Lambda_N \subset \mathbb{Z}^d \), where the subindex \( N \) indicates the linear size of \( \Lambda_N \); we put, for simplicity, \( |\Lambda_N| = N^d \). Let us fix \( \rho \in (\rho_l, \rho_h) \) and ask what are the typical geometric properties of particle configurations \( n \) under the conditional measure \( \Gamma(\cdot | \sum_{i \in \Lambda_N} n_i = \rho N^d) \). In other words, we fix the total number of particles \( \rho N^d \) in such a way that it falls in between the two stable values \( \rho_l N^d \) and \( \rho_h N^d \).

The prototype law of large numbers result we have in mind is schematically:
Thus, with an overwhelming $P(\cdot | \Sigma_{i \in \Lambda} n_i = \rho N^d)$-probability particle configurations $n$ on $\Lambda_N$, $n \in \{0,1\}^{\Lambda_N}$, obey the following phase segregation pattern: $\Lambda_N$ splits into two regions, $\Lambda^h_N = \Lambda^h_N \cup \Lambda^l_N$, where $\Lambda^h_N$ is occupied by the high-density phase, and, respectively, $\Lambda^l_N$ by the low-density one. The relative volume of $\Lambda^h_N$ can be recovered from the canonical constraint,

$$
\rho_h | \Lambda^h_N | + \rho_l | \Lambda^l_N | = \rho N^d,
$$

and the shape of $\Lambda^h_N$ is asymptotically Wulff.

There is a long way even toward making the above statement precise—we should define the microscopic models, quantify the notion of phases, in particular, of phases over finite volumes, and explain how the surface tension is produced in the large $N$ limit.

B. Microscopic models

1. Models with finite-range ferromagnetic two-body interactions

We want to introduce mathematically precise realizations of the models discussed in Sec. I A 5. As described there, our interest lies in models of lattice gases. For simplicity we restrict our attention to a particular subclass of such models, which enjoy several nice properties, the Ising models with finite-range ferromagnetic two-body interactions.

It is rather convenient to work with another, equivalent, formulation of these models, in which the symmetries present are more transparent; this is the magnetic interpretation. To do this, we introduce a new family of random variables $\sigma_i$, $i \in \mathbb{Z}^d$, defined by

$$
\sigma_i = 2n_i - 1.
$$

The random variables $\sigma$, therefore take values in $\{-1,1\}$; $\sigma_i$ is called the spin at the site $i$. Expressed in these variables, the model is defined through the following Gibbs measure in $\Lambda$ with boundary conditions $\vec{\sigma} \in \{-1,1\}^\mathbb{Z}^d$:

$$
\mu^{\beta}_{\Lambda,\vec{\sigma},h}(\sigma) = \begin{cases} 
\frac{1}{Z^\beta_{\Lambda,\vec{\sigma},h}} \exp \left( \beta \sum_{i \in \Lambda} h_i \sigma_i + \beta \sum_{\{i,j\} \cap \Lambda \neq \varnothing} J_{ij} \sigma_i \sigma_j \right), & \text{if } \sigma_i = \vec{\sigma}_i, \text{ for all } i \not\in \Lambda, \\
0, & \text{otherwise,}
\end{cases}
$$

where $h_i \in \mathbb{R}$ are called the magnetic fields and the coupling constants $J_{ij} = J_{||i-j||}$ satisfy $J_{ij} \geq 0$ and $J_{ij} = 0$ if $||i-j|| > r$. A configuration $\sigma$ such that $\sigma_i = \vec{\sigma}_i$, for all $i \not\in \Lambda$, is said to be compatible with b.c. $\vec{\sigma}$ in $\Lambda$: the set of all such configurations is denoted by $\Omega_{\Lambda,\vec{\sigma}}$. We are particularly interested in the $+$ and $-$ b.c. corresponding, respectively, to $\vec{\sigma} = 1$ and $\vec{\sigma} = -1$. The Gibbs measure in $\Lambda$ with free b.c. is the probability measure on $(\{-1,1\}^\Lambda, \mathcal{F}_\Lambda)$, defined by

$$
\mu^{\beta}_{\Lambda,h}(\sigma) = \frac{1}{Z^\beta_{\Lambda,h}} \exp \left( \beta \sum_{i \in \Lambda} h_i \sigma_i + \beta \sum_{\{i,j\} \subset \Lambda} J_{ij} \sigma_i \sigma_j \right).
$$

Expected value w.r.t. these measures are denoted with brackets notations, $\langle \cdot \rangle^{\beta}_{\Lambda,\vec{\sigma},h}, \ldots$. 
In the magnetic formulation, the canonical ensemble corresponds to fixing the value of the magnetization (density) \( m(\sigma) = 1/|\Lambda| \left( \sum_{i \in \Lambda} \sigma_i \right) \),

\[ \mu^\beta_{\Lambda, \sigma, h}(\cdot | m(\sigma) = \bar{m}) , \]

where \( \bar{m} \in \text{Range}(m) \). If \( h_i = h \) for all \( i \), then the (infinite-volume) Gibbs states \( \mu^\beta_{\sigma, h} \) for \( +, - \) and free b.c. can be shown to exist; it is always unique when \( h \neq 0 \). The phase transition statement takes now the following form: There exists \( \beta_c > 0 \) such that

(i) for all \( \beta < \beta_c \), the Gibbs state is unique and \( \langle m \rangle^\beta_{\sigma, 0} = 0 \),

(ii) for all \( \beta > \beta_c \), \( m^*(\beta) = \langle m \rangle^\beta_{\sigma, 0} > 0 \langle m \rangle^0_{\sigma, 0} = -m^*(\beta) \).

We will use the terminology Ising models to refer to the lattice gases in the magnetic formulation. When \( h = 0 \), we will generally omit it from the notations.

Ferromagnetic models are particularly well suited for nonperturbative analyses. Indeed, they enjoy several very useful qualitative properties, most of which taking form of correlation inequalities. Of particular importance for us are the following statements (\( \sigma_A = \prod_{i \in A} \sigma_i \)):

\[ \langle \sigma_A \rangle^\beta_{\Lambda, h} \geq 0 , \]

\[ \langle \sigma_A \sigma_B \rangle^\beta_{\Lambda, h} \geq \langle \sigma_A \rangle^\beta_{\Lambda, h} \langle \sigma_B \rangle^\beta_{\Lambda, h} , \]

provided \( h_i \geq 0 \) for all \( i \) (first and second Griffiths’, or GKS, inequalities\(^{14,15} \)); also,

\[ \frac{\partial^2}{\partial h_i \partial h_j} \langle \sigma_k \rangle^\beta_{\Lambda, h} \leq 0 , \]

for all \( i, j, k \), provided \( h_i \geq 0 \) for all \( i \) (GHS inequalities\(^{16} \)); finally,

\[ \langle f \rangle^\beta_{\Lambda, h} \geq \langle f \rangle^\beta_{\Lambda, h} \langle g \rangle^\beta_{\Lambda, h} , \]

for any increasing functions \( f \) and \( g \), and any \( h \in \mathbb{R}^\Lambda \) (FKG inequality\(^{17} \)). [A function \( f : \{-1,1\}^\mathbb{Z} \rightarrow \mathbb{R} \) is increasing if \( f(\sigma) \geq f(\sigma') \) as soon as \( \sigma_i \geq \sigma_i' \), for all \( i \); it is called decreasing if \( -f \) is increasing.] Observe that any b.c. can be obtained, starting with free b.c. and applying suitable magnetic fields on the spins on the inner boundary of \( \Lambda \), where the inner boundary of a set \( \Lambda \subseteq \mathbb{Z}^d \) is defined as

\[ \partial_\mu \Lambda = \{ i \in \Lambda : \exists j \notin \Lambda, i \sim j \} , \]

where \( i \sim j \) means that \( J_{ij} \neq 0 \). Similarly, we define the (exterior) boundary of \( \Lambda \) by

\[ \partial \Lambda = \{ i \in \Lambda : \exists j \in \Lambda, i \sim j \} . \]

2. **Two-dimensional nearest neighbors ferromagnetic Ising model**

A particularly simple member of the above-mentioned class of models is the two-dimensional nearest neighbors Ising model, in which \( J_{ij} = 0 \) if \( i \) and \( j \) are not nearest neighbors, and \( J_{ij} = 1 \) if they are. This model still has additional remarkable features. First, even though this only plays a very marginal role in this review, it is the only one for which it is possible to compute explicitly various quantities (free energy, surface tension, correlations,...). Of more importance for our purposes is the property of self-duality that it enjoys. (The fact that this model is self-dual is very convenient, but is not required anywhere. What we need is to be able to control precisely the dual of the model; for example, the Ising model on the hexagonal lattice is not self-dual, but it would be possible to prove the same kind of statements for this model as for the one on the square lattice.)
The maximal connected components of these dual edges, seen as closed line segments in \( \mathbb{R}^2 \), are called contours. A contour is said to be open, otherwise it is closed if \( \partial \gamma = \emptyset \). A set \( \Lambda \subseteq \mathbb{Z}^2 \) is simply connected if \( \bigcup_{i \in \Lambda} \{ x \in \mathbb{R}^2 : \| x - i \|_\infty \leq \frac{1}{2} \} \) is a simply connected subset of \( \mathbb{R}^2 \).

Given \( \Lambda \subseteq \mathbb{Z}^2 \), its dual is \( \Lambda^* = \{ i \in \mathbb{Z}_2^2 : \exists j \in \Lambda, \| j - i \|_\infty = \frac{1}{2} \} \). A family of contours is said to be \( \Lambda^* \) compatible if they are disjoint (as sets of bonds and sites) and are included in \( \Lambda^* \). A family of contours \( \gamma \) is said to be \( (\Lambda, \sigma) \) compatible if there exists a configuration \( \sigma \in \Omega_{\Lambda, \sigma} \) such that \( \gamma(\sigma) = \gamma \). It is easy to show that for simply connected \( \Lambda \), \( \Lambda^* \) compatibility of a family of closed contours is equivalent to \( (\Lambda, +) \) compatibility.

The measure \( \mu_{\Lambda, \sigma}^\beta \) can be easily written in terms of these objects; for any \( \sigma \in \Omega_{\Lambda, \sigma} \),

\[
\mu_{\Lambda, \sigma}^\beta(\sigma) = \frac{1}{Z_{\Lambda}^\beta(\Lambda)} \exp \left( -2 \beta \sum_{\gamma \subseteq \gamma(\sigma)} |\gamma| \right),
\]

where \( |\gamma| \) is the number of edges in \( \gamma \) and

\[
Z_{\Lambda}^\beta(\Lambda) = \sum_{\gamma \subseteq \gamma(\Lambda)} \exp \left( -2 \beta \sum_{\gamma \subseteq \gamma} |\gamma| \right) = \sum_{\gamma \subseteq \gamma(\Lambda)} \prod_{\gamma \subseteq \gamma} w(\gamma; \beta).
\]

We now discuss the property of self-duality. Let \( \Lambda \subseteq \mathbb{Z}^2 \) be simply connected. We consider the model at inverse temperature \( \beta^* \) in the box \( \Lambda^* \subseteq \mathbb{Z}_2^*, \) with free boundary conditions. There exists another graphical representation for this model, the high-temperature representation, which results from writing

\[
e^{\beta^* \sigma_j} = \cosh(\beta^*(1 + \sigma_j \sigma_j \tanh(\beta^*)),
\]

opening all the brackets and expanding. After a simple summation over \( \sigma \), this yields

\[
Z_{\Lambda}^\beta(\Lambda) = C(\Lambda) \sum_{\gamma \subseteq \gamma(\Lambda)} \exp \left( -2 \beta \sum_{\gamma \subseteq \gamma} |\gamma| \right) = C(\Lambda) \sum_{\gamma \subseteq \gamma(\Lambda)} \prod_{\gamma \subseteq \gamma} w^*(\gamma; \beta^*) = C(\Lambda) Z_{\Lambda}^{\beta^*}(\Lambda^*),
\]

where \( C(\Lambda) \) is some constant that only depends on the set \( \Lambda \). Setting \( \tanh(\beta^*) = e^{-2\beta} \), we see from (1.2.2) and (1.2.3) that \( Z_{\Lambda}^\beta(\Lambda) = Z_{\Lambda}^{\beta^*}(\Lambda^*) \), since \( \Lambda \) is simply connected. In the same way, we can expand the two-point function, for example, and get the following very useful identity:

\[
\langle \sigma_i \sigma_j \rangle_{\Lambda^*}^{\beta^*} = \sum_{\lambda^* \in \Lambda^*} q_{\Lambda^*}^{\beta^*}(\lambda^*),
\]
where the sum is over all open contours $\lambda$ such that $\partial \lambda = \{i, j\}$, and

$$q^{\beta^*}_{\Lambda^*}(\lambda) = w^*(\lambda; \beta^*) \frac{Z^{\beta^*}(\Lambda^*|\lambda)}{Z^{\beta^*}(\Lambda^*)},$$

$$Z^{\beta^*}(\Lambda^*|\lambda) = \sum_{\gamma \text{ closed}}_{(\gamma, \lambda) \text{ } \Lambda^* \text{ comp.}} \prod_{\gamma \in \gamma} w^*(\gamma; \beta^*).$$

Identity (1.2.4) is the so-called random-line representation for the two-point function of the Ising model, and plays a basic role in the approach to the DKS theory of Sec. III (see Refs. 18, 19 for much more details on this topic). What is particularly useful is that the weights $q^{\beta^*}_{\Lambda^*}$, which we have defined for an open contour, can be immediately extended to any family of $\Lambda^*$-compatible contours (closed or open). In particular, if $\gamma$ is a family of $\Lambda^*$-compatible closed contours, then the following identity holds:

$$q^{\beta^*}_{\Lambda^*}(\gamma) = \mu^{\beta^*}_{\Lambda^*}(\gamma \subseteq \gamma(\cdot)).$$

Applications and further results about the random-line representation are given in Sec. III D and in Sec. IV. The results stated above also hold when the coupling constants are allowed to vary from edge to edge, provided they remain ferromagnetic; if we denote by $J(e)$ the coupling constant at edge $e$, then the duality relation takes the form

$$\tanh(\beta^* J^*(e^*)) = e^{-2\beta^* |e^*|}.$$  \hfill(1.2.5)

3. Kac models

In the original van der Waals theory, the occurrence of phase transitions is due to long range attractive forces between molecules. In its statistical mechanics formulation, these forces are described by Kac potentials that depend on a positive scaling parameter $\epsilon$ that controls the strength and the range of the potential (see Ref. 20). The first probabilistic approach to this model was made in the celebrated paper of Lebowitz and Penrose.  \hfill(21)

In dimension $d$, Ising systems with Kac potentials are defined by Gibbs measures with potentials depending on a scaling parameter $\epsilon > 0$,

$$\forall i, j \in \mathbb{Z}^d, \quad J^*_e(i, j) = \epsilon^d J(\epsilon|\|i - j\|_2),$$

and $J$ is a non-negative, smooth function supported by $[0, 1]$ and normalized so that

$$\int_{\mathbb{R}^d} dr J(\|r\|_2) = 1.$$
**Theorem 1.2.1:** For any $\beta > 1$, there is $\epsilon_0 > 0$ such that for any $\epsilon$ smaller than $\epsilon_0$ a phase transition occurs and there are at least two distinct pure phases, $\mu_x^+$ and $\mu_x^-$. If $\beta > 1$, there is a breaking of symmetry and the spontaneous magnetization is denoted by $\mu_x^+ (\sigma_0) = m_x^+$. Define $m_*^+ = \lim_{\epsilon \to 0} m_x^+$. This theorem was proven via a renormalization procedure that we shall describe in Sec. II C 1.

4. Surface tension

We fix $n$ a vector in $S^{d-1}$ and consider an orthonormal basis $(e_1, \ldots, e_{d-1}, n)$. Let $\Lambda(N,M)$ be the parallelepiped of $R^d$ centered at 0 with side length $N$ for the sides parallel to $(e_1, \ldots, e_{d-1})$ and side length $M$ for the sides parallel to $n$. The microscopic counterpart of $\Lambda(N,M)$ is denoted by $\Lambda(N,M)$. The boundary $\partial \Lambda(N,M)$ is split into two sets:

$$\partial_n^+ \Lambda(N,M) = \{ i \in \partial \Lambda(N,M) \mid i \cdot n \geq 0 \},$$

$$\partial_n^- \Lambda(N,M) = \{ i \in \partial \Lambda(N,M) \mid i \cdot n < 0 \}.$$

We fix the boundary conditions outside $\Lambda(N,M)$ to be equal to 1 on $\partial_n^+ \Lambda(N,M)$ and to $-1$ on $\partial_n^- \Lambda(N,M)$. The corresponding partition function on $\Lambda(N,M)$ is denoted by $Z_{\Lambda(N,M)}^{\beta}$. Notice that any configuration $\sigma$ contributing to the partition function $Z_{\Lambda(N,M), n, \pm}^{\beta}$ contains a $\pm$-contour $\gamma$ that crosses $\Lambda(N,M)$ under the “averaged” direction orthogonal to $n$ (Fig. 5).

**Definition:** The surface tension in the direction $n \in S^{d-1}$ is defined by

$$\tau_{\beta}(n) = \lim_{N \to \infty} \lim_{M \to \infty} \frac{1}{N^{d-1} \log Z_{\Lambda(N,M), n, \pm}} \log Z_{\Lambda(N,M), n, +}^{\beta}.$$  

**Remark:** Notice that surface tension is sometimes defined with an extra multiplicative factor $1/\beta$.

The proof of the existence of the surface tension can be found in many papers (Refs. 25 and 26, to mention a few). A general approach has been developed by Messager, Miracle-Sole, and Ruiz. The core of their proof is the subadditivity of the sequence of finite-volume approximation to $\tau_{\beta}(n)$ that is obtained by means of FKG inequality. The proof is also valid for a wide range of models like Ising models with finite range interactions, Potts and SOS models. Furthermore, they
showed that surface tension can be defined with parallelepipeds \( \Lambda(N, M_N) \), where \( M_N \) is a function of \( N \) that diverges as \( N \) goes to infinity. More general domains can also be considered provided they contain a parallelepiped of the type \( \Lambda(N, M_N) \).

The convexity of the homogeneous extension of \( \tau_\beta \) [see (1.1.3)] is a consequence of the pyramidal inequality proven in Theorem 3 of Ref. 27: Let \( A_0, \ldots, A_d \) be \( d+1 \) points of \( \mathbb{R}^d \) and denote by \( (\Delta_i)_{i=0}^d \) the simplex defined by these points. Let \( \mathbf{n}_i \) be the unit normal to \( \Delta_i \) and \( |\Delta_i| \) its area. Then, the pyramidal inequality says

\[
|\Delta_0| \tau_\beta(\mathbf{n}_0) \leq \sum_{i=1}^d |\Delta_i| \tau_\beta(\mathbf{n}_i).
\]

Note also that the homogeneous extension of \( \tau_\beta \) is continuous because it is locally bounded and convex. Furthermore, \( \tau_\beta \) is uniformly positive on \( \mathbb{S}^{d-1} \). This follows from the fact that the surface tension \( \tau_\beta(\mathbf{n}_0) \) in the direction \( \mathbf{n}_0=(1,0,\ldots,0) \) is strictly positive as \( \beta \) is larger than \( \beta_c \) (see Lebowitz and Pfister\textsuperscript{28}).

### C. Scope of the theory

The key notion behind the attempts to give a rigorous meaning to the type of the phase segregation phenomena, which have been vaguely discussed in Sec. I A 5, is that of renormalization or coarse graining. The energy (probability) competes with the entropy (number) of microscopic configuration in the corresponding energy shells. Macroscopic quantities like surface tension are produced in the aftermath of the entropy/energy cancellation, which is to say that in order to derive large-\( N \) \((N\text{-linear size of the system})\) asymptotics one should renormalize appropriate microscopic objects. The appropriate objects here are, of course, microscopic phase boundaries, which decouple between different “large” microscopic phase regions. These renormalization procedures could follow two different trends, depending on whether the renormalized (mesoscopic) structures keep track of the microscopic or macroscopic state of the system.

#### 1. Dobrushin–Kotecký–Shlosman theory

The coarse graining of the DKS theory closely follows microscopic phase segregation patterns. Basic tools comprise a fluctuation analysis of the microscopic phase boundaries and sharp uniform local limit estimates over domains encircled by such boundaries. Thus, the notion of finite volume phases is quantified by the rate of the relaxation of the statistics of microscopic observables inside the microscopic phase regions toward the corresponding equilibrium values.

The theory has been developed using the low-temperature cluster expansions in the seminal monograph.\textsuperscript{9} Our exposition in Sec. III is nonperturbative and follows the works of Refs. 26, 29, 30, 18, 31 and 32. By and large the existing results are confined to the simplest two-dimensional models (percolation and nearest neighbor Ising).

#### 2. \( L_1 \)-theory

The renormalization approach of the \( L_1 \)-theory is, in a sense, opposite to that of DKS. In the latter case the principal coarse grained objects (skeletons; see Sec. III) are built upon underlying families of large microscopic contours. Such information is waved out in the \( L_1 \)-approach, and the basic renormalization objects here are the local (mesoscopic) order parameters or, in the spin language, locally averaged magnetization on various length scales. The idea is that on sufficiently large scales local averages of the magnetization are, with an overwhelming probability, close to one of the two equilibrium values \( \pm m^* \). Thus, under the renormalization, configurations are characterized by their phase labels on different mesoscopic blocks. The objective of the \( L_1 \)-theory is to describe typical mesoscopic magnetization profiles (or their phase labels) under a relaxed canonical constraint of shell type. Unlike in the DKS case, the mesoscopic phase labels are classified by their proximity to various macroscopic states. Combinatorial complexity of this approximation is reduced by an exponential tightness property of the mesoscopic phase labels (for a general claim of this sort see Theorem 2.2.1), which enables us to restrict attention only to...
L^1-compact subsets of feasible macroscopic states, namely to the phase sets of finite perimeter. The core of the compactness estimates is based on the renormalization decoupling techniques introduced in Ref. 33 and on the methods developed to control the phase of small contours by Refs. 30, 18, 31 and 32. These techniques are robust enough to be applied on a renormalized scale in any dimensions in a nonperturbative setting.

Our exposition in this review is based on the work of Ref. 34 with, though, one exception—we specifically stress that all the relevant estimates of the L^1-theory are obtained on appropriate finite scales. The validity of Lemma 2.4.1 up to the slab percolation threshold follows from the results of Ref. 35.

3. Boundary phenomena

In Secs. II and III we provide a derivation of Wulff construction from the basic principles of equilibrium statistical mechanics. In Sec. IV we are concerned with a study of the effect of the boundary conditions on the macroscopic geometry of the phase separation. In particular, it is shown how the interaction with the boundary of the vessel can be analyzed, and used to provide a derivation of the Winterbottom construction. The relationship between the macroscopic geometry in this case and the wetting transition is also discussed. The presentation follows Ref. 18 for the two-dimensional (2D) case, and Ref. 11 for the higher-dimensional ones.

4. Bibliographical review

The rigorous investigation of the macroscopic geometry of phase separation under a canonical constraint certainly started with two seminal papers of Minlos and Sinai in 1967–1968. In these papers, the authors considered nearest neighbor very low temperature Ising models in arbitrary dimensions d ≥ 2, even though they only wrote down the proof explicitly in the case d = 2. Their results could be roughly stated in the following way: At sufficiently low temperatures, typical configurations of the Ising model in the exact canonical ensemble over finite vessels of linear size N consist of a single large contour whose shape is ‘‘nearly a square,’’ whereas the rest of the contours are small, that is at most of the order log N. This is the picture of low-temperature excitations of canonical ground states, and it has been treated by the authors as such. In particular, the entropic factor has been frequently suppressed by the microscopic energy cost. However, exact asymptotic results on the level of a microscopic justification of the Wulff construction depend, even at very low but still nonzero temperatures, on a nontrivial entropy/energy competition, and, hence, could not be derived in this way.

Then there followed 15–20 years of a relative stagnation, the only contributions to the area being confined to generalizations of Refs. 36 and 37 to more complicated models. A popular interest to the problem has been revived toward the mid-1980s in the framework of an ongoing mingle between probability and statistical mechanics.

A breakthrough occurred around 1989, when Dobrushin, Kotecký, and Shlosman found a way to derive the Wulff shape in a scaling limit of the low-temperature 2-D Ising model. They found much more: Essentially the monograph sets up a comprehensive mathematical theory of phase segregation. This theory happened to be an intrinsically probabilistic one. The DKS approach is, above all, to quantify the phenomenon of phase separation in terms of probabilistic limit theorems and, accordingly, to study the probabilistic structures related to the canonical states. Thus, in a sharp contrast with most of the preceding works, the ideology of Ref. 9 has been from the start a very robust one and, actually, pertained to the whole of the phase transition region. It could be implemented, however, only at very low temperatures, since the authors used low-temperature cluster expansions as the principal tool for proving the corresponding probabilistic theorems.

The ideas of Ref. 9 did not wait long to inspire a wave of investigations, even before the draft of the work started to circulate. Two subsequent works of a fundamental importance are Ref. 26, where an alternative simplified proof of parts of the DKS results has been given using techniques, which are specific to the 2D Ising model, like self-duality, and Ref. 43, where the Wulff construction has been derived in the context of the 2D Bernoulli percolation, but in a completely nonper-
turbative fashion, that is down to the percolation threshold $\frac{1}{2}$. In both instances the exact canonical setting has been substituted by shell-type integral constraints, and, respectively, softer integral-type limit results have been used instead of the local estimates of the original DKS theory.

The results and techniques of Refs. 43 and 26 have been combined with the profound renormalization ideas of Ref. 33 and lead to an extension of this weak integral approach to the Wulff construction in the whole of the 2D Ising phase coexistence limit. Simpler proofs of some of the basic estimates of these two works (e.g., estimates in the phases of small contours or skeleton lower bounds) have been found in Refs. 44 and 45, and the integral version of the two-dimensional DKS theory has been essentially completed in Ref. 18, the estimates of the latter work being already optimal along the lines of the integral approach. Furthermore, Pfister and Velenik investigated the effect of boundary conditions, and in particular, studied the effect of an arbitrary boundary magnetic field, thus providing a derivation of the Winterbottom construction.

In spite of these successes, a nonperturbative treatment of the full DKS theory was still out of reach, because a key ingredient was missing: only rough estimates were available in the phase of small contours. By proving a local limit theorem in the phase of small contours, Ioffe and Schonmann were finally able to provide a nonperturbative version of the strong Wulff theory. The techniques of Ref. 32 are based on improved versions of asymptotic expansions in metastable cutoff phases developed in Ref. 31.

In principle, the two-dimensional DKS theory should lead to exact expansions of canonical partition functions up to zero-order terms. This, however, requires a superb control over the statistical behavior of microscopic phase boundaries, which is currently beyond the reach for the Ising model at moderately low temperatures. A certain progress, though, has been reported at very low temperatures or either in the case of simplified models. Finally, it should be noted that at moderately low temperatures the success of the DKS theory in two dimensions has been by and large confined to the Ising and percolation models, and that there are serious technical and possibly theoretical challenges to extend it to more general two-dimensional models (see Sec. III F for more on this).

On the other hand, as it has been communicated to us, an appropriate version of the low-temperature DKS theory (as originally developed in Ref. 9), should apply to any two-phase model in the realm of the Pirogov–Sinai theory. There is a strong interplay between dynamical properties of the Ising model and its behavior in equilibrium: in the absence of phase transition, the correlations at equilibrium are related to the exponential relaxation of the system; instead, as a phase transition occurs, the dynamics is driven by the evolution of droplets (nucleation, motion by mean curvature...). We will not enter into details and simply refer to the seminal paper on metastability by Schonmann and Shlosman and to the lecture notes by Martinelli and references therein for a survey of the recent works. Let us just mention that, as far as phase coexistence is considered, many dynamical results are only valid in dimension 2 because of the absence of a precise description of the equilibrium properties in higher dimensions.

If the 2D case was subject to rapid progress, the best results for higher dimensions remained for a long time those of Minlos and Sinai. The turning point of the latest developments should be traced back to the seminal works by Pisztora and by Cassandro and Presutti, where crucial renormalization decoupling estimates have been established in the case of the nearest neighbor Ising and, respectively, Kac interactions. The basic philosophy of the $L_1$-approach has been originally developed in the works of Refs. 52–55 in the context of the Ising systems with Kac potentials, and, in a less explicit way, elements and ideas of the theory already appeared in Refs. 43, 33, 30 and 18.

Using an embedding of the renormalized observables into a continuum setting, Alberti, Bellettini, Cassandro and Presutti emphasized the appropriateness of geometric measure theory setting, introduced relevant analytic approximation procedures (see Sec. II F 1) and proved large deviation bounds for the appearance of a droplet of the minority phase in a scaling limit when the size of the domain diverges not much faster than the range of the Kac potentials. In this scaling the
system can be controlled by a continuum limit via the $\Gamma$ convergence of functionals associated to the spin system$^{52}$ and by compactness arguments.$^{53}$

The approach of Refs. 52 and 53 has been extended by Benois, Bodineau, Butta, and Presutti$^{54,55}$ to the case when the range of the interaction remains fixed and does not change with the size of the system. The latter works are, already, structured in a way very similar to the one we expose here. Thus, the main steps of Refs. 54 and 55 comprise the coarse graining of the rescaled magnetization profiles by the $L_1$ proximity to various continuum sets of finite perimeter, surgery procedures to confine interfaces to tubes around the boundaries of such sets and exponential tightness arguments to reduce the combinatorial complexity of the rescaled problem. The essential model-related input has been provided by the decoupling estimates on the renormalized magnetization$^{22,23}$ and by the result on the instanton structure of Kac interfaces.$^{56,57}$ The latter structure, however, yields only approximate bounds at each fixed finite interaction range. Consequently, the exact (van der Waals) surface tension could be recovered only when the range of the interaction tends to infinity, that is, only in the Lebowitz–Penrose limit. Nevertheless, at long but finite range interactions one could say that the typical mesoscopic configurations concentrate on droplets with $L_1$-almost spherical shapes.

A complete picture of the higher-dimensional $L_1$-Wulff construction has been, for the first time, grasped and worked out in a recent remarkable work,$^{58}$ where the corresponding results have been established in the context of the supercritical three-dimensional Bernoulli bond percolation. Using novel and unusual renormalization procedures based on the decoupling results of Ref. 33, Cerf has essentially rediscovered all the main steps of the $L_1$-approach, as described above. The main turning point of Ref. 58 was the introduction of an alternative ingenious definition of the surface tension, which happened to be compatible with the setup of $L_1$-renormalization procedures. (It should be noted, though, that despite relative technical simplicity of this observation, the work$^{58}$ most certainly prompted the completion of the $L_1$-theory by many years.)

The work of Cerf$^{58}$ triggered a wave of new investigations. In Ref. 34 his ideas on how to define and treat the surface tension have been combined with an appropriate adjustment of the renormalization approach of Refs. 54 and 55, which lead to a relatively short proof of the $L_1$-Wulff construction for the nearest neighbor Ising model in three and higher dimensions and at sufficiently low temperatures. Most recently, a similar construction has been established up to the FK slab percolation threshold in Ref. 35. In the latter article new and important techniques have been developed in order to go around mixed boundary conditions via bulk relaxation properties of the FK measures.

Although the techniques of the $L_1$-theory might look “soft” when compared to the local limit setting of the DKS approach, one should bear in mind that there is always a “hard” step needed to initialize the $L_1$ machinery: The renormalized mesoscopic phase labels have to possess sufficiently good decoupling properties. For the case of Kac models the corresponding estimates have been established in Refs. 22, 23, and 59, and in the case of percolation (including FK for the nearest neighbor Ising model) models in dimension $d \geqslant 3$ in Ref. 33, on which both Ref. 58, 35, and 34 rely in a fundamental way.

Higher-dimensional Winterbottom-type shapes have been recovered in the context of effective interface models$^{60–63}$ following the original two-dimensional model defined and studied in Ref. 64.

The results of these works have been also formulated in terms of $L_1$ concentration properties, but the corresponding approach is quite different from the one we expose here. Thus, the analysis of Ref. 61. heavily relies on specific properties of Gaussian interactions. It should be noted, though, that, unlike in the nearest neighbor higher-dimensional Ising case, there is better insight into the fluctuation and relaxation properties of higher-dimensional microscopic interfaces.$^{65,62}$ On the other hand, the shapes produced by the effective interface models are much less “physical,” in particular, the equilibrium shapes are not scale invariant, and the corresponding surface tension is not convex.
II. L_1-THEORY

On the macroscopic level the phenomenon of phase segregation is studied in terms of concentration properties of the locally averaged magnetization. Statistical properties of the microscopic phase boundaries are waved out, and the backbone of the L_1-theory are hard model-oriented renormalization estimates, which enable a sharp surface order analysis of the mesoscopic magnetization profiles. An example of such coarse graining procedures in the case of Kac, percolation, and Ising models are given in Sec. II C.

The averaging is performed on various mesoscopic scales:

Mesoscopic notation. All the intermediate scales are of the form 2^k, k \in \mathbb{N}. For any M = 2^k fixed we split the unit torus \hat{T}^d into the disjoint union of the corresponding mesoscopic boxes,

\[ \hat{T}^d = \bigvee_{x \in \hat{T}^d} \hat{B}_k(x), \]  

(2.0.1)

where \hat{T}_k is the scaled embedding of the discrete torus \hat{T}_M = \{1,...,M\}^d into \hat{T}^d as

\[ \hat{T}_k = \hat{T}^d \cap \left( \frac{1}{M} T_M \right), \]

and, given x \in \hat{T}^d the box \hat{B}_k(x) \in \hat{T}^d is defined via

\[ \hat{B}_k(x) = x + \left[ -\frac{1}{2 \pi \tau}, \frac{1}{2 \pi \tau} \right]^d. \]

Let us use \mathcal{F}_k to denote the (finite) algebra of the subsets of \hat{T}^d generated by the partition (2.0.1). Given the size of the system \( N = 2^n \), the local magnetization \( \mathcal{M}_k \) on the \( M = 2^k \leq N \) scale is always an \( \mathcal{F}_{n-k} \)-measurable function. This notation should not be confusing: the subindex \( k \) in \( \mathcal{M}_k \) measures the “coarseness” of the mesoscopic magnetization profile. Thus, \( \mathcal{M}_0 \) corresponds to the microscopic configuration, and \( \mathcal{M}_n \) identically equals to the averaged total magnetization. In general, the local magnetization \( \mathcal{M}_k \) is a piecewise constant function on \( \hat{T}^d \) defined as

\[ \forall x \in \hat{T}_{n-k}, \quad \forall y \in \hat{B}_{n-k}(x), \quad \mathcal{M}_k(\sigma, y) = \frac{1}{M^d} \sum_{j \in B_M(2^n x)} \sigma_j. \]

Notice that the microscopic counterpart of the box \( \hat{B}_{n-k}(x) \) is the box \( B_M(2^n x) \) of side length \( M \) centered in \( 2^n x \).

We formulate all the results of Sec. II A for the nearest neighbor Ising model. Along with the supercritical Bernoulli percolation this is the only instance when a relatively complete L_1-theory has been developed. In both instances, the validity of the L_1-theory hinges in a crucial way on the validity of Pisztora’s coarse graining, which is by far the most profound model related fact employed. Nevertheless, the approach itself is rather robust, and in subsequent sections we shall try to distinguish between specific model-dependent properties and more general results. In particular, compactness properties of local magnetization profiles are discussed in Sec. II B without any reference to specific models. Instead we briefly indicate how the conditions of the corresponding general exponential tightness theorem could be verified in several particular cases.

A. Results and the strategy of the proof

1. Main results

For simplicity, we restrict to the case of the torus \( \mathbb{T}_N \) and denote by \( \mu_N \) the Gibbs measure with periodic boundary conditions. Define the total magnetization \( M_{\mathbb{T}_N} \) as
\[
M^N_t = \frac{1}{N^d} \sum_{i=1}^{N} \sigma_i.
\]

Let us define also the set \( \mathcal{B}_p \) as
\[
\mathcal{B}_p = \{ \beta : \text{Pisztora's coarse-graining hold for the Ising model at inverse temperature } \beta \}.
\]

We refer to the original article, Refs. 33 and 35, for the precise relevant definitions (see also the remark at the end of Sec. II C 3). It is known that \( \mathcal{B}_p \) contains all except for, at most, countably many points of the interval \( [\bar{\beta}_c, \infty) \), where \( \bar{\beta}_c \) is the so-called slab percolation threshold, which is conjectured to coincide with \( \beta_c \).

A compact way to state the main result of the \( L_1 \)-theory is the following.

**Theorem 2.1.1**: For any \( \beta \in \mathcal{B}_p \) and \( m \) in \( [\bar{m}, m^*] \),
\[
\lim_{N \to \infty} \frac{1}{N^d} \log \mu_N(|M_{N\beta}^N| \leq m) = -\mathcal{W}_\beta(\mathcal{K}_m),
\]

where \( \bar{m} = \bar{m}(\beta) \) and \( \mathcal{K}_m \) were defined in Sec. IA 2.

**Remark**: The above theorem has been established for \( \beta \gg 1 \) in Ref. 34. The only additional ingredient required for an extension of the results of the latter paper to the whole of the temperature range \( \bar{\beta}_c \) was the validity of Lemma 2.4.1. Such a statement happens to be highly nontrivial, and it has been proven in Ref. 35 along with an alternative derivation of the claim of Theorem 2.1.1.

Theorem 2.1.1 looks like a surface order large deviation principle. Such an appellation, however, would not help to explain the structure of the underlying phenomena. In fact, Theorem 2.1.1 is essentially equivalent to a seemingly stronger statement on the macroscopic geometry of the phase segregation of local magnetization profiles under the conditional measure \( \mu_N(\cdot | |M_{N\beta}^N| \leq m) \).

For any function \( v \in L^1([\bar{v}_d, \infty) - 1/m^*, 1/m^*) \), the \( \delta \) neighborhood of \( v \) is denoted by \( \mathcal{V}(v, \delta) \),
\[
\mathcal{V}(v, \delta) = \left\{ v' \in L^1([\bar{v}_d, \infty) - 1/m^*, 1/m^*) \mid \int_{\bar{v}_d}^{v_d} |v'_x - v_x| dx \leq \delta \right\}.
\]

The \( L_1 \)-theorem on the phase separation says that for \( \beta \) large enough with \( \mu_N(\cdot | |M_{N\beta}^N| \leq m) \) probability converging to 1, the function \( M_\delta \) is close to some translate of the Wulff shape \( m^* |_{\mathcal{K}_m} \).

More precisely, fix a number \( \nu < 1/d \).

**Theorem 2.1.2**: For any \( \beta \in \mathcal{B}_p \) and \( m \) in \( [\bar{m}, m^*] \), the following holds: For every \( \delta > 0 \), one can choose a scale \( k_0 = k_0(\beta, \delta, \nu) \), such that
\[
\lim_{N \to \infty} \min_{k_0 \leq k \leq m} \mu_N \left( \frac{M_k}{m^*} \in \bigcup_{x \in \bar{v}_d} \mathcal{V}(1_{\mathcal{K}_m}, \delta) \right| |M_{N\beta}^N| \leq m \right) = 1,
\]

where \( \bar{m} \) and \( \mathcal{K}_m \) were defined in Sec. IA 2.

The proofs of Theorems 2.1.1 and 2.1.2 are similar and are divided into two steps. The first step amounts to prove a compactness theorem and the second one to derive precise logarithmic asymptotics.
2. Exponential tightness

Recall that for any $a$ positive, the set
\[ K_a = \{ v \in BV(\tilde{\mathbb{R}}^d,\{\pm 1\}) \mid \mathcal{P}(\{v = 1\}) \leq a \}, \]
is compact with respect to convergence in $L^1(\tilde{\mathbb{R}}^d)$.

**Proposition 2.1.1**: Let $\beta$ be in $\mathbb{B}_p$. Then there exists a constant $C(\beta) > 0$ such that for all $\delta$ positive one can find $k_0(\delta)$,
\[ \forall \delta > 0, \quad \limsup_{N \to \infty} \frac{1}{N^d} \max_{k_0(\delta) \leq k \leq m} \log \mu_N \left( \frac{M_k}{m^*} \in \mathcal{V}(K_a,\delta)^c \right) \leq -C(\beta)a, \]
where $\mathcal{V}(K_a,\delta)$ is the $\delta$ neighborhood of $K_a$ in $L^1(\mathbb{R}^d)$.

This proposition tells us that only the configurations close to the compact set $K_a$ have a contribution which is of the surface order. This statement reduces the complexity of the problem: as $K_a$ is compact, it is enough to derive the leading terms in the logarithmic asymptotics for the probability of a finite number of events.

In Sec. II-B, we prove that the analog of Proposition 2.1.1 holds for a broad class of models.

3. Precise logarithmic asymptotics

As the minimizers are known, it is sufficient to derive a lower bound for configurations concentrated close to $K_m$.

**Proposition 2.1.2**: Let $\beta$ be in $\mathbb{B}_p$ and let $m$ be in $]\bar{m},m^*[,$
\[ \liminf_{N \to \infty} \frac{1}{N^d} \min_{k_0(\delta) \leq k \leq m} \log \mu_N \left( \frac{M_k}{m^*} \in \mathcal{V}(K_m,\delta)^c \right) \geq -\mathcal{W}_\beta(K_m) - o(\delta), \]
where the function $o(\cdot)$ depends only on $\beta$ and vanishes as $\delta$ goes to 0.

According to Proposition 2.1.1, we will prove the upper bound only for a restricted class of events.

**Proposition 2.1.3**: Let $\beta$ be in $\mathbb{B}_p$. Then for all $v$ in $BV(\tilde{\mathbb{R}}^d,\{\pm 1\})$ such that $\mathcal{W}_\beta(v)$ is finite, one can choose $\delta_0 = \delta_0(v)$, such that uniformly in $\delta < \delta_0$,
\[ \limsup_{N \to \infty} \frac{1}{N^d} \max_{k_0(\delta) \leq k \leq m} \log \mu_N \left( \frac{M_k}{m^*} \in \mathcal{V}(v,\delta) \right) \leq -\mathcal{W}_\beta(v) + o(\delta), \]
where the function $o(\cdot)$ depends only on $\beta$ and $v$ and vanishes as $\delta$ goes to 0.

The propositions above ensure that given a precision $\delta$, there is a finite scale $k_0(\delta)$ after which the phases are uniformly segregated with this precision.

4. Scheme of the proof

The scheme of the proof is well known in the soft context of large deviations: one first proves an exponential tightness property and then a weak large deviation principle (Proposition 2.1.2 holds also for any bounded variation function with a finite perimeter). To be sure, the proof itself has nothing to do with the theory of large deviations: the central tools here are the renormalization estimates leading to Peierls-type bounds and estimate in the phase of small contours, and, of course, the identification methods to produce the macroscopic surface tension in the precise logarithmic asymptotics.

Thus, Proposition 2.1.1 tells us that, under the appropriate renormalization, the occurrence of many small contours or of very large contours is unlikely. It is a straightforward consequence of
the general exponential tightness, Theorem 2.2.1, which we state in Sec. II B. The statement is reminiscent to the results proven in Ref. 55, but the proof itself is based on the analysis of the phase of small contours developed in Refs. 30, 44, and 18.

To prove Propositions 2.1.2 and 2.1.3, we first consider the macroscopic event \( \{ \mathcal{M}_2/m^* \in \mathcal{V}(v, \delta) \} \) and by using several localization procedures, we reduce to compute the probability of microscopic events from which, adopting the procedure developed in Ref. 58, we can derive the exact surface tension factor. This enables us to avoid the computations related to the microscopic phase boundaries at, however, a principal cost of losing track of the latter.

Since the most likely configurations in \( \{ \mathcal{M}_2/m^* \in \mathcal{V}(v, \delta) \} \) are those for which both phases coexist along the boundary of \( \partial^* v \), we would like to prove that a microscopic interface is localized close to the boundary. To derive the lower bound (Proposition 2.1.2), one can enforce such a microscopic interface and then recover the surface tension factor. This is not the case for the upper bound (Proposition 2.1.3) because the \( L_1 \) constraint \( \{ \mathcal{M}_2/m^* \in \mathcal{V}(v, \delta) \} \) imposed on the magnetization is not strong enough to localize the interface close to \( \partial^* v \): there might be mesoscopic fingers of one phase percolating into the other. To circumvent this problem, we follow an argument developed in Ref. 54 and first prove a weak localization on a mesoscopic level. This involves a surgery procedure called the minimal section argument. This procedure ensures that one can chop off the mesoscopic fingers without changing too much the probability of the event and therefore localize the interface on a mesoscopic level. The renormalization is an essential feature of this proof. Once the interface is localized on the mesoscopic level, it remains to identify surface tension.

We now proceed by first defining a coarse graining and deducing the exponential tightness from Theorem 2.2.1. Then we compute the logarithmic asymptotics.

**B. Coarse graining and mesoscopic phase labels**

At every mesoscopic scale \( M = 2^k \) the local magnetization \( \mathcal{M}_k \) gives a coarse grained representation of the system. Statistical properties of the microscopic configurations are waved out, and instead one keeps track only of the local order parameters over the corresponding mesoscopic blocks. These are quantified by three values \( \pm 1 \) and 0 according to whether they are sufficiently close to one of the two equilibrium values \( \pm m^* \) or not. Here 0-blocks play the role of the mesoscopic phase boundaries, and the \( \pm 1 \) blocks of the corresponding mesoscopic phase regions. Thus, the outcome of the renormalization could be schematically represented as the following two-step diagram:

\[
\begin{array}{ccc}
\text{Microscopic configurations} & \rightarrow & \text{Local magnetization} \\
& & \rightarrow & \text{Mesoscopic phase labels}
\end{array}
\]

There are two principal results to be discussed in this section: we show that the \( L_1 \) difference between the local magnetization and the corresponding phase labels vanishes on the exponential scale, and we give a general exponential tightness criterion for families of \( \{\pm 1,0\} \)-valued phase label functions. In Sec. II C, we will indicate how to construct phase labels in the case of Kac, percolation, and nearest neighbor Ising models.

**Definition:** A \( \{\pm 1,0\} \)-valued function \( u \) on \( \mathbb{Z}^d \) is called a mesoscopic phase label, if there exists \( k \in \mathbb{N} \), such that \( u \) is an \( \mathcal{F}_k \)-measurable function.

**1. Tightness theorem for mesoscopic phase labels**

We fix now a sequence of non-negative numbers \( \{p_k\} \), such that

\[
\lim_{k \to \infty} p_k = 0.
\]

The following compactness result holds uniformly in the microscopic scales \( N = 2^n \).
Theorem 2.2.1: (Tightness of mesoscopic phase labels) Let $N = 2^n$ and assume that \( \{u_k(\omega, x)\} \) is a sequence of random mesoscopic phase label functions defined on the common probability space \((\Omega_N, A_N, P_N)\), such that the realizations of \( u_k \in \mathcal{F}_{n-k}, \; k = 1, ..., n \), and for every \( k \) the following two conditions hold.

(a) The distribution of the family of random variables \( \{u_k(\omega, x)\}_{x \in \mathbb{T}^d_{n-k}} \) is stochastically dominated by the Bernoulli site percolation measure \( P_{\text{perc}}^\delta \) on \( \mathbb{T}^d_{n-k} \). In particular,
\[
P_N(u_k(x_1) = 0, ..., u_k(x_t) = 0) \approx (p_k)^t, \tag{2.2.2}
\]
for all \( k \geq k_0 \).

(b) If for two different points \( x, y \in \mathbb{T}^d_{n-k} \) the corresponding \( u_k \)-phase labels have opposite signs, that is if \( u_k(x)u_k(y) = -1 \), then on any finer scale \( k' \equiv k \) any \(*\)-connected chain of \( \mathbb{B}_{n-k'} \) blocks joining \( \mathbb{B}_{n-k}(x) \) to \( \mathbb{B}_{n-k}(y) \) contains at least one block with a zero \( k' \)-label. Then for every \( a > 0 \) and \( \delta > 0 \) there exists a finite scale \( k_0 = k_0(\delta) \), such that
\[
\frac{1}{N^{d-1}} \log P_N(u_k \in \mathcal{V}(K_n, 2\delta)^c) \leq -c_1(d) \min \left\{ 2^{-d-k_0}, \frac{a}{2^{(d-1)k_0} \cdot \delta^{2-n-dk_0}} \right\}, \tag{2.2.3}
\]
for all \( k \geq k_0 \).

Remark: The proof of this general theorem is given in Appendix A. Notice that for \( N \) sufficiently large we obtain a simpler surface order estimate, which, for every \( \nu < 1/d \) fixed, holds uniformly in all mesoscopic scales \( k_0(\delta) \leq k \leq \nu \log N \):
\[
\frac{1}{N^{d-1}} \log P_N(u_k \in \mathcal{V}(K_n, 2\delta)^c) \leq -c_1(d) \frac{a}{2^{(d-1)k_0}}, \tag{2.2.4}
\]
Also, an inspection of the proof shows that the tightness of the phase labels on a certain scale \( k \) does not depend on the validity of Assumptions (a) and (b) of Theorem 2.2.1 on the successive scales \( k' > k \). In particular, the estimate (2.2.4) is valid on fixed (large) finite scales \( k = k_0 \), once assumption (a) is satisfied, and once any \(*\)-connected sign changing chain of \( k_0 \)-blocks necessarily contains a 0-block. This simplified version of Theorem 2.2.1 is used in the case of Kac potentials that we discuss in Sec. II C 1.

2. Relation to magnetization profiles

The original Gibbs measure is related to the above abstract setting in the following way: For every \( N = 2^n \), one constructs a (possibly enlarged) probability space \((\Omega_N, A_N, P_N)\), on which both the spin variables \( \sigma \in \{-1, +1\}^V \) and various indexed families \( \{u_k^\xi\} \) of mesoscopic phase labels are defined. Such construction should enjoy the following set of properties.

(C1) The marginal distribution of \( \sigma \) under \( P_N \) is precisely \( \mu_N \).

(C2) For every \( \xi > 0 \) the family \( \{u_k^\xi\} \) of mesoscopic phase labels satisfies assumption (a) of Theorem 2.2.1 with the corresponding sequence \( \{\rho_k, c^k\} \) of site percolation probabilities obeying (2.2.1).

(C3) For every \( k \in \{0, ..., n\} \) and \( \xi > 0 \) the local magnetization profile \( M_k \) and the phase label \( u_k^\xi \) are related as follows: \( P_N \) a.s.,
\[
|M_k(x) - m\ast u_k^\xi(x)| \leq \xi, \quad \text{whenever} \quad |u_k^\xi(x)| = 1. \tag{2.2.5}
\]
Notice that both functions above are \( \mathcal{F}_{n-k} \) measurable, that is, (2.2.5) should be verified over the mesoscopic boxes indexed by the points \( x \in \mathbb{T}^d_{n-k} \).

Under conditions (C1)–(C3), given any \( \delta > 0 \) one can choose the accuracy \( \xi \) of the coarse graining, a finite scale \( k_0 = k_0(\delta, \beta) \), and a sequence of mesoscopic phase labels \( \{u_k^\xi\} \), such that for every \( \nu < 1/d \) fixed,
\[
\frac{1}{N^{d-1}} \log P_N \left( \max_{k=0}^n \| \mathcal{M}_k - m^* u_k^x \|_1 > \delta \right) \equiv - c_2 2^{(1-d)N}.
\] (2.2.6)

Notice that (2.2.6) holds uniformly in the size of the system \(N = 2^n\), once assumptions (C1)–(C3) do so.

Let us check (2.2.6). By the very construction,

\[
\| \mathcal{M}_k - m^* u_k^x \|_1 \leq \frac{2}{|T^d_{n-k}|} \sum_{x \in T^d_{n-k}} 1_{u_k^x(x) = 0}.
\]

Consequently, using the domination by the Bernoulli site percolation [assumption (a)],

\[
P_N (\| \mathcal{M}_k - m^* u_k^x \|_1 > \delta) \leq P_N \left( \frac{1}{|T^d_{n-k}|} \sum_{x \in T^d_{n-k}} 1_{u_k^x(x) > 0} \right) \leq \exp \left( -c_1 2^{(n-k)} \log \frac{\delta - \xi}{2} \right).
\]

The latter estimate is of the super-surface order once \( \rho_{L,t} \approx (\delta - \xi)/2 \) and \( k < n/d \).

C. Examples of mesoscopic phase labels

We show that mesoscopic phase labels can be constructed in the case of Kac, percolation, and Ising models.

1. Kac potentials

For this model mesoscopic phase labels are defined on the original space of spins \( \sigma \in \{-1, +1\} \tau x \); the coarse graining is obtained by averaging locally the magnetization. Recall that we are using dyadic length scales \( N = 2^n \).

Phase labels are constructed in three steps. First, for any integer \( k \) and \( \xi > 0 \), we introduce the block spin variables \( \bar{u}_k^x \) that label the boxes \( \bar{B}_{n-k} \) according to the averaged magnetization over the boxes of the linear size \( M = 2^k \). These \( \bar{u}_k^x \) are constant on each of the blocks \( \bar{B}_{n-k}(x) \) with \( x \in T^d_{n-k} \),

\[
\bar{u}_k^x (\sigma, x) = \begin{cases} 
\pm 1, & \text{if } \frac{1}{M^2} \sum_{i \in B_{M}(2^n x)} | \sigma_i - m^* | < \xi,
0, & \text{otherwise}.
\end{cases}
\]

In the Kac case we do not use Theorem 2.2.1 in its full generality; the object of the coarse graining is to choose a finite scale \( k_0 \), such that the family of mesoscopic phase labels is exponentially tight in \( L_1 \). Recall that the scaling parameter is chosen such that \( e = 2^{-m} \) with \( m \) large but fixed. Eventually finite renormalization scales \( k_0 \) are going to satisfy \( k_0 = m + a_0 \), where \( a_0 \) depends on \( \beta \) and \( \xi \), but not on \( m \). The sign of the \( k_0 \) label over a box \( \bar{B}_{n-k_0}(x) \) depends on a more refined information on the fluctuations of the magnetization inside the box: we choose another scale \( l_0 = l_0 - b_0 \), where, as in the case of \( a_0 \), the scale \( b_0 \) will eventually depend only on \( \beta \) and \( \xi \), and define the family of modified block spins \( \{ \bar{u}_k^x \} \) on the \( k_0 \) scale as
Finally, we define the mesoscopic phase label functions \( \{ \mu^x_{k_0}(\sigma, x) \} \). If \( \mu^x_{k_0}(\sigma, x) = 0 \), we set \( u^x_{k_0}(\sigma, x) = 0 \). If \( x, y \in \mathbb{T}^d_{n-k_0} \) are *-neighbors, but the corresponding modified blocks spins satisfy \( a^x_{k_0}(\sigma, y) < 0 \), then \( u^x_{k_0}(\sigma, x) = u^y_{k_0}(\sigma, y) = 0 \). Otherwise, we set \( u^x_{k_0}(\sigma, x) = a^x_{k_0}(\sigma, x) \).

A consequence of the Peierls estimate proven in Refs. 22 and 23 is that assumption (a) is satisfied, namely the following.

**Theorem 2.3.1:** For any \( \beta > 1 \), there exists \( \xi_0 = \xi_0(\beta) > 0 \), such that the following holds: For any \( \xi < \xi_0 \) one can choose \( \varepsilon_0 = \varepsilon_0(\xi) \), \( a_0 = a_0(\xi) \) and \( b_0 = b_0(\xi) \), such that uniformly in the interaction parameters \( \varepsilon = 2^{-m} < \varepsilon_0 \),

\[
\mu_{\varepsilon, N}(\mu^x_{k_0}(x_1) = 0, \ldots, u^x_{k_0}(x_s) = 0) \leq \exp \left( -\frac{c_0}{\varepsilon^3} \right),
\]

where, for every fixed \( \varepsilon = 2^{-m} < \varepsilon_0 \), the mesoscopic phase labels \( u^x_{k_0} \) are constructed on the scales \( k_0 = m + b_0(\xi) \) and \( l_0 = m - b_0(\xi) \).

**Remark:** A more refined statement implying exponential decay of correlations was proven in Ref. 59. Notice that conditions (C1)–(C3) of the previous section are satisfied by the definition of the mesoscopic phase label functions. Notice also that assumption (b) of Theorem 2.2.1 is automatically satisfied on the \( k_0 \)-scale. Thus, the family \( \{ u^x_{k_0} \} \) is exponentially tight in \( l_0 \).

A similar renormalization procedure was carried out by Lebowitz, Mazel, and Presutti for a system of point particles in \( \mathbb{R}^d \) interacting with Kac potentials. In this case the study of phase transition in the continuum is much more involved. Beyond a proof of the liquid–vapor phase transition, their results provide an accurate description of the system in terms of mesoscopic phase labels that represent the liquid and the gaseous phases. Such a coarse graining should be helpful to obtain further results on phase coexistence in the continuum.

### 2. Bernoulli bond percolation

Bernoulli bond percolation exhibits features similar to the Ising model as phase transition and surface order behavior in a regime of phases coexistence. Nevertheless, as the setting is different from the Ising model, we briefly recall some notation. The set of edges is \( \mathcal{E} = \{ (x, y) \mid x \sim y \} \), where \( x \sim y \) means that the vertices are nearest neighbors. An edge \( b \in \mathcal{E} \) is open if \( \omega_b = 1 \) and closed otherwise. To any subset \( \Lambda \in \mathbb{Z}^d \), we associate \( [\Lambda] \), the set of edges in \( \Lambda \). The space of bonds configurations in \( \Lambda \) is \( \Omega_\Lambda = \{ 0, 1 \}^{[\Lambda]} \). For a given \( p \) in \( [0, 1] \), we define the Bernoulli bond percolation measure on \( \Omega_\Lambda \) by

\[
\Phi^p_\Lambda(\omega) = \prod_{b \in [\Lambda]_p} (1 - p)^1 - \omega_b p^{\omega_b}.
\]

For simplicity, \( \Phi^p_\Lambda \) denotes the measure on \( \Omega_N = \Omega_T N \).

Let \( \omega \) be a configuration in \( \Omega \), an open path \( (x_1, \ldots, x_n) \) is a finite sequence of distinct nearest neighbors \( x_1, \ldots, x_n \) such that on each edge \( \omega_{\{ x_i, x_{i+1} \}} = 1 \). We write \( \{ A \rightarrow B \} \) for the event such that there exists an open path joining a site of \( A \) to one of \( B \). The connected components of the set of open edges of \( \omega \) are called \( \omega \)-clusters.

A phase transition is characterized by the occurrence of an infinite cluster. Define \( \Theta_\beta \) by

\[
\Theta_\beta = \lim_{N \to \infty} \Phi^{\beta}_\Lambda(\{ 0 \to \partial T_N \}),
\]

(2.3.1)
Applying a Hölder inequality, we get

\[ T \]

probability

\[ T \]

as the vertices in other phase by the other clusters.

From Ref. 69

A straightforward way to recover the previous statement is to partition \( \hat{T}_{n-k} \), the following events depend only on configurations in the box \( B_{M^r}(2^x) \):

\[ U_i = \{ \omega \in \Omega_N | \text{there is a unique crossing cluster } C^* \text{ in } B_{M^r}(2^x) \}. \]

A crossing cluster is a cluster that intersects all the faces of the box. Let \( l \) be an integer smaller than \( k \) that will be fixed later:

\[ R_i = U_i \cap \{ \omega \in \Omega_N | \text{every open path in } B_{M^r}(2^x) \text{ with diameter larger than } 2^l \text{ is contained in } C^* \}, \]

where the diameter of a subset \( A \) of \( \mathbb{Z}^d \) is \( \sup_{x,y \in A} \| x-y \|_1 \). Finally, we consider an event that imposes that the density of the crossing cluster in \( B_M(2^x) \) is close to \( \Theta_p \) with accuracy \( \xi > 0 \),

\[ V_k = U_i \cap \{ \omega \in \Omega_N | \| C^* \cap B_M(2^x) \| \leq [\Theta_p - \xi, \Theta_p + \xi] M^d \}, \]

where \( | \cdot | \) denotes the number of vertices in a set.

Each box \( \hat{B}_{n-k}(x) \) is labeled by the variable \( \tilde{u}_k^i(\omega, x) \),

\[ \forall x \in \hat{T}_{n-k}, \quad \tilde{u}_k^i(\omega, x) = \begin{cases} 1, & \text{if } \omega \in R_i \cap V_k^c, \\ 0, & \text{otherwise}. \end{cases} \]

Let \( \{x_1, \ldots, x_l\} \) be vertices in \( \hat{T}_{n-k} \) not \( * \)-neighbors of \( x \), then Ref. 33 implies that for every \( p > p_c \), there exists \( k_0(p, \xi) \), and \( l_0(p) \) such that for all \( k \geq k_0 \) and \( k \geq l \geq l_0 \),

\[ \Phi_0^s(\tilde{u}_k^i(x_1) = 0, \ldots, \tilde{u}_k^i(x_l) = 0) \leq \exp(-c_12^l) + \exp(-c_2(\xi)2^k). \]

From Ref. 69 (Theorem 1.3), we deduce that for \( k \) and \( l \) large enough, the random variables \( \{\tilde{u}_k^i(x)\} \) are dominated by a Bernoulli site percolation measure \( \mathbb{P}_{\mathbb{P}}^* \)

\[ p_k \leq \exp(-c(\xi)2^k). \quad (2.3.2) \]

A straightforward way to recover the previous statement is to partition \( \hat{T}_{n-k}^d \) into \( c(d) \) sublattices \( (\hat{T}_{n-k-1,J}^d)_{J \leq c(d)} \), that are translates of \( \hat{T}_{n-k-1}^d \). Any collection of vertices \( \{x_1, \ldots, x_l\} \) in \( \hat{T}_{n-k}^d \) can be rearranged into \( c(d) \) subsets \( \{x_1^{(i)}, \ldots, x_l^{(i)}\} \), such that each \( \{x_1^{(i)}, \ldots, x_l^{(i)}\} \) belongs to \( \hat{T}_{n-k-1,J}^d \). Applying a Hölder inequality, we get

\[ \Phi_0^s(\tilde{u}_k^i(x_1) = 0, \ldots, \tilde{u}_k^i(x_l) = 0) \leq \prod_{i=1}^{c(d)} \Phi_0^s(\tilde{u}_k^i(x_1^{(i)}) = 0, \ldots, \tilde{u}_k^i(x_l^{(i)}) = 0)^{1/c(d)}. \]

As the vertices in \( \hat{T}_{n-k-1,J}^d \) are not \( * \)-neighbors in \( \hat{T}_{n-k}^d \), the domination by a Bernoulli product measure follows.
We say that a block \( \tilde{B}_{n-k}(x) \) is regular if \( \tilde{u}_i^j(x) = 1 \). Finally, we define the mesoscopic phase labels \( u_i^j \) to be equal to 1 on the regular blocks connected to the largest cluster and to \(-1\) on the regular blocks disjoint from the largest cluster. Otherwise, we set \( u_i^j(\omega,x) = \tilde{u}_i^j(\omega,x) = 0 \). From (2.3.2), the mesoscopic phase labels satisfy assumption (a). Notice that if \( x \) and \( y \) are \(*\)-neighbors in \( \tilde{H}_n \), the boxes \( B_M(2^nx) \) and \( B_M(2^ny) \) overlap. Choosing the parameter \( l \leq k - 3 \) we ensure that if the boxes \( \tilde{B}_{n-k}(x) \) and \( \tilde{B}_{n-k}(y) \) are both regular, then the crossing clusters in these boxes are connected. This implies that assumption (b) is satisfied: two blocks with \( k \)-clusters of different signs cannot be \(*\)-connected.

The Bernoulli bond percolation model is precisely described by Pisztora's coarse graining, namely on a sufficiently large scale \( 2^k \), the typical configurations have a unique crossing cluster surrounded by small islands of size smaller than \( 2^l \). According to Theorem 2.2.1, the family \( \{u^j_i\} \) is exponentially tight in \( L^1 \).

### 3. Ising nearest neighbor

An extension of the preceding renormalization procedure applicable to the Ising model has been also introduced in Ref. 33. Unlike the Ising model with \( K \) potentials, this coarse graining is defined on an enlarged phase space via the FK representation. For a review of FK measures, we refer the reader to Refs. 33, 70, and 71.

Let us recall the definition of the random cluster measures (or FK measures) that are a generalization of the Bernoulli bond percolation measures with correlated bond distribution. To any subset \( \Lambda \) of \( L^d \) and \( \pi \) included in \( \partial \Lambda \), we associate a set of edges,

\[
[\Lambda]^\pi = \{\{x,y\}|x \sim y, x \in \Lambda, y \in \Lambda \cup \pi\},
\]

and the space of configurations in \( \Lambda \) is \( \Omega^\pi = \{0,1\}^{[\Lambda]^\pi} \). The first step is to introduce a measure on \( \Omega^\Lambda \). A vertex \( x \) of \( \Lambda \) is called \( \pi \)-wired if it is connected by an open path to \( \pi \). We call \( \pi \) clusters the clusters defined with respect to the boundary condition \( \pi \). A \( \pi \)-cluster is a connected set of open edges in \( \Omega^\Lambda \) and we identify to be the same cluster all the clusters that are \( \pi \)-wired, i.e., connected to \( \pi \). For a given \( p \) in \([0,1]\), we define the FK measure on \( \Omega^\pi \) with boundary conditions \( \pi \) by

\[
\Phi^\pi_\Lambda(p) = \frac{1}{Z^\pi_\Lambda} \prod_{b \in [\Lambda]^\pi} (1 - p)^{1-a_{b,p}^{\pi} a_{b}} 2^e_{\pi}(\omega),
\]

where \( Z^\pi_\Lambda \) is a normalization factor and \( e_{\pi}(\omega) \) is the number of clusters that are not \( \pi \)-wired. If \( \pi = \partial \Lambda \) [resp \( \pi = \phi \)] then the boundary conditions are said to be wired [resp. \( \rho_{\text{free}} \)] and the corresponding FK measure on \( \Omega^\Lambda_\pi \) is denoted by \( \Phi^\pi_\Lambda \) [resp. \( \phi^{\rho_{\text{free}}} \)]. Finally, the periodic measure on the torus \( T_N \) is denoted by \( \Phi^{\pi,p}_{N} \) and the phase space by \( \Omega^p_N \).

In order to recover the Gibbs measure \( \mu^\Lambda \), we fix the percolation parameter \( p_{\rho_{\text{free}}} = 1 - \exp(-2\beta) \) and generate the edges configuration \( \omega \) in \( \Omega^p_N \) according to the measure \( \Phi^{\pi,p}_{N} \). Given \( \omega \), we equip randomly each \( \omega \)-cluster with a color \( \pm 1 \) independently from the others. This amounts to introducing the measure \( P^\pi_N \) on \( \{-1,1\}^N \) such that the spin \( \sigma_x \) has the color of the cluster attached to \( x \). The Gibbs measure \( \mu_N^\Lambda \) can be viewed as the first marginal of the coupled measure \( \Gamma_N^\Lambda(\sigma,\omega) = P^\pi_N(\sigma)\Phi^{\pi,p}_{N}(\omega) \) on the space \( \{-1,1\}^N \otimes \Omega^p_N \). In the case of \( \pi \)-wired boundary conditions, the spins attached to the \( \pi \)-wired cluster are equal to 1.

As a consequence of this representation, one has, for any increasing sequence of sets \( \Lambda_N \),

\[
m^\pi = \lim_{N \to \infty} \mu^\Lambda_N(\sigma_0) = \lim_{N \to \infty} \Phi^{\pi,p}_{N}(\{0,0,0\}) = \Theta^{\pi,p}_{\rho_{\text{free}}},
\]

In the following, we use \( m^\pi \) or \( \Theta^{\pi,p}_{\rho_{\text{free}}} \) depending on the context. Furthermore, we suppose that
This property is satisfied for all $\beta$ outside a subset of $\mathbb{R}$ that is, at most, countable (see Lebowitz\textsuperscript{72} and Pfister\textsuperscript{73}).

On the scale $M = 2^k$, we define, in the same way as for Bernoulli bond percolation, the variables $\hat{u}^k_1(\omega, x)$ that are piecewise constant on each box $\hat{B}_{n-k}(x)$ with $x$ in $\hat{\mathbb{Z}}_{n-k}$. The mesoscopic phase labels depend on the averaged magnetization in regular blocks. Define the label of $\hat{B}_{n-k}(x)$ by

$$u^k_1(\sigma, \omega, x) = \begin{cases} \text{sign}(C^*), & \text{if } \hat{u}^k_1(\omega, x) = 1 \text{ and } |M_k(\sigma, x) - \text{sign}(C^*)m^*| < 2\xi, \\ 0, & \text{otherwise,} \end{cases}$$

where $C^*$ is the crossing cluster in $B_M(2^nx)$.

In a regular box $\hat{B}_{n-k}(x)$ [i.e., $\hat{u}^k_1(x) = 1$], the averaged magnetization is controlled by the random coloring of the small clusters included in $B_M(2^nx)$, so that the averaged magnetization in a regular box is independent of the configurations in the neighboring boxes. In the case of the Ising model, the additional parameter $l = l(k)$ is tuned in order to control the fluctuations of the magnetization over the small clusters. As a consequence of this, assumptions (a), (b) and (C1)--(C3) are satisfied for $p_\beta$ above a certain nontrivial slab percolation threshold $p_\beta$, which is conjectured to coincide with $p_{\beta_c}$ (see Ref. 33 for details), and Theorem 2.2.1 holds.

Remark: Using the notations of this section, the set $\mathfrak{B}_\beta$ introduced in Sec. II A 1 could be defined as

$$\mathfrak{B}_\beta = \{\beta: \beta > \beta_c \text{ and (2.3.3) holds}\}.$$

### D. Surface tension

We are going to derive Propositions 2.1.2 and 2.1.3 for the Ising model with nearest neighbor interaction. As explained before, the philosophy of the proof is to start from the macroscopic level and to localize successively on finer scales with the help of a coarse graining. The approach itself is quite general. Nevertheless, the coarse graining is model dependent; therefore we will need first to state an alternative representation of the surface tension in terms of the FK representation in order to use the estimates that will be obtained from Pisztora’s coarse graining. The idea of such definitions has been introduced by Cerf in Ref. 58.

#### 1. FK representation

We fix $n$ a vector in $\mathbb{Z}^{d-1}$ and study $\tau_\beta(n)$. Following the notation of Sec. I B 4, we consider, for any $\varepsilon$ positive, the parallelepiped $\hat{\Lambda}(N, \varepsilon N)$ of $\mathbb{R}^d$ oriented according to $n$. Namely, the basis of $\hat{\Lambda}(N, \varepsilon N)$ with side lengths equal to $N$ is orthogonal to $n$ and the other sides have lengths equal to $\varepsilon N$. For simplicity its microscopic counterpart $\hat{\Lambda}(N, \varepsilon N) \cap \mathbb{Z}^d$ will be denoted by $\Lambda_N^\varepsilon$.

By using the correspondence between the Ising model and the FK representation, one can rewrite $\tau_\beta$ in terms of the bond model. Let $\{\hat{\sigma}^+ \Lambda_N^\varepsilon \rightarrow \hat{\sigma}^- \Lambda_N^\varepsilon\}$ be the event such that there is no open path inside $\Lambda_N^\varepsilon$ joining $\hat{\sigma}^+ \Lambda_N^\varepsilon$ to $\hat{\sigma}^- \Lambda_N^\varepsilon$. Then,

$$\tau_\beta(n) = \lim_{N \to \infty} -\frac{1}{N^{d-1}} \log \Phi_{\Lambda_N^\varepsilon}^{\mu, \beta}(\{\hat{\sigma}^+ \Lambda_N^\varepsilon \rightarrow \hat{\sigma}^- \Lambda_N^\varepsilon\}).$$

Notice that the event $\{\hat{\sigma}^+ \Lambda_N^\varepsilon \rightarrow \hat{\sigma}^- \Lambda_N^\varepsilon\}$ takes only into account the paths inside $\Lambda_N^\varepsilon$ and not the identification produced by wired boundary conditions. The relation above will be useful only in the proof of Proposition 2.1.2.
We are now going to state an approximate expression of the surface tension that is weakly dependent on the boundary conditions. It will be used in the derivation of Proposition 2.1.3. Let $\Lambda_N^t(\varepsilon)$ be the parallelepiped,

$$\Lambda_N^t(\varepsilon) = \left\{ i \in \Lambda_N(\varepsilon) \mid i \cdot n = \left\lfloor -\frac{E}{4N}, \frac{E}{4N} \right\rfloor \right\},$$

(2.4.2)

and denote by $\partial^{top} \Lambda_N^t(\varepsilon)$ [resp. $\partial^{bot} \Lambda_N^t(\varepsilon)$] the face of $\partial^+ \Lambda_N^t(\varepsilon)$ [resp. $\partial^- \Lambda_N^t(\varepsilon)$] orthogonal to $n$. Let $\{\partial^{top} \Lambda_N^t(\varepsilon), \partial^{bot} \Lambda_N^t(\varepsilon)\}$ be the event such that there is no open path inside $\Lambda_N(\varepsilon)$ connecting $\partial^{top} \Lambda_N^t(\varepsilon)$ to $\partial^{bot} \Lambda_N^t(\varepsilon)$. One has the following.

**Lemma 2.4.1:** [Ref. 34, $\beta > 1$; Ref. 35, $\beta \in \mathbb{B}_p$. For any $\beta \in \mathbb{B}_p$,

$$\tau_{\beta}(n) = -\frac{1}{Nd^{-1}} \log \Phi^{\partial^{top} \Lambda_N(\varepsilon), \partial^{bot} \Lambda_N(\varepsilon)}(\{\partial^{top} \Lambda_N^t(\varepsilon), \partial^{bot} \Lambda_N^t(\varepsilon)\}) + c_{\varepsilon,N}(\pi),$$

(2.4.3)

where the function $c_{\varepsilon,N}$ goes to 0 as $N$ tends to infinity and $\varepsilon$ goes to 0, uniformly over the boundary conditions $\pi$ and $n \in \mathbb{S}^{d-1}$.

As it will be explained in Sec. IV on the wetting phenomenon, the system is, in fact, extremely sensitive to boundary conditions. Nevertheless, in the above lemma, the interface is constrained to be in $\Lambda_N^t(\varepsilon)$, so that it does not feel the influence of the boundary; the boundary conditions are screened because the system relaxes to equilibrium in the region $\Lambda_N(\varepsilon) \setminus \Lambda_N^t(\varepsilon)$.

Let us first examine the influence of the boundary conditions $\pi$ on the faces of $\Lambda_N(\varepsilon)$ orthogonal to $n$. As $\{\partial^{top} \Lambda_N^t(\varepsilon), \partial^{bot} \Lambda_N^t(\varepsilon)\}$ is a decreasing event, the FKG inequality implies that it is enough to check that

$$\tau_{\beta}(n) = \lim_{N \to \infty} -\frac{1}{Nd^{-1}} \log \Phi^{\partial^{top} \Lambda_N(\varepsilon), \partial^{bot} \Lambda_N(\varepsilon)}(\{\partial^+ \Lambda_N^t(\varepsilon), \partial^- \Lambda_N^t(\varepsilon)\}),$$

(2.4.4)

where $\Phi^{\partial^{top} \Lambda_N(\varepsilon), \partial^{bot} \Lambda_N(\varepsilon)}$ is the FK measure with free boundary conditions on the faces orthogonal to $n$ and wired on the others. This can be proved by means of a Peierls argument for $\beta$ large enough or by an analysis of the relaxation of the clusters density for $\beta$ in $\mathbb{B}_p$.

As already noticed in Ref. 58 in the context of percolation, the influence of the boundary conditions on the sides of $\Lambda_N(\varepsilon)$ parallel to $n$ is negligible as $\varepsilon$ goes to 0. This explains that the factor $c_{\varepsilon,N}(\cdot)$ vanishes uniformly over the boundary conditions.

**2. Extended representation**

We would like to stress that the previous treatment of the surface tension is not satisfactory and a more coherent approach would be to consider a more general definition independent of the model in terms only of mesoscopic phase labels. In fact, a definition of surface tension valid in an abstract setting would be difficult to use because the surgical procedure of the minimal section argument requires a precise knowledge of how the microscopic system is related to the mesoscopic phase labels.

**E. Lower bound: Proposition 2.1.2**

The proof is divided into three steps. We first start by approximating the surface $\partial^\delta K_m$ by a regular surface $\partial \hat{K}$ and imposing the condition that a mesoscopic interface exists close to $\partial \hat{K}$. Then, using the definition of surface tension (2.4.1), we derive Proposition 2.1.2.

**1. Step 1: Approximation procedure**

A polyhedral set has a boundary included in the union of a finite number of hyperplanes. The surface $\partial^\delta K_m$ can be approximated as follows (see Fig. 6).

**Theorem 2.5.1:** For any $\delta$ positive, there exists a polyhedral set $\hat{K}$ such that
For any \( h \) small enough there are \( l \) disjoint parallelepipeds \( \hat{R}^1, \ldots, \hat{R}^l \) with basis \( \hat{B}^1, \ldots, \hat{B}^l \) included in \( \partial K \) of side length \( h \) and height \( \delta h \). Furthermore, the sets \( \hat{B}^1, \ldots, \hat{B}^l \) cover \( \partial K \) up to a set of measure less than \( \delta \) denoted by \( \hat{U} \). The proof is a direct application of Reshtnyak's theorem and can be found in the paper of Alberti and Bellettini.  

Using Theorem 2.5.1, we can reduce the proof of Proposition 2.1.2 to the computation of the probability of \( M_k/m^* \in \mathcal{V}(1_k, \delta) \). According to (2.2.6) the estimates can be restated in terms of the mesoscopic phase labels. For any \( \delta > 0 \), there exists \( \zeta = \zeta(\delta) \) and \( k_0 = k_0(\delta) \) such that Proposition 2.1.2 will be implied by

\[
\lim inf_{N \to \infty} \frac{1}{N^{d-1}} \min_{k(\delta) \in k \leq k_0} \log P_N(\mathbf{u}_k^* \in \mathcal{V}(1_k, \delta)) \geq -W_P(\hat{K}) - o(\delta). \tag{2.5.1}
\]

2. **Step 2: Localization of the interface**

The images of \( \hat{K}, \hat{R}^i, \) and \( \hat{U} \) in \( T_N \) will be denoted by \( K_N, R_N^i, \) and \( U_N^\delta \). In order to enforce a mesoscopic interface across each \( R_N^i \), we define the event

\[
A = \bigcap_{i=1}^l \{ \delta^+ R_N^i \cap \delta^- R_N^i \}. \nonumber
\]

We consider also \( B \) the set of configurations such that the bonds at a distance less than 10 of \( U_N^\delta \) are closed. Notice that these events depend only on bonds variables. One has

\[
P_N(\mathbf{u}_k^* \in \mathcal{V}(1_k^*, \delta)) \geq P_N(\{ u_k^* \in \mathcal{V}(1_k^*, \delta) \} \cap A \cap B). \tag{2.5.2}
\]

The interface imposed by the event \( A \cap B \) decouples \( K_N \) from its complement, therefore the system is in equilibrium in \( K_N \) and \( K_N^c \): a proof similar to the one of Theorem 2.2.1 implies that one can choose \( \zeta' = \zeta'(\delta) \) and \( k_0' = k_0'(\delta) \) such that
where \( H^0 \) stands for \( H/\delta \). Of the surface tension factor.

\[ \text{copic level via the minimal section argument.} \]

\[ \text{Finally, the last step is devoted to the computation of surface tension factors in small regions.} \]

\[ \text{Then in each region we localize the interface on the mesoscopic scale.} \]

\[ \text{We approximate the interface on the mesoscopic scale.} \]

\[ \text{The proof is divided into three steps.} \]

\[ \text{First we decompose the event \( B \).} \]

\[ \text{We have also used the fact that the event \( B \) is supported by, at most, \( c(d, \delta) N^{d-1} \) edges, where \( c(d, \delta) \) vanishes as \( \delta \) goes to 0. Therefore the probability of \( B \) is negligible with respect to a surface order.} \]

\[ \text{F. Upper bound: Proposition 2.1.3} \]

\[ \text{The proof is divided into three steps.} \]

\[ \text{First we decompose \( \partial^* v \) in order to reduce the proof to local computations in small regions.} \]

\[ \text{Then in each region we localize the interface on the mesoscopic scale.} \]

\[ \text{Finally, the last step is devoted to the computation of the surface tension factor.} \]

\[ \text{1. Step 1: Approximation procedure} \]

\[ \text{We approximate \( \partial^* v \) with a finite number of parallelepipeds (see Fig. 7).} \]

\[ \text{Theorem 2.6.1: For any \( \delta \) positive, there exists \( h \) positive such that there are \( l \) disjoint parallelepipeds \( \hat{R}^1, \ldots, \hat{R}^l \) included in \( \mathbb{R}^d \) with basis \( \hat{B}^1, \ldots, \hat{B}^l \) of size \( h \) and height \( \delta h \).} \]

\[ \text{The basis \( \hat{B}^i \) divides \( \hat{R}^i \) in two parallelepipeds \( \hat{R}^{i,+} \) and \( \hat{R}^{i,-} \) and we denote by \( n_i \) the normal to \( \hat{B}^i \). Furthermore, the parallelepipeds satisfy the following properties:} \]

\[ \int_{\hat{R}^i} |\mathcal{X}_{\hat{B}^i}(x) - v(x)| \, dx \leq \delta \, \text{vol}(\hat{R}^i) \quad \text{and} \quad \left| \sum_{i=1}^l \int_{\hat{B}^i} \tau_{\beta}(n_i) \, d\mathcal{H}^{d-1} - \mathcal{W}_\beta(v) \right| \leq \delta, \]

\[ \text{where} \quad \mathcal{X}_{\hat{B}^i} = 1_{\hat{R}^{i,+}} - 1_{\hat{R}^{i,-}} \quad \text{and} \quad \text{the volume of} \quad \hat{R}^i \quad \text{is} \quad \text{vol}(\hat{R}^i) = \delta h^d. \]
This theorem is a rather standard assertion of the geometric measure theory. A variation of it has been formulated and applied in the context of the \( L_1 \)-theory of phase segregation in Ref. \( 52 \) along with a sketch of the proof, which, however, contained a gap (see Ref. \( 34 \) for a detailed proof along the lines of Ref. \( 52 \)). A very clean alternative derivation of a similar result has been given by Cerf\( ^{55} \) using the Vitali covering theorem.

Theorem 2.6.1 enables us to decompose the boundary into regular sets (see Fig. \( 7 \)) so that it will be enough to consider events of the type

\[
\left\{ \frac{M_k}{m^n} \in \bigcap_{i=1}^l \mathcal{V}(\tilde{R}^i, \delta \text{ vol}(\tilde{R}^i)) \right\},
\]

where \( \mathcal{V}(\tilde{R}^i, \varepsilon) \) is the \( \varepsilon \) neighborhood of \( X_{\tilde{R}^i} \),

\[
\mathcal{V}(\tilde{R}^i, \varepsilon) = \left\{ v' \in L^1(\mathbb{R}^d) \left| \int_{\tilde{R}^i} |v'(x) - X_{\tilde{R}^i}(x)| \, dx \leq \varepsilon \right. \right\}.
\]

Using (2.2.6), we see that to derive Proposition 2.1.3, it is equivalent to prove the following statement for any \( \delta \) positive and \( k_0 = k_0(\delta) \), \( \zeta = \zeta(\delta) \),

\[
\limsup_{N \to \infty} \frac{1}{N^d-1} \max_{k_0(\delta) \leq k \leq \varepsilon N} \log P_N \left( u^N_k \in \bigcap_{i=1}^l \mathcal{V}(\tilde{R}^i, \delta \text{ vol}(\tilde{R}^i)) \right) \leq -\mathcal{W}_\#(v) + C(\beta, v) \delta.
\]

2. Step 2: Minimal section argument

The microscopic domain associated to \( \tilde{R}^i \) is \( R_N^i = \tilde{R}^i \cap \mathbb{T}_N \). We also set \( R_N^{i,+} = R_N^i \cap \mathbb{T}_N^+ \) and \( R_N^{i,-} = R_N^i \cap \mathbb{T}_N^- \). At the scale \( M = 2^k \), we associate to any configuration \( (\sigma, \omega) \) the set of \( \text{bad} \) boxes that are the boxes \( B_M^i \) intersecting \( R_N \) labeled by \( 0 \) and the ones intersecting \( R_N^{i,+} \) (resp. \( R_N^{i,-} \)) labeled by \( -1 \) (resp. \( 1 \)). Let \( c_d \) be a constant depending on the dimension. For any integer \( j \), we set \( \tilde{B}_N^{i,j} = \tilde{B}^i + j c_d 2^{-n+j} \mathbb{N}^n \), and define

\[
B_N^{i,j} = \{ j' \in R_N^i \mid \exists x \in \tilde{B}_N^{i,j}, \| j' - N x \|_1 \leq 10 \}.
\]

Let \( B^{i}_N \) be the smallest connected set of boxes \( B_M^i \) intersecting \( B_N^{i,j} \). For \( c_d \) large enough, the \( B^{i}_N \) are disjoint surfaces of boxes. For \( j \) positive, let \( n^+_j(j) \) be the number of \( \text{bad} \) boxes in \( B^{i}_N \) and define

\[
n^+_j(j) = \min \left\{ n^+_j(j) : 0 < j < \frac{\delta h}{2 c_d 2^{n-k}} \right\}.
\]

Call \( j^+ \) the smallest location where the minimum is achieved and define the minimal section in \( R_N^{i,+} \) as \( B_N^{i,j^+} \). For \( j \) negative, we denote by \( B_N^{i,-} \) the minimal section in \( R_N^{i,-} \) and \( n^-_j \) the number of \( \text{bad} \) boxes in \( B_N^{i,-} \) (see Fig. \( 8 \)).

For any configuration \( (\sigma, \omega) \) such that \( u^N_k(\sigma, \omega) \) belongs to

\[
\bigcap_{i=1}^l \mathcal{V}(\tilde{R}^i, \delta \text{ vol}(\tilde{R}^i)),
\]

one can bound the number of \( \text{bad} \) boxes in the minimal sections by

\[
\sum_{i=1}^l n^+_i + n^-_i = \delta C_1(\nu) 2^{(d-1)(n-k)}.
\]

Such an estimate implies that a mesoscopic interface is mainly located between the two minimal sections and that only some mesoscopic fingers attached to the interface may percolate. As these
fingers will cross the minimal sections through bad boxes, the strategy is therefore to modify the configuration $\omega$ on the bad boxes so that no fingers can percolate in the new configuration. More precisely, we introduce the set $A = \{ \omega \in \Omega_N^{\text{per}} \mid \exists \sigma \text{ such that } u_\omega^i(\sigma, \omega) \in \bigcap_{i=1}^l \nu(\hat{R}^i, \delta \text{ vol}(\hat{R}^i)) \}$, and for any $\omega$ in $A$ define $\tilde{\omega}$ the configuration with closed edges on the boundary of the bad blocks in the minimal sections and equal to $\omega$ otherwise. Inequality (2.6.1) implies that $\omega$ and $\tilde{\omega}$ differ only on, at most $C_3^N(v) N^{d-1}$ edges, so that we can control precisely the cost of the surgical procedure that consists in isolating the bad blocks in the minimal sections by closing the edges around them:

$$P_N\left( u_\omega^i(\sigma, \omega) \in \bigcap_{i=1}^l \nu(\hat{R}^i, \delta \text{ vol}(\hat{R}^i)) \right) \leq \Phi_N^{\text{per}, \beta}(A) \leq \exp(\delta C_3(v, \beta) N^{d-1}) \Phi_N^{\text{per}, \beta}(\tilde{A}),$$

(2.6.2)

where $\tilde{A} = \{ \tilde{\omega} | \omega \in A \}$.

3. Step 3: Surface tension estimates

Let $\hat{R}_i$ be the parallelepiped included in $\hat{R}^i$ with basis $\hat{B}^i$ and height $(\delta/2)h$. Its microscopic counterpart is $R_i$. We are going to check now that $\tilde{A}$ is included in $\cap_{i=1}^l \{ \hat{\partial}^{\text{per}} R^i_N \implies \hat{\partial}^{\text{per}} R^i_N \}$. This amounts to saying that not only the minimal section argument enables us to find a mesoscopic interface in $R_i$ but that, in fact, this interface exists on the microscopic level. To see this, choose any configuration $\omega$ in $A$ that contains an open path $C$ joining $\hat{\partial}^{\text{per}} R^i_N \implies \hat{\partial}^{\text{per}} R^i_N$ and suppose that $C$ crosses the minimal sections without intersecting a bad box. Then $C$ intersects two regular boxes $B_M(2^x)$ and $B_M(2^x)$ in $\hat{B}^i$ and $\hat{B}^i$. According to the definition of the coarse graining, this would imply that the crossing clusters of $B_M(2^x)$ and $B_M(2^x)$ are connected to $C$, so that $\tilde{u}_\omega^i(\pm) = \tilde{u}_\omega^i(\pm)$. Therefore one of these boxes has to be a bad box.

From (2.6.2), we get
\[
P_N \left( \bigcap_{i=1}^{I} \nu(\tilde{R}^i, \delta \text{vol}(\tilde{R}^i)) \right) \leq \exp(\delta C_3(\nu, \beta) N^{d-1}) \Phi^\text{per,\beta}_N \left( \bigcap_{i=1}^{I} \nu(\tilde{R}^i, \delta \text{vol}(\tilde{R}^i)) \right).
\]

Conditioning outside each domain \( R^i_N \) and using (2.4.3), we derive
\[
\limsup_{N \to \infty} \frac{1}{K^{d-1}} \max_{k \in \{0,2\}} \log \left( \bigcap_{i=1}^{I} \nu(\tilde{R}^i, \delta \text{vol}(\tilde{R}^i)) \right) \leq -\sum_{i=1}^{I} \int_{\tilde{R}^i} \tau_d(n) d\mathcal{H}_\alpha + C_d(\beta, \nu) \delta.
\]
This concludes the proposition.

G. Open problems

We would like mention some open questions related to the \( L_1 \)-theory.

(1) Extention of the \( L_1 \)-theory to general finite range models and to the context of Pirogov–Sinai theory.

(2) Proof of the Wulff construction for continuum models in an \( L_1 \) setting.

(3) Upgrade of the concentration properties to the Hausdorff distance, based on more delicate versions of the minimal section argument; some results of this sort should appear in Ref. 11.

(4) A more challenging problem would be to provide an accurate description of phase segregation à la DKS. In particular, one should understand how to control phase boundaries and prove local limit results with boundary conditions that are only statistically pure.

III. DOBRUSHIN–KOTECKÝ–SHLOSMAN (DKS) THEORY IN TWO DIMENSIONS

In this part we review and explain the results on phase separation in the two-dimensional nearest neighbor Ising model as enforced by the canonical constraint on the magnetization.9,32 The theory is built upon sharp local estimates over finite volume vessels \( \Lambda_N \) and on the probabilistic analysis of the random microscopic phase separation line. We focus here on the “free” spatial geometry of the phase segregation, that is, disregarding the boundary effects. These effects could enter the picture in two different ways: in terms of the boundary conditions on \( \partial \Lambda_N \) and in terms of the geometry of \( \Lambda_N \). In the former case the minority phase could be absorbed by part of the boundary \( \partial \Lambda_N \). This and related phenomena are discussed in Sec. IV. In the second case the finite vessel \( \Lambda_N \) might not be able to accommodate the corresponding optimal crystal shape. Such a geometric constraint is, from the point of view of the microscopic theory, merely a technical nuisance, though, on the macroscopic level, it might lead to formidable variational problems. We go around this domain geometry issue by choosing \( \Lambda_N \) to be of the Wulff shape itself,
\[
\Lambda_N = N K_1 \cap \mathbb{Z}^2,
\]
where \( K_1 \) is the unit area Wulff shape. Thus, \( \Lambda_N \) accommodates any optimal shape of area smaller than \( N^2 \).

The corresponding finite volume canonical Gibbs measure is then defined by
\[
\mu^\beta_{N,-}(\cdot|\Sigma_{i=1}^{\Lambda_N} \sigma_i) = -N^2 m^* + a_N,
\]
where \( \Sigma_{i=1}^{\Lambda_N} \sigma_i \) is the total spin, \( m^* = m^*(\beta) \) is the spontaneous magnetization, and \( a_N \) points inside the phase transition region, \( a_N \in (0, 2N^2 m^*) \). In the sequel we shall use the shortcut \( \mu^\beta_{N,-} \) for the finite volume measure \( \mu^\beta_{N,-} \).

Notation: The values of positive constants \( c_1, c_2, \ldots \), are updated with each section.
A. Main result

DKS theory gives a comprehensive solution to the following problem of phase separation.

Problem 1: For $\beta > \beta_c$ and $a_N \in (0, 2N^2m^*)$ characterize typical spin configurations $\sigma$ under the canonical measure (3.0.3).

An ostensibly simpler problem is the following.

Problem 2: For $\beta > \beta_c$ and $a_N \in (0, 2N^2m^*)$ find sharp local asymptotics of

$$\mu^\beta_N(\{M_N = -m^*N^2 + a_N\}).$$

In fact, both problems are equivalent. In particular, the phenomenon behind the shift of the magnetization is inside the phase transition region not a bulk one (and hence is not in the realm of the usual theory of large deviations), and the crucial role is played by the spatial geometry of symmetry breaking.

1. Heuristics

Under the finite volume pure state $\mu^\beta_N$, the typical maximal size of $\pm$ contours is of order $\log N$. One could then visualize a typical microscopic configuration $\sigma$ on $\Lambda_N$ in terms of an archipelago of small (that is, of the maximal size $\sim \log N$) \textquoteleft\textquoteleft plus\textquoteright\textquoteright\ islands that could contain smaller \textquoteleft\textquoteleft plus\textquoteright\textquoteright\ lakes, etc. This archipelago spreads out uniformly over $\Lambda_N$, and the density of the plus \textquoteleft\textquoteleft soil\textquoteright\textquoteright\ that spells out in terms of the magnetization $M_N(\sigma)$ as $\langle|\Lambda_N| + M_N(\sigma)\rangle/2|\Lambda_N|$, is close to its equilibrium value,

$$\frac{|\Lambda_N| + \langle M_N \rangle^\beta_{N,-}}{2|\Lambda_N|} \sim 1 - m^*.$$

Thus, one could think of two different competing patterns behind the $a_N$-shifts, $a_N \approx 0$, of the magnetization $M_N$ from its equilibrium value $\langle M_N \rangle^\beta_{N,-} \sim -m^*|\Lambda_N|$

1. The density of the archipelago increases in a spatially homogeneous fashion without, however, altering the typical sizes of the islands.
2. Spatial symmetry is broken, and an abnormally huge island of the \textquoteleft\textquoteleft plus\textquoteright\textquoteright\ phase of excess area $\sim a_N/2m^*$ appears.

Heuristically, the first scenario corresponds to Gaussian fluctuations, and its price, in terms of probability, should be of order

$$\exp(-c_1(\beta)a_N^2/N^2).$$

Phase segregation manifests itself in the second scenario, and the probabilistic price for creating such a huge island is proportional to the length of its boundary,

$$\exp(-c_2(\beta)\sqrt{a_N}).$$

A comparison between the two expressions above suggests that the first scenario should be prefered whenever $a_N \ll N^{4/3}$, whereas large shifts $a_N \approx N^{4/3}$ should result in the phase segregation picture described in the second scenario. This indeed happens to be the case, and we refer to Refs. 75 and 32 for a complete rigorous treatment. (The critical case of $a_N \sim N^{4/3}$ is still an open problem.)

For the sake of the exposition, we shall stick here to the possibly most interesting case of $a_N \sim N^{3/2}$, which corresponds also to the microscopic type of scaling discussed in Sec. II. The DKS theory gives then the following sharp characterization of the phase segregation in the canonical ensemble: under $\mu^\beta_N(\{M_N = -m^*N^2 + a_N\})$ a typical spin configuration $\sigma$ contains exactly one abnormally large contour $\gamma$ that decouples between the \textquoteleft\textquoteleft plus\textquoteright\textquoteright\ phase (inside $\gamma$) and the \textquoteleft\textquoteleft minus\textquoteright\textquoteright\ phase...
(outside $\gamma$). In particular, the average magnetization inside (respectively, outside) $\gamma$ is close to $m^*$ (respectively, $-m^*$), and the area encircled by $\gamma$ can be thus recovered from the canonical constraint,

$$m^*|\text{int}(\gamma)| - m^*(N^2 - |\text{int}(\gamma)|) \approx -m^*N^2 + a_N \Rightarrow |\text{int}(\gamma)| \approx \frac{a_N}{2m^*}.$$ 

Under the scaling of $\Lambda_N$ by $1/N$, that is into the normalized continuous shape $K \subset \mathbb{R}^2$, the microscopic phase boundary $\gamma$ sharply concentrates around a shift of the Wulff shape of the corresponding scaled area $a_N/2m^*N^2$ (Fig. 9).

### 2. DKS theorem

More precisely, for any $r \in \mathbb{R}_+$ let $K_r$ to denote the Wulff shape of the area $r$. Also given a number $s \in \mathbb{R}_+$, let us say that a microscopic contour $\gamma$ is $s$-large, if $\text{diam}_{s}(\gamma) > s$.

**Theorem 3.1.1:** [Ref. 9; Ref. 32]. Let the inverse temperature $\beta > \beta_c$ be fixed, and let the sequence $\{a_N\}, -m^*N^2 + a_N \in \text{Range}(M_N)$, be such that the limit

$$a = \lim_{N \to \infty} \frac{a_N}{N^2} \in (0,2m^*(\beta))$$

exists. Then,

$$\log \mu^\beta_{N,N}(M_N = -m^*N^2 + a_N) = -\mathcal{W}_\beta(\partial K_{a_N/2m^*})(1 + O(N^{-1/2}\log N)).$$

Moreover, if $K = K(\beta)$ is large enough, with $\mu^\beta_{N,N}(|M_N = -N^2m^* + a_N)$ probability converging to 1 as $N \to \infty$, we have the following.

1. **There is exactly one $K(\beta)\log N$–large contour $\gamma$.**
2. **This $\gamma$ satisfies**

$$\min_{x} \frac{1}{N} d_H(x + \partial K_{a_N/2m^*}) \leq c_1(\beta)N^{-1/4}\sqrt{\log N}, \quad (3.1.1)$$

and

$$\min_{x} \frac{1}{N^2} \text{Area}(\text{int}(\gamma)(x + K_{a_N/2m^*})) \leq c_2(\beta)N^{-3/4}\sqrt{\log N}. \quad (3.1.2)$$

**Remark:** In the original monograph the corresponding results have been derived in the context of the Ising model with periodic boundary condition.
3. DKS theory

The DKS theory views the production of the event \(\{M_N = -m^* N^2 + a_N\}\) in terms of a two-step procedure: On the first stage a length scale \(s = s(N)\) is chosen, and all the microscopic \(s\)-large contours \((\gamma_1, \ldots, \gamma_n)\) are fixed. If the total area inside these \(s(N)\)-large contours is smaller than \(a_N/2m^*\), then the total magnetization \(M_N\) still has to be steered toward the imposed value \(M_N = -m^* N^2 + a_N\), but already under the constraint that all the \((\gamma_1, \ldots, \gamma_n)\) are \(s(N)\) small. The probability \(\mu_{\beta, s(N)}^N(\{M_N = -m^* N^2 + a_N\})\) reflects the price of the optical strategy along these lines.

We record the two steps of the DKS theory as follows.

1. Study the statistics of \(s(N)\)-large contours under \(\mu_{\beta, s(N)}^N\).
2. Give local limit estimates on the magnetization in the \(s(N)\)-restricted phases.

The introduction of \(s(N)\)-cutoffs leads to the separation of the length scales, which has a double impact on the problem: it sets up the stage for the renormalization analysis of microscopic phase boundaries, and it improves the control over the bulk magnetization inside the corresponding microscopic phase regions. Let us try to explain this in more detail: As far as the statistics of the \(s(N)\)-large contours is considered, we are interested in giving sharp estimates on the \(\mu_{\beta, s(N)}^N\)-probability of the events of the type

\[\{s(N)\text{-large contours of } \sigma \text{ encircle a certain prescribed area}\}\]

The point is that the contribution of any particular microscopic contour to the probability of such an event is negligible. In other words, one also has to take into account the entropy (number) of all the contributing contours. The required entropy cancellation (and hence the production of the relevant limiting thermodynamic quantity—surface tension) is achieved by means of a certain coarse graining procedure, the so-called skeleton calculus, which we describe in Sec. III D. Roughly, instead of studying the probabilities of individual microscopic contours one considers the packets of all contours passing through the vertices of a given \(''s(N)\text{-skeleton}''\) \(S = (u_1, u_2, \ldots, u_n)\) and staying within a distance of the order \(s(N)\) from the closed polygonal line \(\text{Pol}(S)\) (Fig. 10). The distance between successive vertices of \(S\) complies with the length scale \(s(N)\). \(\|u_{i+1} - u_i\|_\infty \sim s(N)\). Surface tension is produced on the level of skeletons. In fact, the probability of observing a \(\sigma\) contour compatible with a given skeleton \(S\) admits an asymptotic (with \(s(N) \rightarrow \infty\) description,

\[\mu_{\beta, s(N)}^N(S) \sim \exp\{-\gamma_{\beta, s(N)}(\text{Pol}(S))\}\]  

(3.1.3)

We quote the precise result in Sec. III D, which we devote to a general exposition of the skeleton calculus.

FIG. 10. Two microscopic contours \(\gamma_1\) and \(\gamma_2\) are compatible with the same skeleton \(S = (u_1, \ldots, u_n)\).
Since the vertices of $S$ are $s(N)$ apart, and the surface tension $\tau_\beta$ is strictly positive for all $\beta > \beta^c_\gamma$, the energy $W_\beta(\text{Pol}(S))$ controls the number $\#(S)$ of vertices of $S$ as

$$
\#(S) \leq c(\beta) \frac{W_\beta(\text{Pol}(S))}{s(N)}. \quad (3.1.4)
$$

When combined with (3.1.3) this leads to the reduction of the combinatorial complexity of the problem: the number of different skeletons of a fixed energy $\overline{W}_N$ does not compete with the approximate probability $\exp\{-\overline{W}_N\}$ to observe any such skeletons. Thus, the study of $\{M_N = -m^* N^2 + a_N\}$ reduces, in terms of skeletons, to the maximal term estimation. It should be stressed, however, that unlike the coarse graining procedures of the L1-theory, the mesoscopic objects (skeletons) of the DKS theory closely follow the microscopic structure of phase boundaries.

The local limit estimates in the $s(N)$-restricted phases are, therefore, required uniformly over finite lattice domains whose boundaries are carved with $s(N)$-large contours compatible with not too costly skeletons. This imposes a natural restriction on the length of these boundaries, and we shall describe the appropriate family of domains in Sec. III B along with the exposition of the corresponding uniform local limit results. Intuitively, long contours are responsible for long range dependencies between spins, and, therefore, the $s(N)$-cutoff constraint improves the mixing properties of the system and helps to extend the validity of classical (Gaussian) behavior of moderate deviations. In Sec. III C we quote the corresponding relaxation and decay properties that lie in the heart of the local limit estimates. In Sec. III E we give an outline of the proof of the DKS theorem.

Finally, the (long) list of open problems is briefly addressed in Sec. III F.

**B. Estimates in the phases of small contours**

As it has been mentioned, the estimates in the phase of small contours should be derived uniformly over a family of lattice domains whose boundaries are composed of not too costly $s(N)$-large contours.

**Definition:** Basic family $D_N$ of subsets $A \subseteq \Lambda_N$: We fix two numbers $a$ (small) and $R$ (big):

$$
A \in D_N \iff a N^2 \leq |A| \quad \text{and} \quad |\partial A| \leq R N \log N.
$$

We fix a basic scale $s(N) = K \log N$ of large contours, where $K = K(\beta)$ is a sufficiently large number, so that $K \log N$ contours are highly improbable under the pure state $\mu^\beta_{\Lambda_{-}}$. Of course, exactly the same number $K$ appears in the statement of Theorem 3.1.1. The upper bound on $|\partial A|$ in the definition of the family $D_N$ states that the configurations with a total length of $K \log N$ large contour exceeding $RN \log N$ are ruled out. This conclusion is explained in more detail in Sec. III D (see the remark following Lemma 3.4.1).

**1. Structure of local limit estimates**

Let us turn now to the structure of local limit estimates in the $s(N)$-restricted phases. First of all, given any $A \subseteq \mathbb{Z}^2$, the $s$-restricted phase on $A$ is defined via

$$
\mu^\beta_{\Lambda_{-}}(\cdot) \equiv \mu^\beta_{\Lambda_{-}}(\cdot | \text{All } \pm \text{ contours are } s\text{-small}).
$$

We would like to study the probabilities of deviations $a_N \geq 0$ of the total magnetization $M_A$ from the corresponding averaged value $(M_A)_{\Lambda_{-}}^{\beta_{\Lambda_{-}}}$. Let us define the set of feasible values of such deviations as

$$
M_A^+ = \{a_N \geq 0: (M_A)_{\Lambda_{-}}^{\beta_{\Lambda_{-}}} + a_N \in \text{Range}(M_A)\}.
$$

Roughly, the cutoff $s$ extends the validity of Gaussian moderate deviations for the following reason: The price of shifting the magnetization by $a_N$ on the expense of $s(N)$-small contours is of
the order \((a_N/s^2)s \sim a_N/s(N)\). This should be tested against the Gaussian moderate deviation exponent of the order \(a_N^2/N^2\). Thus, the Gaussian behavior should prevail once \(a_N \approx N^2/s(N)\). Of course, the latter constraint on \(a_N\) becomes less stringent as \(s(N)\) decreases. On the rigorous mathematical part the classical approach to estimating

\[ \mu^\beta_s \left( M_A = \langle M_A \rangle^\beta_s + a_N \right), \]

amounts to first finding the value of the magnetic field,

\[ g = g(A, s(N), a_N), \]

such that the expected magnetization under the \(g\)-tilted state is precisely what we want,

\[ \langle M_A \rangle^\beta_s = \langle M_A \rangle^\beta_s + a_N, \quad (3.2.1) \]

and, then, to rewrite the \(\mu^\beta_s\)-probability in terms of the \(\mu^\beta_s\),\(g\) one:

\[ \mu^\beta_s(M_A = (M_A)^\beta_s + a_N) = \exp\left\{-\left( (M_A)^\beta_s + a_N \right)^g + \log(e^{sM_A})^{\beta_s} \right\} \mu^\beta_s(M_A = (M_A)^\beta_s) \]

\[ = \exp\left\{- \int_0^g \int_r \langle M_A; M_A^\beta_s \rangle dh \, dr \right\} \mu^\beta_s(M_A = (M_A)^\beta_s)^g. \quad (3.2.2) \]

One then tries to derive sufficiently precise estimates on the semi-invariants of \(\mu^\beta_s\) and to prove a local CLT under \(\mu^\beta_s\). Thus, it is extremely important to understand how the magnetization \(\langle M_A \rangle^\beta_s\) and other semi-invariants of \(\mu^\beta_s\) change with the magnetic field \(g\) in the phase of \(s(N)\)-small contours.

Breaking of the classical limit behavior in the \(s(N)\)-restricted phase manifests itself by the jump of the magnetization that is related to the appearance of abnormally large \(\pm\) contours. Without cutoffs this jump occurs for \(g \sim 1/N\), and imposing the \(s(N)\) constraint would delay such a jump.\(^{31}\) It is easy to imagine what should be the critical order of the magnetic field \(g\), at which those large contours should start to be favored in the \(s\)-restricted phase: for \(a \approx \pm\) contour of the linear size \(s(N)\) one wins \(\sim s^2/g\) on the level of magnetization and loses \(\sim s\) on the level of surface energy. These two terms start to be comparable when \(sg \sim 1\). Therefore no particular deviation from the classical behavior should be expected as far as \(gs(N) \approx 1\). We refer to Ref. 32, where all these heuristic considerations have been made precise.

2. Basic local estimate on the \(K \log N\) scale

Actually,\(^{32}\) it is enough to consider only the basic \(K \log N\) scale.

Lemma 3.2.1 (Ref. 32): Assume that a sequence of numbers \(\{b_N\}\) satisfies

\[ \lim_{N \to \infty} \frac{b_N \log N}{N^2} = 0. \]

Then, on the basic scale \(s(N) = K \log N\), the estimate

\[ \mu^\beta_s(M_A = (M_A)^\beta_s + a_N) = \frac{1}{\sqrt{2\pi \chi(\beta) |A|}} \exp\left\{ - \frac{a_N^2}{2 \chi(\beta) |A|} + \mathcal{O}\left( \frac{a_N^2}{N^2} \log N \sqrt{\frac{a_N}{N}} \right) \right\} (1 + o(1)), \quad (3.2.3) \]

holds uniformly in domains \(A \in D_N\) and in \(a_N \in \mathbb{M}_+^s \cap [0, b_N]\), where \(\chi(\beta)\) is the susceptibility under the pure state \(\mu^\beta\).
3. Super-surface estimates in the restricted phases

 Moderate deviations on the intermediate scales $s(N) \gg \log N$ are, for the purposes of the theory, controlled by the following super-surface order estimate in the phase of small contours (cf. Lemma 2.5.1 in Ref. 32).

Lemma 3.2.2: Let the large contour parameter $s(N) \gg \log N$ be fixed. There exists a constant $c_1 = c_1(\beta) > 0$, such that for all $N > 0$, $A \in \mathcal{D}_N$ and all $a_N \in \mathbb{M}_a^+$,

$$
\mu_{a_N}^A = (M_A)^{a_N} \equiv \exp \left( -c_1 \frac{a_N}{N^2} \frac{a_N}{s(N)} \right).
$$

(3.2.4)

The idea of the proof is simple: either an area of order $a_N/2m^*$ is exhausted by the $K \log N$ large contours, which, in the $(\mathbb{M}_a^*)$-restricted phase, should have a surface tension price with the exponent of the order $a_N/s(N)$, or $K \log N$ large contours cover an area much less than $a_N/2m^*$, which means that the remaining deficit of the magnetization should be compensated in the basic $K \log N$ restricted phase, where we can use Lemma 3.2.1.

C. Bulk relaxation in pure phases

The term relaxation is used here in the equilibrium setting in order to describe the approximation of local finite volume statistics by the infinite volume ones. We successively describe the relaxation properties of pure "+" states with nonpositive and small positive magnetic fields and in the restricted phases of small contours.

1. Nonpositive magnetic fields $h \leq 0$

The crucial property of low-temperature pure phases could be stated as follows: Let us say that sites $i$ and $j$ are +*-neighbors if $|\{i=j\}|=1$. Given a spin configuration $\sigma$ on $\{-1,+1\}^Z$, let us say that sites $i$ and $j$ are +*-connected, if there exists a +*-connected chain of sites $i_1, \ldots, i_n, i_1 = i$ and $i_n = j$, such that $\sigma(i_k) = 1$ for every $k = 1, \ldots, n$.

Theorem 3.3.1: (Ref. 42) For every $\beta > \beta_\ast$, there exists $c_1 = c_1(\beta) > 0$, such that uniformly in subsets $A \subseteq \mathbb{Z}^2$, $i, j \in A$ and in magnetic fields $h \leq 0$,

$$
\mu_{A, n}^i \equiv \exp(-c_1(\beta)|i-j|). \quad \text{(3.3.1)}
$$

Remark: Of course, since $\{i \leftrightarrow j\}$ is a nondecreasing event, the uniformity follows from the FKG ordering, once (3.3.1) is verified for the infinite volume zero-field measure $\mu_\beta^0$.

Corollary 3.3.1: (Relaxation of local observables) Fix $k \in \mathbb{Z}$. Uniformly in $\mathbb{A} \subseteq \mathbb{Z}^2$, magnetic fields $h \leq 0$ and local observables $f$ with $|\supp(f)| = k$,

$$
|\langle f \rangle_{A, n}^i - \langle f \rangle_{A, n}^j| \equiv c_2(k) \exp(-c_3(\beta) \text{dist}_a(\supp(f), \partial A)). \quad \text{(3.3.2)}
$$

Furthermore, we have the following.

Corollary 3.3.2: (Relaxation and decay of semi-invariants) Fix $n \in \mathbb{Z}$. Uniformly in $\mathbb{A} \subseteq \mathbb{Z}^2$, magnetic fields $h \leq 0$ and sites $i_1, \ldots, i_n \in A$,

$$
|\langle \sigma(i_1) \ldots \sigma(i_n) \rangle_{A, n}^i - \langle \sigma(i_1) \ldots \sigma(i_n) \rangle_{A, n}^j| \equiv c_4(n) \exp(-c_5(\beta) \text{dist}_a(\{i_1, \ldots, i_n\}, \partial A)). \quad \text{(3.3.3)}
$$

and

$$
|\langle \sigma(i_1) \ldots \sigma(i_n) \rangle_{A, n}^i| \equiv c_6(n) \exp(-c_7(\beta) \frac{\text{diam}_a(i_1, \ldots, i_n)}{n}). \quad \text{(3.3.4)}
$$

Finally, we have the following.

Corollary 3.3.3: (Asymptotic expansions) Fix $n \in \mathbb{Z}$. Uniformly in $\mathbb{A} \subseteq \mathbb{Z}^2$ and in $i \in A$,
\[
\left( \sigma(i) \right)_{A,-h}^\beta = \left( -m^*(\beta) + \sum_{k=1}^n s_k^h \right) = c_5(n) |h|^{n+1} + c_6(n) \exp(-c_1(\beta) \text{dist}_A(i, \partial A)),
\]
(3.3.5)

where \( s_k \) is the \( k \)th semi-invariant of the zero-field infinite volume measure \( \mu^\beta \).

\[
\begin{align*}
s_k & \triangleq \sum_{i_1, \ldots, i_k \in \mathbb{Z}^2} \langle \sigma(0); \sigma(i_1); \ldots; \sigma(i_k) \rangle^\beta. 
\end{align*}
\]

Remark: It is possible (and straightforward) to formulate (3.3.3), (3.3.4), and (3.3.5) in the general case of \( n \) local observables \( f_1, \ldots, f_n \).

2. Positive magnetic fields \( h > 0 \)

Modifying ‘‘−’’ states by negative magnetic fields \( h < 0 \) amounts to moving away from the phase transition region. Relaxation properties of \( \mu_{A,-h}^\beta \) with \( h > 0 \) are radically different—uniformity is lost, and the size of the domain \( A \) starts to play a crucial role. Indeed, the unique infinite volume measure \( \mu_{\beta,h}^\beta = \mu_h^\beta \) stochastically dominates \( \mu_{\beta,h}^\beta \) whatever small \( h > 0 \) is. Thus, for large domains \( A \), the configuration in the bulk is flipped under \( \mu_{A,-h}^\beta \) into the ‘‘+’’ dominated state. It is easy to understand on the heuristic grounds what should be the order of the critical size of \( A \) for such a ‘‘flip’’ to occur: given \( h > 0 \), the surface energy of a ± contour \( \gamma \) is of the order \( |\gamma| \) and it competes with the bulk gain inside the contour, which, in its turn, is proportional to \( h \text{area}(\gamma) \). The latter factor wins (loses), once the linear size of \( \gamma \) is much larger (respectively, much smaller) than \( 1/h \). Thus the sign of the dominant spin under \( \mu_{A,-h}^\beta \) should depend on whether \( A \) can accommodate large enough contours, or, in other words, on how the linear size of \( A \) relates to \( 1/h \).

The important and remarkable fact is that exponential relaxation properties of finite volume ‘‘−’’ states are uniformly preserved for domains of the subcritical size.

**Theorem 3.3.2** (Refs. 31 and 32): There exists a constant \( a = a(\beta) > 0 \) such that for any \( h > 0 \) fixed,
\[
\mu_{A,-h}^\beta (i^{+*} \rightarrow j) \leq \exp(-c_1(\beta) \|i-j\|_\infty),
\]
(3.3.6)
uniformly in domains \( A \subset \mathbb{Z} \) such that any connected component of \( A \) has diameter bounded above by \( ah \). As a consequence, exponential decay of semi-invariants (3.3.4) and the asymptotic expansion estimate (3.3.5) hold uniformly in such domains as well.

3. Phases of small contours

Theorem 3.3.2 explains how the cutoff parameter \( s(N) \) upgrades the regular behavior of ‘‘−’’ states with positive magnetic fields \( h \): By the definition of the restricted phase \( \mu_{A,-h}^{\beta,s} \), the diameter of any relevant microscopic domain is, at most, of the order \( s(N) \).

**Theorem 3.3.3:** (Refs. 31 and 32) There exists a constant \( a = a(\beta) > 0 \) such that for any \( h > 0 \) and \( s \) satisfying \( hs \leq a(\beta) \),
\[
\mu_{A,-h}^{\beta,s} (i^{+*} \rightarrow j) \leq \exp(-c_1(\beta) \|i-j\|_\infty),
\]
(3.3.7)
uniformly in domains \( A \subset \mathbb{Z} \).

Furthermore, the expectations in restricted phase are controlled as follows: for every \( k \in \mathbb{Z} \),
\[
|\langle f \rangle_{A,-h}^{\beta,s} - \langle f \rangle_{A \setminus A_{\beta,0},-h}^{\beta}| \leq c_2(k) e^{-c_3(\beta)s},
\]
(3.3.8)
uniformly in $A \subseteq \mathbb{Z}^2$ and in local functions $f$, $|\text{supp}(f)| = k$, where we have used the following notation: $\Lambda_s(f) = \{ i : \text{dist}_*(i, \text{supp}(f)) \leq s \}$. Finally, the decay of the semi-invariants is controlled in the restricted phases as

$$\langle \sigma(i_1) : \ldots : \sigma(i_n) \rangle_{A_{s,h}} \leq c_s(n) \exp \left( - c_s(n) \frac{\text{diam}_s(i_1, \ldots, i_n)}{n} \right).$$

(3.3.9)

D. Calculus of skeletons

The renormalization analysis of large $\pm$ contours is performed on various cutoff scales $s$, the appropriate choice of $s$ typically depending on the linear size $N$ of the system $s = s(N)$. We shall state coarse graining estimates uniformly in finite domains $A \subseteq \mathbb{Z}^2$ and in the cutoff scales $s$.

1. Definition

A $\pm$ contour $\gamma$ is said to be $s$ large if $\text{diam}_s(\gamma) \geq s$. Given a cutoff scale $s \in \mathbb{N}$ and an $s$-large $\pm$ contour $\gamma$, we say that a collection $S = (u_1, \ldots, u_n)$ is an $s$-straight skeleton of $\gamma$, $\gamma \sim S$ if the following occurs.

1. All vertices of $S$ lie on $\gamma$.
2. $s / 2 \leq \| u_i - u_{i+1} \|_s \leq 2s$, $\forall i = 1, \ldots, n$, where we have identified $u_{n+1} = u_1$.
3. The Hausdorff distance $d_H$ between $\gamma$ and the polygonal line $\text{Pol}(S)$ through the vertices of $S$ satisfies

$$d_H(\gamma, \text{Pol}(S)) \leq s(N).$$

Similarly, given the collection $(\gamma_1, \ldots, \gamma_n)$ of all $s$-large contours of a configuration $\sigma \in \Omega_{A,-}$, let us say that a collection $\mathcal{S} = (S_1, \ldots, S_n)$ of $s$-large skeletons is compatible with $\sigma$, $\sigma \sim \mathcal{S}$, if $\gamma_i \sim S_i$ for all $i = 1, \ldots, n$.

Of course, a configuration $\sigma \in \Omega_{A,-}$ has, in general, many different compatible collections of $s$-sketches. Nonetheless, for each particular $\mathcal{S}$, the probability

$$\mu_{A,-}^\beta(\mathcal{S}) \equiv \mu_{A,-}^\beta(\sigma : \sigma \sim \mathcal{S}),$$

(3.4.1)

is well defined.

2. Energy estimate

As the renormalization scale $s$ grows, the probabilities (3.4.1) start to admit a sharp characterization in terms of the energies $W_{\beta}(\mathcal{S})$,

$$W_{\beta}(\mathcal{S}) \equiv \sum_{i=1}^{n} W_{\beta}(\text{Pol}(S_i)).$$

for a collection $\mathcal{S} = (S_1, \ldots, S_n)$. Below we give a precise version of this crucial statement in terms of the upper and lower bounds on the corresponding probabilities. The first important renormalization energy estimates could be formulated as follows.

Lemma 3.4.1: (Ref. 26) On every skeleton scale $s$ and independently of $A \subseteq \mathbb{Z}^2$,

$$\mu_{A,-}^\beta(\mathcal{S}) \equiv \exp \{ - W_{\beta}(\mathcal{S}) \}.$$  

(3.4.2)

Furthermore, uniformly in $A \subseteq \mathbb{Z}$, $r > 0$ and cutoff parameters $s$,

$$\mu_{A,-}^\beta(W_{\beta}(\mathcal{S}) \geq r) \equiv \exp \left( - r \left( 1 - \frac{c_1 \log |A|}{s} \right) \right).$$

(3.4.3)

Energy estimate (3.4.2) provides an upper bound on the probability of observing $\pm$ contours in the vicinity of a skeleton. Before going to a complementary lower bound, let us dwell on the sample path structure of the contours that is hidden behind these renormalization estimates.
3. Calculus of skeletons

By definition, a contour is a self-avoiding closed path of nearest neighbor bonds of $\mathbb{Z}^2$. For every set $A \subseteq \mathbb{Z}^2$, the Ising measure $\mu^\beta_{A,-}$ induces a weight function $q^\beta_{A,*}$ on the space of such self-avoiding polygons (see Sec. 1B 2),

$$q^\beta_{A,*}(\gamma) = \mu^\beta_{A,-}(\sigma \in \Omega: \gamma \text{ is a } \pm \text{ contour of } \sigma).$$

In terms of these weights the probability of observing a certain skeleton $S = \{ u_1, \ldots, u_n \}$ could be written as

$$\mu^\beta_{A,-}(S) = \sum_{\gamma \in S} q^\beta_{A,*}(\gamma).$$

Each microscopic contour $\gamma$ compatible with $S$, $\gamma \subseteq S$, splits into the union of disjoint open self-avoiding lattice paths $\gamma_k: u_k \to u_{k+1}$, $k = 1, \ldots, n$. The analysis of limit properties of $\mu^\beta_{A,-}$ comprises two main steps that could be loosely described as follows.

1. As the renormalization scale $s$ grows, the statistical behavior of different pieces $\gamma_k$ decouple under $q^\beta_{A,*}$, that is,

$$\sum_{\gamma \in S} q^\beta_{A,*}(\gamma) = \prod_{k=1}^n \left( \sum_{\gamma_k: u_k \to u_{k+1}} q^\beta_{A,*}(\gamma_k) \right). \quad (3.4.4)$$

2. The $k$th term ($k = 1, \ldots, n$) in the above product corresponds to a $\pm$ interface stretched in the direction of the vector $u_{k+1} - u_k \in \mathbb{R}^2$, in other words,

$$q^\beta_{A,*}(\gamma_k) = \exp(-\tau_\beta(u_{k+1} - u_k)). \quad (3.4.5)$$

Thus, the skeleton calculus resembles a refined version of the sample path large deviation principle for genuinely two-dimensional random curves. At very low temperatures, a very precise local analysis of the phase separation line has been developed in Refs. 9 and 75 using the method of cluster expansions. Our approach here pertains to the whole of the phase transition region $\beta > \beta_c$, but is strongly linked to the very specific self-duality properties of the two-dimensional nearest neighbor Ising model. We refer to Sec. 1B 2 and, eventually, to Refs. 18 and 19 for a comprehensive description and study of the relevant properties of the duality transformation. The output of these techniques could be recorded in the following form.

Lemma 3.4.2: (Probabilistic structure of the phase separation line)\(^{18}\) Given any $A \subseteq \mathbb{Z}^2$ and any two compatible self-avoiding paths $\lambda_1$ and $\lambda_2$,

$$q^\beta_{A,*}(\lambda_1 \cup \lambda_2) = q^\beta_{A,*}(\lambda_1) q^\beta_{A,*}(\lambda_2). \quad (3.4.6)$$

Furthermore,

$$\exp(-c_1(\beta) \mid \gamma_2) \leq \frac{q^\beta_{A,*}(\lambda_1 \cup \lambda_2)}{q^\beta_{A,*}(\lambda_1)} \leq \exp(-c_2(\beta) \mid \lambda_2). \quad (3.4.7)$$

On the other hand, given any $A \subseteq \mathbb{Z}^2$ and any three points $u, v, w \in A^\ast$, the $q^\beta_{A,*}$ weight of the paths going from $u$ to $v$ through $w$ is bounded above as\(^{18}\)

$$\sum_{w \in \lambda} q^\beta_{A,*}(\lambda) \leq \left( \sum_{\lambda_1: u \to v} q^\beta_{A,*}(\lambda_1) \right) \left( \sum_{\lambda_2: v \to w} q^\beta_{A,*}(\lambda_2) \right). \quad (3.4.8)$$
Finally, the weights \( q_{A^s}^\beta \) are nonincreasing in \( A \), and are related to the dual connectivities as

\[
\sum_{\lambda: u \to v} q_{A^s}^\beta(\lambda) = (\sigma(u) \sigma(v))_{A^s}^\beta.
\]  

(3.4.9)

Relation (3.4.9) is the link to the surface tension: first of all, the impact of a particular set \( A \) exponentially diminishes with the distance to \( \partial A \),

\[
(\sigma(u) \sigma(v))_{f}^\beta - \exp\{c_3(\beta) \text{dist}_u(\{u,v\}, \partial A)\} \leq (\sigma(u) \sigma(v))_{A^s}^\beta \leq (\sigma(u) \sigma(v))_{f}^\beta - \exp\{c_3(\beta) \log \|u-v\|_\infty\},
\]  

(3.4.10)

uniformly in \( A^s \subseteq \mathbb{Z}^2 \) and any \( u,v \in A^s \). Moreover, the following Ornstein–Zernike-type correction formula\(^{35}\) holds uniformly in \( u,v \in \mathbb{Z}^2 \):

\[
\exp\{ -\tau_\beta(u-v) - c_3(\beta) \log \|u-v\|_\infty\} \leq (\sigma(u) \sigma(v))_{f}^\beta \leq \exp\{ -\tau_\beta(u-v)\}.
\]  

(3.4.11)

4. Skeleton lower bound

The energy estimate (3.4.2) is an immediate consequence of the (iterated) submultiplicative property (3.4.8), the representation formula (3.4.9), and the rightmost inequalities in (3.4.10) and (3.4.11). In order to prove a lower bound, one essentially needs to reverse the inequality in (3.4.10). An indirect way to do so is to use the FK representation (see Refs. 45 and 32). We shall briefly present here a more direct approach that has been developed in Refs. 29 and 18. Qualitatively it gives the same order of corrections as the FK one, but has a clear advantage of being explicitly related to the statistics of the microscopic phase boundaries at different length scales. The basic idea is that the phase separation line has rather strong mixing properties, in particular, paths \( \lambda_1 \) and \( \lambda_2 \) on the right-hand side of (3.4.8) should interfere, in the case of \( (u,v,w) \) being in a general position, only in a vicinity of \( w \). Thus, at a price of lower-order corrections (as we shall see these corrections are logarithmic with the skeleton scale \( s \)) the inequality (3.4.8) could be reversed using the supermultiplicativity property (3.4.6). The notion of ”general position” simply means that \( u,w \), and \( v \) do not form too small an angle and live on the same length scale, and it is quantified by the following.

**Definition:** Given a skeleton scale \( s \in \mathbb{N} \) and a number \( \epsilon > 0 \), let us say that a triple \( (u,w,v) \) of \( \mathbb{Z}^2 \)-lattice points is \((s, \epsilon)\)-compatible, if

\[
\frac{s}{2} \leq \min\{\|w-u\|_\infty, \|v-w\|_\infty\} \leq \max\{\|w-u\|_\infty, \|v-w\|_\infty\} \leq 2s,
\]

whereas \( \cos(w-u,v-w) \geq -1 + \epsilon. \)

We shall state the lower bound in terms of the limiting weights \( q_{A^s}^\beta(\cdot) \equiv \lim_{s} \sum_{\lambda: u \to v} q_{A^s}^\beta(\lambda) \) (which exist by Lemma 3.4.2).

**Lemma 3.4.3:** Fix \( \epsilon > 0 \). Then there exists a scale \( s = s(\epsilon) \), such that

\[
\sum_{\lambda: u \to v \atop w \in \lambda} q_{A^s}^\beta(\lambda) \geq \exp\{ - (\tau_\beta(w-u) + \tau_\beta(v-w)) - c_1(\beta) \log s \},
\]  

(3.4.12)

uniformly in all skeleton scales \( s \geq s(\epsilon) \) and in all \((s, \epsilon)\)-compatible triples \((u,w,v)\).

We sketch the proof of this lemma in Appendix B. Iterating (3.4.12), we arrive at the following lower bound on the probability of observing a certain regular skeleton.

**Definition:** A skeleton \( S = \{ u_1, \ldots, u_n \} \) is said to be \((s, \epsilon)\)-regular, if any triple \((u_{i-1}, u_i, u_{i+1})\) of successive points of \( S \) is \((s, \epsilon)\)-compatible, and the distance between any two non-neighboring
intervals $[u_j, u_{j+1}]$ and $[u_j, u_{j+1}]$ exceeds $\epsilon$.

Lemma 3.4.4: For every $\epsilon > 0$, there exists a number $c_2 = c_2(\epsilon, \log s)$, such that uniformly in the skeleton scales $s$ and in all $(s, \delta)$-regular skeletons $S$,

$$
\mu_{\nu, -}^0(\exists a \pm \text{ contour } \gamma: d_H(\gamma, \text{Pol}(S)) \leq K(\beta) \sqrt{s} \log s) \geq \exp \left[ -W_\beta(\text{Pol}(S)) + c_2(\epsilon)\#(S)\log s \right].
$$

(3.4.13)

where $\#(S)$ denotes the number of vertices in $S$, and the last inequality follows from (3.1.4).

In fact, we need lower bounds only for a very specific set of $s$-skeletons, namely on those approximating the Wulff shape $K_\nu$.

E. Structure of the proof

In order to give a probabilistic characterization of the microscopic canonical state $\mu_{\nu, -}^0(M_N = -m^* N^2 + a_N)$ one first derives a sharpest possible lower bound on the probability $\mu_{\nu, -}^0(M_N = -m^* N^2 + a_N)$, and then rules out those geometric events (in terms of skeletons, but with an eventual translation to the language of microscopic spin configurations), which happen to qualify as improbable when compared with this lower bound.

1. Lower bound

The best lower bound comes as an outcome of the optimal combination of the basic local limit Lemma 3.2.1 and the skeleton lower bound (3.4.13). We choose a skeleton approximation of the corresponding Wulff shape $K_\nu$, and using local limit estimates steer the magnetization toward the desirable value $-m^* N^2 + a_N$. Optimality reflects the choice of the best possible skeleton scale: Notice that the estimate (3.4.13) becomes sharper with the growth of the cutoff parameter $s(N)$. On the other hand, the area of the microscopic phase region is controlled, with respect to the normal $\text{Pol}(S)$, up to a $\sqrt{s(N)} \log s(N)$ correction (see Appendix B or Ref. 32), which, of course, makes the local limit step more expensive for large values of $s(N)$. It happens that the bounds are balanced on the skeleton scale $s(N) \sim \sqrt{a_N}$.

Theorem 3.5.1: (Ref. 32) Uniformly in $a_N \in M^*_N$, that is for all $a_N \geq 0$, such that $-m^* N^2 + a_N \in \text{Range}(M_N)$,

$$
\mu_{\nu, -}^0(M_N = -m^* N^2 + a_N) \geq \exp \left[ -\sqrt{\frac{a_N}{2m^*}} W_\beta(\partial K_1) - c_1(\beta) \sqrt{a_N} \log N \right].
$$

(3.5.1)

2. Upper bounds

First of all, one derives an upper bound on the shift of the magnetization. On any skeleton scale,

$$
\mu_{\nu, -}^0(M_N = -m^* N^2 + a_N) \leq \sum_\delta \mu_{\nu, -}^0(M_N = -m^* N^2 + a_N : \delta).
$$

(3.5.2)

Due to the intrinsic entropy cancellation under the skeleton coarse graining, and in view of the lower bound (3.5.1) and the energy estimate (3.4.2), one could, for example, shoot for the maximal term in the above sum. If the phase volume (see Ref. 9 for the precise definition) of $\delta$ is much less than $\alpha_N/2m^*$, then the deficit of the magnetization should be compensated in the phase of $s(N)$-small contours, which, by Lemma 3.2.2, exerts a super-surface price in the exponent. On the other hand, if the phase volume of $\delta$ is close to $\alpha_N/2m^*$, then by the isoperimetric inequality and by the energy estimate (3.4.2), the best possible price one should be prepared to pay is already
close to $\exp\{-W_\beta(\mathcal{K}_{a_N/2m^*})\}$. Again the resulting estimate is subject to an optimization via a careful choice of the skeleton scale $s(N)$.

**Theorem 3.5.2:** (Ref. 32) Uniformly in $a_N \sim N^2$,

$$\mu_{N, -}^0(M_N = -m^* N^2 + a_N) \leq \exp\left\{-\frac{a_N}{2m^*}W_\beta(\partial \mathcal{K}) + c_1 (\beta)^{\frac{1}{4}} \overline{a_N \log N} \right\}. \quad (3.5.3)$$

A more delicate study of the typical sample properties of the microscopic configuration $\sigma$ under $\mu_{N, -}^0(M_N = -m^* N^2 + a_N)$ is again based on the analysis of (3.5.2). At this point the stability Bonnesen-type estimates (see Sec. 1C) for the Wulff variational problem become important—they enable us to quantify the conclusion that only those collections $\mathcal{S}$, which are close to the shifts of the Wulff shape $\mathcal{K}_{a_N/2m^*}$, have a chance to survive a comparison with the lower bound (3.5.1). A step further, involving local limit estimates of Lemma 3.2.1, is to conclude that all these collections actually contain exactly one large skeleton, which corresponds to the unique large contour, as asserted by the DKS theorem.

**F. Open problems**

There are still important open problems, even in the nearest neighbor Ising case. Notably, one knows how to control precise fluctuations of the phase separation line only at very low temperatures, that is, using the method of cluster expansions. This is a serious gap in the theory, since large scale statistics of microscopic phase boundaries are ultimately responsible for exact (up to zero order terms) expansions of canonical partition functions. So far, qualitative probabilistic results have been obtained either for very low-temperature models, or in the simplified setting of self-avoiding polygons or Bernoulli bond percolation. Another interesting and apparently important problem is to understand sample path properties of spin configurations in a situation when a canonical constraint is imposed in the restricted phase. Apart from giving rise to a potentially fascinating probabilistic structure, this question is closely related to the issue of the dynamical spinodal decomposition.

There is absolutely no matching probabilistic study of the phase separation in multiphase two-dimensional models, for example, $q$-states Potts models. Some results in this direction are reported in Ref. 79, but this issue is almost entirely open, even in the context of the $L_1$-theory. In particular, the corresponding phenomena are still not worked out on the level of macroscopic variational problems, see, however, Refs. 80, 81 and the references therein.

The key issue, however, which we feel is largely misunderstood is that at moderately low temperatures the DKS theory of two-dimensional phase segregation, say in the general context of finite range ferromagnetic models with pair interactions is far from being complete. What currently exists is an example of how these ideas could be implemented in the nearest neighbor case. At least from the mathematical point of view, the nearest neighbor case is a degenerate one, in a sense that it enables a reduction to pure boundary conditions over decoupled microscopic regions even at temperatures only moderately below critical. This should not be the case for a more general range of interactions. In this respect the assertion that low temperature expansions should go through for general interactions much along the same lines as they do for the nearest neighbor model, seems to be rather irrelevant—the real issue is not to kill mixed boundary conditions, but to understand how they should be incorporated into the DKS theory.

**IV. BOUNDARY EFFECTS**

In the previous parts, we explained how the thermodynamical variational problem describing the macroscopic geometry of coexisting phases can be derived in various lattice models of statistical physics. To simplify the analysis, we restricted our attention to periodic boundary conditions or to systems contained in a Wulff-shaped box, avoiding thus a discussion of the effect of a confining geometry on the behavior of the system. In this part, we would like to explain what happens when we take such effects into account. Boundary conditions play a particularly impor-
tant role in the kind of problems presented in this review, since they concern the asymptotic behavior of large but finite systems and therefore the boundary cannot be simply “sent to infinity,” as usually done. We will see that taking care of boundary effects not only provides a complete description of the geometry of these constrained systems, thus allowing a rigorous description of the interaction between an equilibrium crystal and a substrate, but also allows us to study the effect of so-called surface phase transitions.

For simplicity, we only discuss the case of the Ising model with nearest neighbors interaction.

A. Wall free energy

The vessel containing the system has not only the property of confining it, but can also act in an asymmetric way on the various phases inside, favoring some of them; indeed this is what happens typically in real systems. In fact, this is precisely the reason why one introduces boundary conditions in the first place: to impose the equilibrium phase the system realizes. It appears to be convenient to have a parameter allowing a fine tuning of the asymmetry, interpolating between pure + or − boundary conditions. Let us now describe how this is done.

Let \( \Sigma = \{ i \in \mathbb{Z}^d ; i(d) = 0 \} \) and \( \mathcal{L}^d = \{ i \in \mathbb{Z}^d ; i(d) = 0 \} \). The vessel of our system is the box,

\[
\mathcal{D}_{N,M} = \{ i \in \mathbb{Z}^d ; -N \leq i(n) \leq N, \; n = 1,...,d-1, \; 0 \leq i(d) \leq M \},
\]

and the wall is \( \Sigma_N = \mathcal{D}_{N,M} \cap \Sigma \).

Let \( \eta \in \mathbb{R} \); we consider the following Hamiltonian:

\[
\mathbf{H}^\eta_{\mathcal{D}_{N,M}} (\sigma) = - \sum_{\{i,j\} \subseteq \mathcal{L}^d} \sigma_i \sigma_j - \eta \sum_{i \in \Sigma} \sigma_i.
\]

Let \( \bar{\sigma} \in \{-1,1\}^{\mathcal{L}^d} \); the Gibbs measure in \( \mathcal{D}_{N,M} \) with boundary condition \( \bar{\sigma} \) is the following probability measure on \( \{-1,1\}^{\mathcal{D}^d} \),

\[
\mu^{\bar{\sigma}_{\mathcal{D}_{N,M}}} (\sigma) = \begin{cases} 
(\mathbf{Z}^{\bar{\sigma}_{\mathcal{D}_{N,M}}} \eta)^{-1} \exp \left[ - \beta \mathbf{H}^\eta_{\mathcal{D}_{N,M}} (\sigma) \right], & \text{if } \sigma_i = \bar{\sigma}_i, \; \forall i \notin \mathcal{D}_{N,M}, \\
0, & \text{otherwise}.
\end{cases}
\]

Remark: Note that we could equivalently consider \( \mu^{\bar{\sigma}_{\mathcal{D}_{N,M}}} \) as a probability measure on \( \{-1,1\}^{\mathcal{L}^d} \) by extending the b.c. \( \bar{\sigma} \) by \( \bar{\sigma}_i = 1 \) for all \( i \in \mathbb{Z}^d \setminus \mathcal{L}^d \); it is then possible to replace the boundary magnetic field \( \eta \) by a coupling constant: \( \eta \sum_{i \in \Sigma} \sigma_i = \eta \sum_{\{i,j\} \subseteq \mathcal{L}^d} \sigma_i \sigma_j \). This will be used when dealing with a negative boundary field; see Sec. IV D 1.

We will usually use the short-hand notations \( \mu^{\bar{\sigma}_{\mathcal{D}_{N,M}}} \), \( \mathbf{Z}^{\bar{\sigma}_{\mathcal{D}_{N,M}}} \), . . . . As usual, we write \( + \) for \( \bar{\sigma} = 1 \) and \( - \) for \( \bar{\sigma} = -1 \). We therefore distinguish one of the sides of the box \( \mathcal{D}_{N,M}, \Sigma_N \), which we call the “wall.” Notice that instead of usual boundary conditions, a boundary magnetic field \( \eta \) is acting on \( \Sigma_N \); since setting \( \eta = 1 \) produces \( + \) b.c. on the wall, while setting \( \eta = -1 \) results in \( - \) b.c., this provides the promised interpolation parameter. Of course, we could also consider more complicated situations, where (possibly inhomogeneous) boundary magnetic fields act on the whole boundary of the box. However, for simplicity, we restrict our attention to this particular case, which will turn out to be general enough that the basic phenomena induced by the use of boundary fields can already be analyzed.

To quantify the preference of the wall toward one of the phases, it is convenient to introduce a new thermodynamic quantity, the wall free energy,

\[
\tau_{\text{wd}} (\beta, \eta) \triangleq \lim_{N \to \infty \atop M \to \infty} \frac{1}{|\Sigma_N|} \log \frac{\mathbf{Z}^{\bar{\sigma}_{\mathcal{D}_{N,M}}} \eta}{\mathbf{Z}^{\bar{\sigma}_{\mathcal{D}_{N,M}}} \eta}.
\]
The existence of this quantity, and the remarkable fact that the two limits can be taken in any order, has been established in Ref. 82; the proof relies on the simple identity

$$
\tau_{bd}(\beta, \eta) = \lim_{N,M \to \infty} \beta \int_0^{\eta} \frac{1}{\eta |\Sigma_N|} \sum_{\alpha} \sigma^{\beta n}_{N,M,\alpha} \, d\eta'.
$$

(4.1.2)

We will return to this formula in the next section. The heuristics behind the definition of $\tau_{bd}(\beta, \eta)$ is that the free energy $F^{\beta, \eta}_{N,M,+(-)} = -\log Z^{\beta, \eta}_{N,M,+(-)}$ of the + (-) phase can be decomposed in the following way:

$$
F^{\beta, \eta}_{N,M,+} = f_\beta(\beta)|D_{N,M}| + f_\beta^s(\beta)|\partial D_{N,M} \Sigma_N| + f_\beta^w(\beta, \eta)|\Sigma_N| + o(|\partial D_{N,M}||\Sigma_N|),
$$

$$
F^{\beta, \eta}_{N,M,-} = f_\beta(\beta)|D_{N,M}| + f_\beta^s(\beta)|\partial D_{N,M} \Sigma_N| + f_\beta^w(\beta, \eta)|\Sigma_N| + o(|\partial D_{N,M}||\Sigma_N|),
$$

where

$$
f_\beta(\beta) \triangleq \lim_{N,M \to \infty} |D_{N,M}|^{-1} \log Z^{\beta, \eta}_{N,M,\vec \sigma},
$$

$$
f_\beta^s(\beta) \triangleq \lim_{N,M \to \infty} |\partial D_{N,M}|^{-1} \log Z^{\beta, \eta}_{N,M,\vec \sigma} - f_\beta(\beta)|D_{N,M}|,
$$

$$
f_\beta^w(\beta, \eta) \triangleq \lim_{N,M \to \infty} |\Sigma_N|^{-1} \log Z^{\beta, \eta}_{N,M,\vec \sigma} - f_\beta(\beta)|D_{N,M}| - f_\beta^s(\beta)|\partial D_{N,M} \Sigma_N|.
$$

[and similarly for $f_\beta^s(\beta)$ and $f_\beta^w(\beta, \eta)$]. As the notations suggest, $f_\beta(\beta)$ is independent of $\eta$ and $\vec \sigma$, $f_\beta^s(\beta)$ is independent of $\eta$, and by symmetry $f_\beta^w(\beta) = f_\beta^w(\beta, \eta)$. Therefore, we see that $\tau_{bd}(\beta, \eta) = \lim_{N,M \to \infty} (1/|\Sigma_N|)(F^{\beta, \eta}_{N,M,-} - F^{\beta, \eta}_{N,M,+}) = f_\beta^w(\beta, \eta) - f_\beta^w(\beta, \eta)$ is nothing else than the leading-order term of the difference in free energy between the two phases in the presence of the wall.

The ultimate justification of (4.1.1), however, is that this quantity plays exactly the role of its thermodynamical analog in the variational problem describing the macroscopic geometry of phase coexistence; see Theorems 4.3.2 and 4.3.3 below.

The following theorem states basic properties of $\tau_{bd}(\beta, \eta)$: since $\tau_{bd}(\beta, \eta)$ is obviously odd in $\eta$, we just state them for $\eta \geq 0$ [also $\tau_{bd}(\beta, 0) = 0$].

**Theorem 4.1.1:** (Ref. 83) Let $\tau_{\beta}^0 = \tau_{\beta}(\epsilon_d)$ and suppose $\eta \geq 0$. Then we have the following.

1. $\tau_{bd}(\beta, \eta)$ is a non-negative, increasing function of $\beta$ and $\eta$, concave in $\eta$; moreover, if $\eta > 0$,

$$
\tau_{bd}(\beta, \eta) > 0 \Leftrightarrow \beta > \beta_c.
$$

2. For all $\beta$ and $\eta$, $\tau_{bd}(\beta, \eta) \leq \tau_{\beta}^0$.

3. For all $\beta > \beta_c$, there exists $1 \geq \eta_0(\beta) > 0$, such that

$$
\tau_{bd}(\beta, \eta) < \tau_{\beta}^0 \Leftrightarrow \eta < \eta_0(\beta).
$$

In the case of the 2D Ising model, $\eta_0(\beta)$ can be computed explicitly; see Refs. 84, 85 and Fig. 11.

The following terminology is standard (This terminology only makes sense once we have chosen one of the equilibrium phases as a reference; here it is the + phase): when $\eta \geq \eta_0(\beta)$, we say that the system is in the complete drying regime; when $|\eta| < \eta_0(\beta)$, it is in the partial wetting regime; and when $\eta \leq -\eta_0(\beta)$, it is in the complete wetting regime. The reason for this terminology should become clear later.
B. Surface phase transition

In this section, we will see that the boundary magnetic field can trigger surface phase transitions. The behavior of the system in the vicinity of the wall depends dramatically on $|\eta|$ being greater or smaller than $\eta_w(\beta)$. A more detailed discussion of these issues can be found in Ref. 46.

The state of the system in the middle of a big box $D_{N,M}$ is entirely determined by the boundary conditions, and is independent of the value of the boundary field, so that the usual (infinite volume) Gibbs state simply does not provide any information on the behavior of the system close to the wall. To analyze the behavior of the system "in the vicinity" of the wall, it is therefore useful to introduce the notion of surface Gibbs states; these differ from the Gibbs states usually considered in these models by the fact that one does not work with a sequence of boxes converging to $\mathbb{R}^d$, but instead converging only to the half-space $\mathbb{R}^d_+$. More precisely, the surface Gibbs states are the weak limits of the measures $\mu_{N,M,d,\eta}$ when $N,M\to\infty$ (observe that $D_{N,M}/\mathbb{R}^d_+$). Two of them are of particular importance for our discussion, $\mu_{d,+,\eta}$ and $\mu_{d,-,\eta}$, obtained, respectively, by taking weak limits of the measures with + and − boundary conditions. It is not difficult to show that these two measures exist, are extremal, and are invariant under translations parallel to the wall; moreover, there is uniqueness of the surface Gibbs state if and only if $\mu_{d,+,\eta} = \mu_{d,-,\eta}$.

There is a close relation between $\tau_{bd}(\beta, \eta)$ and the behavior of the system near the wall; this can be most easily seen from the following identity, that is a consequence of (4.1.2) and of symmetry:

$$\tau_{bd}(\beta, \eta) = \int_0^\eta (\sigma_0)_{d,+,\eta}^\beta d\eta' = \int_0^\eta ((\sigma_0)_{d,+,\eta}^\beta - (\sigma_0)_{d,-,\eta}^\beta) d\eta'. \tag{4.2.1}$$

Using (4.2.1), it is possible to prove the following theorem, showing that a surface phase transition occurs at $\eta = \eta_w(\beta)$; this is the so-called wetting transition.

**Theorem 4.2.1:** (Ref. 83) There is a unique surface Gibbs state if and only if $|\eta| > \eta_w(\beta)$.

Let us briefly discuss the heuristics behind this result. The + and − boundary conditions fix the phase present in the bulk (i.e., in the middle of a big box $D_{N,M}$). However, Theorem 4.2.1 shows that when $\eta = \eta_w(\beta)$, the surface Gibbs state is unique, and therefore the state of the system near the wall is independent of the boundary condition, i.e., of the phase present in the bulk.

The mechanism responsible for this is the following. Suppose that $\eta<0$ and consider + boundary conditions; then it is natural to regard the boundary field as a negative b.c., and therefore to introduce an open contour with boundary $\partial \Sigma_N$ separating the − phase favored by the wall from the + phase present in the bulk. As long as $\eta > -1$, there is a competition between two effects: On the one hand it is energetically favorable for the open contour to follow the wall; on the other hand, this would lead to a loss in entropy, since there is less room for fluctuations. When $\eta = -\eta_w(\beta)$, the entropy wins: The contour is repelled away from the wall, at a distance diverging with the size of the box; this is the phenomenon of entropic repulsion. The surface Gibbs state then describes the behavior of the system below this surface, i.e., a mesoscopic film of the − phase along the bottom wall. The fact that the contour is sent away from the wall explains why we recover the surface tension, $\tau_{bd}(\beta, \eta) = \tau^0_{bd}$. When $\eta > -\eta_w(\beta)$ energy wins, and this modifies completely the behavior of the microscopic surface: it sticks to the wall, making only small excursions away from it; in this case, the phase in the bulk can reach the wall and the surface Gibbs state depends on the choice of boundary conditions.

Part of these heuristics can be made quite precise in the 2D case. Consider + boundary conditions. When $0 > \eta > -\eta_w(\beta)$, one can prove that the probability that a connected piece $I$ of the wall is not touched by the open contour is bounded above by $K \exp[-(\tau^0_{bd} - \tau_{bd}(\beta, \eta))/\beta]$, showing that the phase separation line really sticks to the wall. The information available when $\eta < -\eta_w(\beta)$ are much less precise; the magnetization profile computed in Ref. 84 shows that there is a film of width of order $\sqrt{N}$ along the wall. A related, much more precise, result, which holds at sufficiently low temperature and for $\eta = -1$ is that the phase separation line, once
suitably rescaled, converges weakly to the Brownian excursion; this should be true for any \( \eta \leq -\eta_w(\beta) \).

In higher dimensions, much less is known. When \( \eta > -\eta_w(\beta) \), one can show that the probability that the open contour touches the middle of the wall is bounded away from 0 uniformly in the size of the box. When \( \eta \leq -\eta_w(\beta) \), very little is known, except in the simpler case of SOS models. Also, if it is known in dimension 2 that \( \eta_w(\beta) < 1 \) [since the exact expression for \( \eta_w(\beta) \) has been computed], this is an open problem in higher dimensions.

Theorem 4.2.1 gives a first explanation of the terminology introduced above: when the system is in the complete drying regime, the equilibrium phase along the wall is the + phase, whatever the phase in the bulk is; when there is complete wetting, it is the − phase; only in the regime of partial wetting can both phases be present near the wall. The fact that the phase transition is determined by \( \eta_w(\beta) \) (i.e., the characterization of the partial wetting regime by \( \tau^w_{\beta} > |\tau_{bd}(\beta, \eta)\) is known as Cahn’s criterion.

C. Derivation of the Winterbottom construction

In this section, we show how Winterbottom construction, describing the equilibrium shape of a crystal in the presence of an attractive substrate, can be recovered from a microscopic theory. To do this, we consider the measure \( \mu_{\eta,N,+}^{\infty} \), for some \( \in \mathbb{R} \), conditioned with some canonical constraint (exact or approximate; see below). Of course, the situation here is more complicated than the one described in the Introduction, since instead of an infinite wall, the system is contained in a finite vessel. This, of course, makes the problem more difficult: When the solution of the Winterbottom variational problem does not fit inside the box

\[
\hat{D}^d = \{ x \in \mathbb{R}^d : |x(n)| \leq 1, n = 1, \ldots, d-1, 0 \leq x(d) \leq r \},
\]

the solution of the constrained problem will differ from the Winterbottom shape. In fact, the general solution of the constrained problem is not known. In the way we state them below, the derivation of this variational problem from statistical mechanics still applies in the case when the solution is not known.

Before stating the main theorems of this part, we briefly describe how the wetting transition manifests itself in the macroscopic geometry of phase separation. To do this, let \( \beta > \beta \), be fixed, and choose a value \( m \) for the canonical constraint so that the corresponding Wulff shape is small enough to be placed inside the box \( \hat{D}^d \). If \( \eta \geq \eta_w(\beta) \), then \( \tau_{bd}(\beta, \eta) = \tau^w_{\beta} \), and the typical configurations will consist of a macroscopic droplet of the + phase, with a Wulff shape, immersed in a background of the − phase; in particular, the shape of the droplet is independent of the value of the boundary field [Fig. 11(a)]. This behavior persists up to the value \( \eta = \eta_w(\beta) \). Notice that as soon as \( \eta < 1 \), it becomes energetically more favorable for the droplet to touch the wall. In dimension 2, however, since \( \eta_w(\beta) < 1 \), the droplet stays away from the wall, because entropy loss is not compensated by energy gain until \( \eta \) reaches the value \( \eta_w(\beta) \). It is an interesting open

FIG. 11. The case of the 2D Ising model. Left: The phase diagram; the region of nonuniqueness of the surface Gibbs state is shaded. In the other region, there is a single surface Gibbs state. Right: A sequence of equilibrium shapes.
problem to decide whether $\eta_\nu(\beta) = 1$ for $d>2$. When $\eta < \eta_\nu(\beta)$, the typical configurations consist of a macroscopic droplet, with a Winterbottom shape, tied to the wall. The shape of the droplet now depends on the value of $\eta$, and decreasing the boundary field amounts to letting the droplet spread more and more [Figs. 11(b)–11(e)]. For some value $\tilde{\eta}$, the droplet covers for the first time the entire wall [Fig. 11(e)]. From this point on, the shape of the droplet is left unchanged when $\eta$ is decreased [Fig. 11(f)]; the dashed line represents part of a possible “true” equilibrium shape for the unconstrained problem.

From this discussion, we see that the wetting transition at $\eta_\nu(\beta)$ has a macroscopic manifestation in the canonical ensemble. Because of the confined geometry, however, the second transition, at $\eta = -\eta_\nu(\beta)$ cannot be seen. To be able to detect it, one has to consider mesoscopic droplets (in the form of large moderate deviations, see the remark after Theorem 4.3.2).

This also explains pretty well the terminology introduced previously: In the complete drying regime, the droplet stays away from the wall, and so the wall is completely dry w.r.t. the $-$ phase; in the partial wetting regime, the droplet touches the wall, and both the $+$ and $-$ phase are in contact with it (provided $\eta > \tilde{\eta}$). The complete wetting regime cannot be distinguished from the partial wetting regime in this setting, but see the remark after Theorem 4.3.2 for a discussion of this issue.

1. 2D Ising model

Let $r \in \mathbb{R}$. Our aim in this section is to describe the typical configurations under the measure

$$
\mu^{\beta, \eta}_{N, r N, 2} (\cdot | M_N = m | D_{N, r N}),
$$

where $m \in (-m^*, m^*)$ and $M_N = \sum_{i \in D_{N, r N}} \sigma_i$; we will simplify the notations further by writing simply $\mu^{\beta, \eta}_{N, r N}$ (r being kept fixed). As in Sec. III, it is possible to obtain precise asymptotics for the large deviations, in the form of the following generalization of the first part of Theorem 3.1.1.

Let $W_{\beta, \eta}^*(m)$ be the infimum of the functional $W_{\beta, \eta}$ on subsets of $D_r^2$ with volume $[(m^*-m)/2m^*][D_r^2]$.

**Theorem 4.3.1:** Let the inverse temperature $\beta > \beta_c$ and the boundary magnetic field $\eta \in \mathbb{R}$ be fixed; let the sequence $\{a_N\} = -m^* | D_{N, r N} | + a_N \in \text{Range}(M_N)$, be such that the limit

$$
m = \lim_{N \to \infty} \frac{a_N}{|D_{N, r N}|} \in (0, 2m^*(\beta))
$$

exists. Then,

$$
\log \mu^{\beta, \eta}_{N, r N} (M_N = m^* | D_{N, r N} | - a_N) = -W_{\beta, \eta}^*(m) (1 + O(N^{-1/2} \log N)).
$$

A version of this theorem, in an approximate canonical ensemble [as in (4.3.1)], has been proven in Ref. 18; this stronger version can be obtained by combining the techniques of Ref. 18 and of Ref. 32; see Sec. IV.D.

In Theorem 4.3.1, we have made no statement about the asymptotic description of the typical configurations under the conditioned measure. The reason is the following: These strong concentration results require the knowledge of stability properties of the variational problem in the form, for example, of a Bonnesen inequality. However, in the present case, one does not always have that much information about the variational problem; in fact, even its solution is not always known. This prevents us from translating the energy estimates on the skeletons [see (4.4.8), (4.4.10), and (4.4.11)] into strong concentration properties of the microscopic contours. Of course, in the situations when such stability properties are known (Ref. 13 contains a simple derivation of such a result for many situations), it is possible to obtain statements of the same kind as those of Sec. III.

This illustrates the fact that although the probabilistic theory in the 2D case is complete, in the sense that all the relevant information on the microscopic scale is available, the sharpness of the
is in the complete wetting regime, i.e., the measure was \( \mu_{N,2}^{B,s} (\cdot | A(m;c)) \), where

\[
A(m;c) = \left\{ \sigma : \left| |D_{N,rN}|^{-1} M_N(\sigma) - m \right| \leq N^{-c} \right\},
\]

with \(-m^* < m < m^*\), and \(c\) is some real number not too large (see Theorem 4.3.2 below). We are going to prove that the phases concentrate near macroscopic droplets that belong to the set \( \mathcal{D}(m) \),

\[
\mathcal{D}(m) = \left\{ V \subset \hat{\Omega}^2_r : |V| = \frac{m^* - m}{2m^{*}} |\hat{\Omega}^2_r|, \ W_{\beta,s}(\partial V) = W^*_{\beta,s}(m) \right\}.
\]

Recall that to each \( V \in \mathcal{D}(m) \), we associate the function \( l_V = 1_{V^c} - l_V \).

To state this phase segregation theorem, we use analogous notation to the mesoscopic setup introduced in Sec. II. Recall that \( N = 2^n \). For any \( a < 1 \), we define a magnetization profile \( M_{[an]}(\sigma,x) \) at the \( 2^{[1/an]} \) scale that is piecewise constant on boxes \( \hat{B}_{n-[an]}(x) \) with \( x \in \mathbb{D}_n \cap 2^{[an]} - n2^d \).

\[
M_{[an]}(\sigma,x) = 2^{-d_{[an]}} \sum_{i \in [1/an]} \sigma_i.
\]

We get the following.

**Theorem 4.3.2:** (Ref. 18) Let \( \beta > \beta_c, \eta \in \mathbb{R}, -m^* < m < m^* \), and \( 1/4 > c > 0 \). Then there exist a function \( \delta(N) \) such that \( \lim_{N \to \infty} \delta(N) = 0 \), a real number \( \kappa > 0 \), and a coarse graining parameter \( 1 > a > 0 \) such that for \( N \) large enough

\[
\mu_{N,2}^{B,s} \left( \frac{M_{[an]}(\sigma,x)}{m^*} \right) \in \bigcup_{V \in \mathcal{D}(m)} \mathcal{V}_{(l_V, \delta(N))} \left| A(m;c) \right| \approx 1 - \exp\{-O(N^\kappa)\}.
\]

**Remark:** In this case, it should also be possible to study the whole range of moderate deviations, combining the techniques of Refs. 32 and 18, although this has not been done explicitly. We briefly describe the results obtained for large deviations sufficiently close to volume order.

As long as \( \eta > -\eta_\eta(\beta) \), the results are similar to those obtained in the setting of Sec. III: The measure concentrates on configurations containing a single large droplet of the \( - \) phase, with the Wulff or Winterbottom shape depending on \( \eta \); in particular, the order of the large moderate deviations is still \( \exp\{-O((a_N)^2N^{-3})\} \). There should not be any problem to extend this to the whole large deviations regime \((a_N) \approx N^{4/3}\).

More interesting is the case \( \eta = -\eta_\eta(\beta) \). For those values of the boundary field, the system is in the complete wetting regime \( [\tau_{\eta}(\beta, \eta) = -\tau_{\eta}^s] \), and the solution of the unconstrained variational problem is degenerate. The solution of the constrained variational problem in \( \hat{\Omega}^2_r \) is, however, still well defined for every \( N \); it is obtained by extracting the cap of a Wulff shape and rescaling it so that the basis of the cap completely covers the wall and the rescaled cap has the required volume. When \( N \) goes to infinity, this droplet spreads out to become a thin film in the limit (covering the entire wall; hence the terminology complete wetting), and the corresponding value of the surface free energy functional goes to zero. As a result of this, the scale of the large moderate deviations is not the same as when \( \eta < \eta_\eta(\beta) \); indeed the leading term of the asymptotics can again be computed explicitly, and is found to be of order \( \exp\{-O((a_N)^2N^{-3})\} \). In particular, we see that the large moderate deviations cannot extend up to \( a_N \approx N^{4/3} \), since \((a_N)^2N^{-3}\) is of order 1 already when \( a_N \approx N^{32} \). This should not be surprising since, in the
complete wetting regime, the volume under the microscopic contour is expected to have typical fluctuations of order \(N^{3/2}\) (this can be shown when \(\eta = -1\) and \(\beta\) is very large using the convergence to Brownian excursion stated in Ref. 86). Therefore, typical fluctuations of magnetization in the complete wetting regime are not governed by bulk fluctuations anymore, but by fluctuations of the microscopic phase separation line. To prove that this behavior is valid up to \(a_N \sim N^{3/2}\) might be a nontrivial task.

2. Ising model in \(d=3\)

Let \(r \in \mathbb{R}\) and let \(\mathcal{D}(m)\) be the set of macroscopic droplets at equilibrium in \(\hat{\mathcal{D}}^d_r\),

\[
\mathcal{D}(m) = \left\{ V \subset \hat{\mathcal{D}}^d_r : |V| = \frac{m^* - m}{2m^*} |\hat{\mathcal{D}}^d_r|, W_{\beta, \eta}(\partial V) = W_{\beta, \eta}^*(m) \right\}.
\]

The rest of the notations were introduced in Sec. II. The main result is the following.

Theorem 4.3.3: (Ref. 11) For any \(\beta \in \mathcal{B}_d\), any \(\eta \in \mathbb{R}\), any \(m \in (-m^*, m^*)\), the following holds: For any \(\delta > 0\), there is \(k_\delta = k_\delta(\delta)\) such that for \(\nu < 1 - d\),

\[
\lim_{N \to \infty} \min_{k_0 \leq k \leq \nu m} \mu_{N, \eta}^{\beta, \eta} \left( \frac{M_k}{m^*} \in \bigcup_{V \in \mathcal{D}(m)} \mathcal{Y}(1_V, \delta) \right) |M_N| = m |\mathcal{D}_{N, rN}| = 1.
\]

D. The tools

In this section, we explain how the procedures described in Secs. II and III have to be modified to take into account the effect of the boundary.

1. 2D Ising model

We describe the main modifications one needs to apply to the proofs of Sec. III in order to get the results stated in Theorems 4.3.1 and 4.3.2. We split this section into two parts, one dealing with the lower bound on \(\mu_{N, \eta}^{\beta, \eta}(\mathcal{A}(m; c))\) or \(\mu_{N, \eta}^{\beta, \eta}(M_N = -m^* |\mathcal{D}_{N, rN}| + a_N)\); the other one with the upper bound.

The lower bound. The constrained variational problem is more difficult than the usual one. In fact, as noted above, the solution (and a fortiori its stability) is not known, in general, although it is in many cases. This prevents us from proceeding as in Sec. III, where the lower bound follows from summing over large contours fluctuating around the Wulff shape. It would then appear necessary to make the same kind of proof, but for any configurations of droplets surrounding the right volume (all potential solutions to the variational problem). This, however, would be tricky; indeed, since we want our results to hold for large, but finite boxes, it is compulsory to obtain estimates uniform over the droplet in the chosen set! Fortunately, properties of the surface tension and wall free energy allow us to restrict our analysis to a small class of well-behaved droplets: The solution of the variational problem is necessarily taken on a single convex droplet. This is a consequence of the convexity of \(\tau_\beta\) (use the Jensen inequality) and the fact that \(\tau_\beta(\beta, \eta) = \tau_\beta^0\), which implies that replacing a droplet by its convex hull cannot increase the surface free energy; rescaling the resulting droplet decreases the energy even more. It is thus enough to prove the following.

Proposition 4.4.1: (Ref. 18) Let \(\beta > \beta_c\) and \(\eta \in \mathbb{R}\). There exists \(N_0 = N_0(\beta, \eta, m, c, r)\) and a constant \(C\) such that, for any simple closed rectifiable curve \(C\) which is the boundary of a convex body of volume \(|\hat{\mathcal{D}}^d_r|(m^* + m)/2m^*\) contained in \(\hat{\mathcal{D}}^d_r\), and for all \(N \geq N_0\),

\[
\mu_{N, \eta}^{\beta, \eta}(\mathcal{A}(m; c)) \geq \exp\left\{ -W_{\beta, \eta}(C)N - \beta CN^{1/2} \log N \right\}.
\]

A completely analogous statement holds in the case of the exact canonical ensemble.

The proof of Proposition 4.4.1 is similar to the proof of Theorem 3.5.1. We sketch now the main changes needed to deal with the boundary conditions. The case \(\eta \leq 0\) requires a slightly more complicated proof than the case \(\eta > 0\) so we first consider the latter.
First case: \( \eta > 0 \). As in the usual case, we want to approximate \( C \) with some polygonal curve with vertices on the dual lattice, and then sum over all contours going through the latter; this would allow us to extract, for each piece of the contour, the surface tension of the corresponding part of the polygonal line. Here, however, we want to be able to extract the wall free energy when the curve \( C \) follows the wall. There are some complications related to this: If two vertices are close to the wall, but do not belong to it, the sum over the corresponding piece of contour might not yield simply \( \tau_\beta \) or \( \tau_{\text{bd}}(\beta, \eta) \), but some complicated mixture, since typical such contours might first go down to the wall, then follow it on some length, and only then go up to the other vertex, see Fig. 12; this kind of behavior has been studied in detail in Ref. 19. Consider, for example, a family of curves \( C \) getting closer and closer to the wall; since we need estimates uniform in all such curves, one has to be able to deal with such a situation.

It turns out that it is possible to construct a polygonal approximation to the curve \( C \) whose surface tension is not too large in comparison with that of \( C \), while removing these possible pathologies. The idea is the following. Let \( N = N^{-1/2} \log N \), and set

\[
D_r^2(N) = \{ x \in D_r^2 : \min\{ ||y - x||_1 \geq \delta_N \} \}
\]

Let \( V \) be the convex body with boundary \( C \) and set \( C_N = \partial(V \cap D_r^2(N)) \). We first construct a polygonal approximation for each of the components of \( C_N \cap D_r^2(N) \) with segments of length \( \delta_N \) (apart from at most eight of them, which may be shorter). Set \( [x,y] = \{ z \in C_N : \delta(2) = \delta_N \} \). If \( [x,y] \neq \emptyset \), we connect the two corresponding pieces of polygonal lines by a broken line from \( x \) to \( (x(1),0) \), then to \( (y(1),0) \), and finally to \( y \); we divide the segment between \( (x(1),0) \) and \( (y(1),0) \) into segments of length \( \delta_N/2 \) (except possibly for the last one, which can be shorter). We repeat this construction for the three other sides of the box. The resulting closed polygonal line is denoted by \( \hat{P}_N \) (see Fig. 13). Notice that by construction there exists an absolute constant \( C \), such that
\[ \mathcal{W}_{\beta, \eta}(\mathcal{C}) = \mathcal{W}_{\beta, \eta}(\hat{\mathcal{P}}_N) - C\beta \delta_N. \]

\[ |\text{vol}(\mathcal{C}) - \text{vol} \hat{\mathcal{P}}_N| \leq C|\hat{\mathcal{L}}^2| \delta_N. \]

We then rescale the polygonal line \( \hat{\mathcal{P}}_N \) by a factor \( N \) and if necessary move slightly the rescaled vertices so that they belong to the dual lattice; the rescaled polygons is denoted by \( \mathcal{P}_N \). We then define a class \( \mathcal{G} \) of closed contours going through the vertices of \( \mathcal{P}_N \) (in the right order), and staying in some small boxes along its edges. For all edges of length smaller than \( N \delta_N \), as well as for the (up to eight) pieces we added above to join \( \mathcal{C}_N \) to the boundary, we impose that the corresponding piece of the contour is a fixed length-minimizing path between the vertices.

The rest of the argument proceeds in a similar way as in the standard case. The estimates in the phase of small contours carry over without any problems, since in that case the effect of the boundary field cannot propagate far away from the wall.

We still have to explain how one can extract the correct surface tension for \( \hat{\mathcal{P}}_N \) from the sum over contours in the class \( \mathcal{G} \) introduced above. To do this, we use several results about the random-line representation, proved in Refs. 18 and 19. To lighten the notation, we simply write \( q^\beta \cdot \eta \) instead of \( q^\beta_{N,N^*} \cdot \eta^* \); \( \beta^* \) and \( \eta^* \) are the dual of \( \beta \) and \( \eta \); see (1.2.5). The first inequality is just the analog of (3.4.6) in our case, which turns out to be valid for arbitrary ferromagnetic coupling constants: The weight of any high-temperature contour \( \gamma \in \mathcal{G} \) satisfies (Ref. 18, Lemma 5.4)

\[ q^\beta \cdot \eta(\gamma) \geq \prod q^\beta \cdot \eta(\gamma_k), \]

where \( \gamma_k \) denotes the piece of the contour \( \gamma \) between the \( k \)th and \( k+1 \)th vertices of \( \mathcal{P}_N \). The next step is to replace \( q^\beta \cdot \eta(\gamma_k) \) by the corresponding infinite-volume quantity. First, for any \( \gamma_k \) joining vertices not belonging to \( \Sigma^*_N \triangleq \{ i \in \mathcal{S}_N; i(2) = -\frac{1}{2} \} \) [note that \( \gamma_k \) stays necessarily at a distance \( O(N \delta_N) \) from \( \Sigma^*_N \)],

\[ q^\beta \cdot \eta(\gamma_k) \geq (1 - e^{-O(N \delta_N)}) q^\beta(\gamma_k); \]

second, for the pieces \( \gamma_k \) joining two sites of \( \Sigma^*_N \), we use

\[ q^\beta \cdot \eta(\gamma_k) \geq q^\beta \cdot \eta(\gamma_k), \]

where \( 1_d \triangleq \{ i \in \mathcal{P}_N^d; i(2) = -\frac{1}{2} \} \) (both results are proved in Ref. 18, Lemma 5.3). Finally, the remaining pieces have a length of, at most, \( 8N \delta_N \), so that their total weight is larger than \( e^{-c \cdot O(N \delta_N)} \).

The last step is to extract the surface free energy. The basic tool to do this is, as in the proof of Theorem 3.4.4, concentration properties for open contours between two fixed dual sites. For the pieces \( \gamma_k \) not touching the boundary, we can use the usual infinite volume results based on (B3), setting \( s = N \delta_N \). For the pieces along the boundary, one can use the following statement (Ref. 19, Lemma 6.10):

\[ \sum_{\lambda, i, j} q^\beta \cdot \eta(\lambda) \geq (\sigma_j \sigma_i^*)^\beta \cdot \eta(\lambda) (1 + o(1)). \quad (4.4.1) \]

where \( \mathcal{N}_N(i,j) \) is defined in Appendix B (with \( s = N \delta_N \)). [In fact, (4.4.1) can be strengthened when \( \eta < \eta_\alpha(\beta) \): in this case, the set \( \mathcal{N}_N(i,j) \cap \mathcal{L}^d \) can be replaced by the set (Ref. 19, Lemma 6.13)
which is compatible with our picture of partial wetting.

The result then follows from lower bounds on the corresponding two-point functions. The only new inputs are the following lower bounds on the boundary two-point function:

\[
\langle \sigma, \sigma \rangle^{\beta^*, \eta^*}_{L_{\beta^*}, \eta^*} \geq C \exp\left(\frac{-\tau_{bd}(\beta, \eta)\|j-i\|}{\|j-i\|^0}\right), \quad \forall \eta \geq \eta_0(\beta), \tag{4.4.2}
\]

\[
\langle \sigma, \sigma \rangle^{\beta^*, \eta^*}_{L_{\beta^*}, \eta^*} \geq C \exp\left(\frac{-\tau_{bd}(\beta, \eta)\|j-i\|}{\|j-i\|^0}\right), \quad \forall \eta \geq \eta_0(\beta). \tag{4.4.3}
\]

for any \(i, j \in \Sigma^* \triangleq \{k \in \mathbb{Z}^d : k(2) = -\frac{1}{2}\}\). (4.4.3) is proved in Ref. 18, Prop. 7.1, while (4.4.2) follows from exact computations in the case \(\eta^* = 1\) (Ref. 85, and Ref. 18, Prop. 7.1).

\[
\langle \sigma, \sigma \rangle^{\beta^*, \eta^*}_{L_{\beta^*}, \eta^*} \geq (\tanh \beta^*)^2 \langle \sigma, \sigma \rangle^{\beta^*, 1}_{L_{\beta^*}, \eta^*}, \quad \forall \eta \geq 0.
\]

Second case: \(\eta = 0\). This is a somewhat marginal case. The apparent difficulty is that in this case \(\eta^* = \infty\). However, this does not create any real complications. One just has to modify the construction of the first case as follows: We replace the polygonal line \(\mathcal{P}_N\) by the (possibly open) polygonal line \(\mathcal{P}_N \setminus \{u \in \mathbb{R}^2 : u(2) = 0\}\); we then sum over contours going through the vertices of this polygonal line (contours that are open if the polygonal line is open). This does not give any contribution for the part of \(\mathcal{C}\) along the wall, which is what we want since \(\tau_{bd}(\beta, 0) = 0\).

Third case: \(\eta < 0\). This is slightly more tricky. In this situation, one may be even more pessimistic, since the duality is simply not defined when nonferromagnetic interactions are present! However, this turns out to be a false problem. Indeed, we can use the following obvious identity to recover ferromagnetic interactions (see remark, p. 1078)

\[
\mu_{\eta, N, i}^{\beta, \eta} = \mu_{\eta, N, i}^{\beta, |\eta|},
\]

where \(\pm\) correspond to the boundary condition \(\bar{\sigma}_i = 1\) if \(i(2) \geq 0\) and \(\bar{\sigma}_i = -1\) otherwise.

We then construct \(\mathcal{P}_N\) as in the first step and set \(I = \mathcal{P}_N \cap \{x \in \mathbb{Z}^2 : x(2) = 0\}\). If \(I = \emptyset\), then we subdivide the set \(\{x \in \mathbb{Z}^2 : x(2) = 0\}\) into segments of length \(\delta_N/2\) (except possibly for the last one, which might be shorter); this defines a second (open) polygonal line \(\mathcal{P}'_N\) (with all its vertices along the wall) (see Fig. 14). We then introduce a class of pair of contours \((\gamma, \gamma')\), \(\gamma\) going through the vertices of \(\mathcal{P}_N\) and defined as before, and \(\gamma'\) following the wall, going through the vertices of \(\mathcal{P}'_N\) and staying inside small boxes along its edges, similarly as for the other one (\(\gamma'\) is open). By construction, \(\gamma\) and \(\gamma'\) are disjoint. Duality then implies the following identity:

**FIG. 14.** The construction for \(\eta < 0\). Left: \(I = \emptyset\) (two polygonal lines: one open and one closed. Right: \(I \neq \emptyset\) (one open polygonal line).
\[ \mu_{N,z}^{\beta,|\eta|}(\{\gamma, \gamma'\} \subset \gamma(\cdot)) = (Z_{N,z}^{\beta,|\eta|})^{-1} \sum_{(\xi, \gamma', \gamma) \Lambda^*\text{-comp.}} w(\xi) w(\gamma') \]
\[ = (1 - e^{-\Omega(N)}) Z_{N,z}^{\beta,|\eta|} q_N^{\beta^*,:|\eta|^*}(\gamma, \gamma'). \tag{4.4.4} \]

The factor \((1 - e^{-\Omega(N)})\) comes from the fact that we can apply duality only to simply connected sets, and the exterior of \(\gamma\) is not simply connected. We must therefore forbid families \(\xi\) for which duality does not hold; since such families must contain at least one contour surrounding \(\gamma\), we get the above correction.

We can now proceed as in the first case. The only additional work to do is to analyze the ratio of partition functions in (4.4.4), but this is easy, since by duality,
\[ \frac{Z_{N,z}^{\beta,|\eta|}}{Z_{N,z}^{\beta^*,|\eta|^*}} = (\langle \sigma_i, \sigma_j \rangle_{\Lambda_{N,r,N}}^{\beta^*,|\eta|^*})^{-1} e^{-\tau_{rd}(\beta, |\eta|)(2N + 1)}, \tag{4.4.5} \]
where \(t_\ell = (-L - \frac{1}{2}, -\frac{1}{2})\) and \(t_r = (L + \frac{1}{2}, -\frac{1}{2})\) are the two dual sites at the lower left and lower right corners of \(\Omega^*_{N,r,N}\), and the last inequality follows from the upper bound (see Ref. 18, for example),
\[ \langle \sigma_i, \sigma_j \rangle_{\Lambda_{N,r,N}}^{\beta^*,|\eta|^*} e^{-\tau_{rd}(\beta, |\eta|)\|j-i\|}, \tag{4.4.6} \]
valid for any \(i, j \in \Sigma^*_N\). We then see that the ratio of the partition function cancels the contribution from the sum over the open contour \(\gamma'\), up to an error term \(\exp(\Omega(N\delta_N))\).

If \(I \neq \emptyset\), the situation is simpler. Let us write \(I = [x, y]\); then we define a new polygonal line \(\partial^*_N : \partial_N^*\) goes from the lower right corner of \(D^*_N\) to \(a\) along the wall, then it follows \(\partial_N^* \setminus \{x \in \hat{D}^N_2 : x(2) = 0\}\) up to \(b\) and finally goes from \(b\) to the lower right corner of \(D^*_N\) (see Fig. 14). We subdivide as usual the part of \(\partial_N^*\) along the wall into segments of length \(\delta_N/2\) and proceed as in the first case, with \(\partial_N^*\) replacing \(\partial_N\), using (4.4.4). Summing over the open contour going through the vertices of \(\partial_N^*\) produces (up to the usual error term) a term \(\exp\{-\mathcal{W}_{\beta,|\eta|}(\partial_N^*)N\}\). Combining this with (4.4.5) and observing that
\[ \exp(2\tau_{rd}(\beta, |\eta|)N) \exp\{-\mathcal{W}_{\beta,|\eta|}(\partial_N^*)N\} = \exp\{-\mathcal{W}_{\beta,|\eta|}(\partial_N)N\}, \]
the conclusion follows as in the usual situation.

**The upper bound.** Let us now turn our attention to the proof of the upper bound. The basic strategy is completely similar to that of the standard case; see Sec. III F 2. The only serious modification concerns the energy estimate, which should now associate the functional \(\mathcal{W}_{\beta,|\eta|}\) to the probability of skeletons. Again, the case \(\eta \geq 0\) is somewhat simpler than the other, so we start with this one.

**First case: \(\eta \geq 0\).** The basic problem we encounter when trying to make the energy estimate is the same we met in the proof of the lower bound. Summing over an open contour connecting two dual sites \(i\) and \(j\) might not yield a decay of order \(\exp(-\tau_{bd}(j-i))\) or \(\exp(-\tau_{rd}(\beta, |\eta|)\|j-i\|)\) if \(i\) and \(j\) are close enough to the wall but not on it (see Ref. 19). However, the following bound, proven in Ref. 18, Lemma 5.1, is sufficient to derive the energy estimate,
\[ \sum_{\lambda, i-j} q_N^{\beta^*,|\eta|^*}(\lambda) \leq \exp\{-\tau_{rd}(j-i)\}, \tag{4.4.7} \]
for any \( \eta \geq 0 \); \( \mathcal{E}(\Sigma_N^v) = \{ x^\ast \subset \Sigma_N^v \} \). The definition of skeletons will be done in such a way as to ensure that the additional constraint \( \lambda \cap \mathcal{E}(\Sigma_N^v) = \emptyset \) is automatically satisfied; see below. We also need to extract the wall free energy when summing over contours joining two dual sites belonging to \( \Sigma_N^v \); this, however, is nothing else as (4.4.6).

Let us now describe the construction of a skeleton \( S = (u_1, \ldots, u_n) \) of a closed contour \( \gamma \). Remember that we have to define the skeletons in such a way as to ensure that (1) the piece of the contour between two dual sites not both on the wall must be edge disjoint from the wall, and (2) the Hausdorff distance between the contour \( \gamma \) and the polygonal line \( \text{Pol}(S) \) is smaller than the cutoff parameter \( s(N) \).

For contours \( \gamma \) that do not touch the wall, the definition of skeletons is the same as in Sec. III. Suppose \( \gamma \cap \mathcal{E}(\Sigma_N^v) \neq \emptyset \). Let us define \((u_1, \ldots, u_{2m})\) as the minimal family of dual sites satisfying the following properties.

1. \( u_k \in \Sigma_N^v \cap \gamma \) for \( k = 1, \ldots, 2m \) and \( u_k(1) < u_k(1) \) if \( k < k' \).
2. \((u_1, \ldots, u_m)\) split \( \gamma \) into pieces \( \gamma_k : v_1 \rightarrow v_2, \ldots, v_{2m} \rightarrow v_1 \), such that \( \gamma_k \cap \mathcal{E}(\Sigma_N^v) = \emptyset \) for all \( k = 1, \ldots, m \); \( d_{\gamma_k}(\{x \in \mathbb{R}^2 : x(2) = -\frac{1}{2}\}) \geq s(N) \) for all \( k = 1, \ldots, m \); and \( d_{\gamma_k}(\{x \in \mathbb{R}^2 : x(2) = \frac{1}{2}\}) \geq s(N) \) for all \( k = 1, \ldots, m \).

We then say that \( S = (u_1, \ldots, u_n) \) is an \( s \) skeleton of \( \gamma \) if all vertices of \( S \) belong to \( \gamma \); \( v_1, \ldots, v_{2m} \) are vertices of \( S \); the only vertices of \( S \) along \( \gamma_{2k+1} \) are \( v_{2k+1} \) and \( v_{2k+2} \), for all \( k = 1, \ldots, m \); the distance between any successive pair of vertices \( u_j, u_{j+1} \) of \( S \) along \( \gamma_{2k} \) satisfies \( s(N) / 2 \leq \|u_j - u_{j+1}\| \leq 2s(N) \), for all \( k = 1, \ldots, m \); \( d_{\gamma_k}(\gamma, \text{Pol}(S)) \leq s(N) \).

This definition has the nice property that either \( u_j \) and \( u_{j+1} \) both belong to \( \Sigma_N^v \), or the part of \( \gamma \) between these two sites is edge disjoint from \( \Sigma_N^v \) (see Fig. 15). This allows us to use the estimates (4.4.6) and (4.4.7). This yields the following extension of (3.4.2) (Ref. 18):

\[
\mu_{\beta, \eta}^{\beta, \eta^*} (\mathcal{G}) \approx \exp \left( -\mathcal{W}_{\beta, \eta}(\mathcal{G}) \right). \tag{4.4.8}
\]

The analog of the energy estimate (3.4.3) then follows easily, since \( \tau_{\eta}(\beta, \eta) = 0 \) when \( \eta \geq 0 \) and therefore it is still possible to control the number of vertices of \( \mathcal{G} \) in terms of \( \mathcal{W}_{\beta, \eta}(\mathcal{G}) \). This gives

\[
\mu_{\beta, \eta}^{\beta, \eta^*} (\mathcal{W}_{\beta, \eta}(\mathcal{G}) \geq r) \leq \exp \left( -r \left( 1 - \frac{C \log N}{s(N)} \right) \right). \tag{4.4.9}
\]

Using this and the estimates in the phase of small contours, which still hold in the presence of a boundary field, the upper bound follows easily.

**Second case:** \( \eta < 0 \). As for the lower bound, we have to deal with the fact that, for \( \eta < 0 \), the duality is not defined. The solution is the same as there: We just change boundary conditions, i.e., we look at the measure \( \mu_{\beta, \eta}^{\beta, \eta^*} \), which was defined when we dealt with the lower bound.
It turns out, however, that in the partial wetting regime, its associated family of polygonal lines satisfies
\[ S_{\text{Pol}}(T) \]
In particular, the following version of the same set of vertices introduce another family \( K \) for some absolute constant \( D \) the variational problem, i.e., with the boundary of a convex body in the associated to the situation similar to the case \( T \), is more subtle, but happens not to give too many problems as long as we consider volume

Once we have done this, the main difference is that the family of low-temperature contours of any configurations compatible with these boundary conditions contains exactly one open contour, with end points \( t_1 = (N - \frac{1}{2}, \frac{1}{2}) \) and \( t_2 = (N + \frac{1}{2}, \frac{1}{2}) \). It is straightforward to generalize the notion of a skeleton introduced in the preceding case to the present situation. What we get by this procedure is a family of skeletons \( \mathcal{S} = (S_0, S_1, \ldots, S_n) \) containing exactly one skeleton, \( S_0 \), with \( \text{Pol}(S_0) \) open with end points \( t_1 \) and \( t_2 \).

Since we want to compare the corresponding families of polygonal lines with the solution of the variational problem, i.e., with the boundary of a convex body in \( \mathbb{D}_2 \), it is convenient to introduce another family \( \mathcal{G} \) of skeletons whose associated polygonal lines are closed; \( \mathcal{G} \) possesses the same set of vertices (except for \( t_1 \) and \( t_2 \)), but with a different set of edges, which is such that its associated family of polygonal lines satisfies
\[ \text{Pol}(\mathcal{G}) = \text{Pol}(\mathcal{G}^\mathbb{Z}) \Delta \{ x \in \mathbb{R}^2 : N/2 - \frac{1}{2} \leq x(1) \leq N/2 + \frac{1}{2}, x(2) = -\frac{1}{2} \} \]
where \( \Delta \) denotes symmetric difference (see Fig. 16).

One then has the following relation:
\[ \mathcal{W}_{\beta, \eta}(\mathcal{G}) = \mathcal{W}_{\beta, \eta}(\mathcal{G}^\mathbb{Z}) - (2N + 1) \tau_{\text{bd}}(\beta, \eta) \]
In particular, the following version of (4.4.8) holds:
\[ \mu_{\mathcal{G}}^{\beta, \eta}(\mathcal{G}^\mathbb{Z}) = K_1 \exp\{-\mathcal{W}_{\beta, \eta}(\mathcal{G})\}, \quad \eta > -\eta_w(\beta). \quad (4.4.10) \]
\[ \mu_{\mathcal{G}}^{\beta, \eta}(\mathcal{G}^\mathbb{Z}) = K_2 N^{3/2} \exp\{-\mathcal{W}_{\beta, \eta}(\mathcal{G})\}, \quad \eta < -\eta_w(\beta). \quad (4.4.11) \]
The energy estimate (4.4.9) is slightly more delicate now, since the wall free energy is negative. It turns out, however, that in the partial wetting regime, \( \eta > -\eta_w(\beta) \), it is easy to reduce ourselves to a situation similar to the case \( \tau_{\text{bd}}(\beta, \eta) \geq 0 \). The case \( \eta \leq -\eta_w(\beta) \), i.e., complete wetting, is more subtle, but happens not to give too many problems as long as we consider volume-order large deviations (or, in fact, deviations close enough to volume order).

Let us first consider the case of partial wetting; this regime is characterized by \( |\tau_{\text{bd}}(\beta, \eta)| < \tau_{\beta}^+ \). Let us write \( \mathcal{W}_{\beta, \eta}(\mathcal{G}) = T^+ + T^- \), where \( T^+, T^- \) is the positive (negative) part of the functional. Then, since \( T^+ \geq (\tau_{\beta}^+/\tau_{\text{bd}}(\beta, \eta))T^- \) and the number of vertices along the wall is at most two-thirds of the total number \( \#(\mathcal{G}) \), we have
\[ \#(\mathcal{G}) \leq \frac{K}{s(N)(\tau_{\beta}^+ + \tau_{\text{bd}}(\beta, \eta))} \mathcal{W}_{\beta, \eta}(\mathcal{G}), \]
for some absolute constant \( K \). This allows us to prove that
\[ \mu_{\mathcal{G}}^{\beta, \eta}(\mathcal{W}_{\beta, \eta}(\mathcal{G}) \geq r) \leq \exp\left\{-r \left(1 - \frac{C \log N}{s(N)}\right)\right\}. \quad (4.4.12) \]
When \( \eta = -\eta_a(\beta) \), one cannot establish as good an upper bound. The best we can do is to use the fact that \( T^* = (2N+1)\tau_{\text{bd}}(\beta, \eta) \), which turns out to be enough to prove the following, weaker, version of the energy estimate:

\[
\mu^\beta_{N, \eta}(\mathcal{W}_{\beta, \eta}(\tilde{\mathcal{S}}) \geq r) \geq \exp\left(-r \left(1 - \frac{C \log N}{s(N)}\right) + C_1 \frac{N \log N}{s(N)}\right). \tag{4.4.13}
\]

The reason why such an estimate is still sufficient to get the desired result is that the relevant values of \( r \) are also of order \( N \), so that the first term can always be made to dominate the second one.

Once we have (4.4.12) and (4.4.13), the proof is concluded as usual, after observing that the estimate in the phase of small contours still applies in the presence of the boundary field \( |\eta| \).

### 2. Ising model in \( d=3 \)

The proof of Theorem 4.3.3 is based on the \( L_1 \) theory introduced in Sec. II. We simply explain how the main ingredients of the proof should be modified and refer to Ref. 11 for details.

The arguments of geometric measure theory can be extended easily to this new setting. In particular, it is straightforward to check that the functional \( \mathcal{W}_{\beta, \eta} \) is lower semicontinuous and that the approximation Theorems 2.5.1 and 2.6.1 hold.

The main problem is to define proper mesoscopic phase labels for the measures with a boundary magnetic field. If \( \eta \geq 0 \), then the mesoscopic phase labels introduced in Sec. II satisfy the Assumptions (a) and (b) as well as Conditions (C1)–(C3) under the measure \( \mu^\beta_{N, \eta} \). Instead, if \( \eta < 0 \), some problems occur because the FK measure loses its ferromagnetic properties and the random coloring measures are more complicated to deal with. Nevertheless, it is still possible to define mesoscopic phase labels and to derive estimates as in Sec. II B.

Other difficulties have to be overcome in order to implement the general philosophy of the \( L_1 \)-theory. In the case of a negative boundary magnetic field, the interface induced by the field prevents us from applying directly the techniques developed to prove the exponential tightness, Theorem 2.1.1. Therefore an alternative approach similar to the one described in Sec. IV D1 is required. The analysis of the surface tension also needs some care. We recall that the computation of surface tension is based on a localization procedure along the boundary of functions of bounded variation. For a given test function either locally its boundary is in the bulk and we recover the usual surface tension term or it intersects the wall and arguments similar to those used in the bulk enable us to derive the wall free energy. In this way the complexity of the problem is reduced because the difficult analysis of the fluctuations of the microscopic interface between the wall and the bulk is replaced by soft \( L_1 \) estimates.

### E. Open problems

As in the previous parts, there are still a lot of open problems. Most of those presented before have natural analogs in the present situation. In the following, we restrict ourselves to problems intrinsically related to the topics discussed in this part.

#### 2D nearest neighbors Ising model

The fact that one is still unable to analyze nonperturbatively the fluctuations of the phase separation line is only strengthened when we would like to study boundary effects. Indeed, a general analysis of typical open paths with end points at general positions with respect to the wall has not been done, even at low temperature. Problems related to this are the following.

1. Give a nonperturbative proof that the probability measure of a suitably rescaled version of an open contour with end points on the wall converges weakly to the measure of Brownian excursion when \( \eta = -\eta_a(\beta) \) (as was sketched in the low-temperature case for \( \eta = -1 \) in Ref. 86). This would provide a way of analyzing the typical fluctuations of magnetization in the complete wetting regime, and would complete the heuristic picture of the wetting transition in the grand-canonical ensemble.
(2) Establish Ornstein–Zernike behavior for the boundary two-point function without having recourse to explicit computations. Even weaker lower bounds, like those given in Ref. 76, have not been proved in such a constrained geometry.

Another open problem is to investigate the full range of moderate deviations. This may require an understanding of point (1) above.

Higher-dimensional nearest neighbors Ising models. If fluctuations of phase separation lines are not yet understood, the situation is only much worse when considering their higher-dimensional counterparts; in fact, even perturbative results are not always available. Here is a far from exhaustive list of related open problems.

(1) Give a microscopic description of the behavior of phase boundaries in the partial and complete wetting regimes in the grand-canonical ensemble to put some flesh on the heuristics given above.
(2) Decide whether \( \eta_\omega(\beta) = 1 \) or not. The corresponding results for the SOS model suggest that \( \eta_\omega(\beta) < 1 \) in any dimension; numerical investigations confirm this in dimension 3.

In fact, even much simpler problems related to the behavior of higher-dimensional interfaces are still open: proof of the existence of a roughening transition in \( d = 3 \), proof of the unstability of the \((1, 1, 1)\) interface, ... .

In some simpler models of the SOS type some (but not all!) of these problems can be solved, but this does not seem to help in solving the original ones.

The wall. Another type of problem concerns properties of the wall. In particular, it might be interesting to answer the following questions.

(1) What happens if the interaction with the wall is more complicated (say, non-nearest neighbor). (2) What happens if the boundary field is not homogeneous (for example, is a “random” configuration of \( \eta_1 \) and \( \eta_2 \) macroscopically equivalent to some well-chosen homogeneous boundary field \( \eta = \tilde{\eta} \)?)

APPENDIX A: PROOF OF THEOREM 2.2.1

Assumption (a) controls the number of zero \( u_k \)-blocks, whereas assumption (b) is used to control the geometry of the mesoscopic phase labels. The dependence of \( k_0 \) on \( \delta \) could be described as follows: we choose \( k_0 \) so large that

\[
\rho_\delta \leq \frac{1}{C(d)} \delta, \quad \text{for every } k \geq k_0, \tag{A1}
\]

where \( C(d) \) is a large enough fixed constant. Three terms on the left-hand side on (2.2.3) correspond to three different exponential estimates:

1. Estimate on the volume of zero \( u_k \)-blocks

The domination by Bernoulli measure (2.2.2) implies that

\[
\Gamma_\beta \left\{ \# \{ x \in \hat{\mathcal{A}}^d_{-k} : u_k(x) = 0 \} \geq \delta \left( \frac{N}{2^\pi} \right)^d \right\} \leq c_2 \exp \left[ -\delta \left( \frac{N}{2^\pi} \right)^d \log \frac{\delta}{\rho_\delta} \right]. \tag{A2}
\]

Each realization of the phase label function \( u_k \) splits \( \hat{\mathcal{A}}^d \) into the disjoint union of three mesoscopic regions:

\[
\hat{\mathcal{A}}^d = \{ x : u_k(x) = 1 \} \lor \{ x : u_k(x) = -1 \} \lor \{ x : u_k(x) = 0 \} \triangleq \mathcal{A}_+ \lor \mathcal{A}_- \lor \mathcal{A}_0.
\]

By the choice of the scale \( k_0 \) in (A1) the estimate (A2) is nontrivial for every \( k \geq k_0 \), and, in view of the target claim (2.2.3), we can restrict attention only to such realizations of \( u_k \) for which
Indeed, if the total volume inside small contours is less than \( \lambda \), the area of the boundary of any regular set \( A \) such that \( A_s \subseteq A \subset \mathcal{T}_d \setminus A_\infty \) is bounded below as

\[
|A_0| = \int_{\mathcal{T}_d} 1_{\{u_0(x) = 0\}} \, dx < \delta. \tag{A3}
\]

This has the following important implication: if \( u_0 \in \mathcal{W}(K_s, \delta)^c \), the area of the boundary of any regular set \( A \) such that \( A_s \subseteq A \subset \mathcal{T}_d \setminus A_\infty \) is bounded below as

\[
|\partial A| \geq a. \tag{A4}
\]

Using assumption (b) of the theorem, we are going to construct such sets \( A \) on the finite \( k_0 \) scale; \( A \in \mathcal{F}_{n-k_0} \) and in such a fashion that all the boundary \( k_0 \)-blocks of \( A \) will necessarily have zero \( u_{k_0} \)-labels. This reduction enables a uniform treatment of all coarser scales \( k \geq k_0 \).

So let \( k \equiv k_0 \), and assume that (A3) holds. We denote by \( A_- \) (resp., \( A_+ \)) the set of all boxes \( \hat{B}_{n-k_0} \) in \( A_- \) (resp., \( A_+ \)). We say that \( x \in \mathcal{T}_{d-n-k_0}^d \) is \( - \leftrightarrow + \) connected to \( A_- \) if \( x \leftrightarrow A_- \), if there exists a \( - \leftrightarrow + \) chain of \( u_{k_0} \)-blocks leading from \( \hat{B}_{n-k_0}(x) \) (and including it) to \( A_- \). Define now the complement \( A^c \) as follows:

\[
A^c = A_+ \cup_{x \in \hat{B}_{n-k_0}} \hat{B}_{n-k_0}(x). 
\]

By the virtue of assumption (b), \( A_s \subseteq A \). Moreover, by construction all the \( k_0 \)-blocks of \( A \) attached to the boundary \( \partial A^c \) have zero \( u_{k_0} \)-labels. With a slight abuse of notation we proceed to denote this collection of boundary \( k_0 \)-blocks as \( \partial A \). By (A4) the number of \( k_0 \)-blocks in \( \partial A \) is bounded below by

\[
\#_{k_0}(\partial A) \geq \frac{c(d)a}{2a^{-1}k_0} N^{d-1}. \tag{A5}
\]

Since, however, the total number of \( k_0 \)-blocks in the corresponding decomposition of \( \mathcal{T}_d \) equal to \( N^d/2^{d_k} \) the estimate (A5) alone is not sufficient for giving the desirable upper bound on the probability \( \mathcal{P}_{\hat{B}}(u_0 \in \mathcal{W}(K_s, \delta)^c) \). The required entropy cancellation stems from the fact that small connected contours of \( \partial A \) cannot surround too much volume.

Let us decompose \( A \) to the disjoint union of its maximal connected components:

\[
A = \biguplus_{i=1}^{l} A_i, \quad \text{respectively,} \quad \partial A = \biguplus_{i=1}^{l} \partial A_i.
\]

We shall quantify contours \( \partial A_i \) according to the size (or the number of \( k_0 \) blocks) in \( A_i \). Namely, the contour \( \partial A_i \) is called small, if

\[
\#_{k_0}(A_i) \leq K(d)\log N \quad \text{or} \quad |A_i| \leq K(d) \frac{2^{d_k}}{N^{d'}} \log N, \tag{A6}
\]

where \( K(d) \) is a sufficiently large constant. Otherwise, the contour \( \partial A_i \) is called large.

We claim that under (A3) the following inclusion is valid:

\[
\{u_k \in \mathcal{W}(K_s, 2\delta)^c\} \subseteq \left\{ \sum_{\partial A_{i} \text{small}} |A_i| > \delta \right\} \cup \left\{ \sum_{\partial A_{i} \text{large}} |\partial A_i| > a \right\}.
\]

Indeed, if the total volume inside small contours is less than \( \delta \), then repainting all the small components \( A_i \) into \( -1 \) and all the large components \( A_j \) into \( +1 \) we produce a \( \{\pm 1\} \)-valued function that is, at most, at the \( L_1 \) distance \( 2\delta \) from \( u_k \) and that, thereby, cannot belong to \( K_s \).
2. Peierls estimate on the size of large contours

\[ P_N \left( \sum_{\partial A_i \text{ large}} |\partial A_i| > a \right) = P_N \left( \sum_{\partial A_i \text{ large}} \# k_0(\partial A_i) > \frac{c(d)a}{2^{d-1}N_k} N^{d-1} \right) \leq \exp \left\{ -c_3(d) \frac{a}{2^{d-1}N_k} N^{d-1} \right\} \]  

This immediately follows from assumption (a), once the constant \( K(d) \) in (A6) has been properly chosen.

3. Estimate in the phase of small contours

The volume of small components \( A_i \) is related to the total number of \( k_0 \)-blocks in these components as

\[ \sum_{\partial A_i \text{ small}} |A_i| = \left( \frac{N}{2^{n_k}} \right)^{-d} \sum_{\partial A_i \text{ small}} \# k_0(A_i). \]

On the other hand, for every \( l \in \{1, \ldots, n-k_0\} \):

\[ \sum_{\partial A_i \text{ small}} \# k_0(A_i) = \sum_{x \in \tau_{n-k_0}} \sum_{\partial A_i \text{ small}} 1_{\{x \in A_i\}} = \sum_{t \in \{0, \ldots, 2^l\}^d} \sum_{x \in \tau_{n-k_0-t}} \sum_{\partial A_i \text{ small}} 1_{\{\theta_{\Delta_0} x \in A_i\}}, \]

where \( \Delta_0 = 2^{k_0-n} \) is the step size on the embedded torus \( \tau_{n-k_0}^d \), and \( \theta_x \) is the shift on this torus. Consequently,

\[ P_N \left( \sum_{\partial A_i \text{ small}} |A_i| > \delta \right) \leq \max_{t \in \{0, \ldots, 2^l\}^d} P_N \left( \sum_{x \in \tau_{n-k_0-t}} \sum_{\partial A_i \text{ small}} 1_{\{\theta_{\Delta_0} x \in A_i\}} > \delta \left( \frac{N}{2^{n_k}} \right)^d \right). \]  

If, however, \( 2^l > K(d) \log N \), then no two distinct points on the torus \( \tau_{n-k_0-t}^d \) (or any shift of it) can belong to the same component \( A_i \). This, in view of the domination by the independent Bernoulli site percolation [assumption (a)] suggests an application of the B–K inequality. Since, by the choice of the scale \( k_0 \) in (A1),

\[ \epsilon_{k_0} \triangleq \frac{N_{k_0}}{\log \epsilon_{k_0}} (\exists \text{ a closed surface of zero } u_{k_0} \text{-blocks around } x) < \delta, \]

for every \( x \in \tau_{n-k_0} \), we readily obtain that the right-hand side of (A9) is bounded above by

\[ c_4(d) \exp \left\{ -\delta \left( \frac{N}{2^{n_k}} \right)^d \log \left( \frac{\delta}{\epsilon_{k_0}} \right) \right\}. \]

The proof of Theorem 2.2.1 is concluded.

\[ \square \]

APPENDIX B: PROOF OF THE THREE-POINT LOWER BOUND LEMMA 3.4.3

The proof of Lemma 3.4.3 is based on the following positive stiffness property of the surface tension\(^9\).
\[
\min_{\theta \in [0, 2\pi]} \left[ \frac{d^2}{d\theta^2} \tau_{\beta}(n(\theta)) + \tau_{\beta}(n(\theta)) \right] = \min_{\theta \in [0, 2\pi]} R_{\beta}(n(\theta)) > 0, \tag{B1}
\]

where the unit normal \(n(\theta)\) is defined via \(n(\theta) = (\cos \theta, \sin \theta)\), and \(R_{\beta}(n)\) is the radius of curvature of \(\partial K\) at the point supporting the tangent line orthogonal to \(n\). An integral version of (B1) is the sharp triangle inequality Ref. 29, 79: For any \(u, v \in \mathbb{R}^2\)
\[
\tau_{\beta}(u) + \tau_{\beta}(v) - \tau_{\beta}(u + v) \geq c_1(\beta)(\|u\|_2 + \|v\|_2 - \|u + v\|_2). \tag{B2}
\]

The latter inequality is used to control the fluctuations of the microscopic phase boundaries (in their random line representation of Sec. III D).

Let now an \((s, \epsilon)\)-compatible triple of points \((u, v, w)\) be given. Fix \(K = K(\beta)\) large enough and define the ‘‘oval’’ neighborhood \(N_K(u, w)\) of \([u, v]\) as
\[
N_K(u, w) = \{z \in \mathbb{R}^2 : \tau_{\beta}(z - u) + \tau_{\beta}(w - z) - \tau_{\beta}(w - u) \leq K \log s\}.
\]

The oval neighborhood \(N_K(w, v)\) is defined exactly in the same fashion. Relations (3.4.8) and (3.4.11) readily imply that the main contribution to \(\langle \sigma_{\beta}(\sigma_{\beta})^* \rangle_{f} \) (respectively, to \(\langle \sigma_{\beta}(\sigma_{\beta})^* \rangle_{f} *\)) comes from the paths \(\lambda_1\) (respectively, \(\lambda_2\)) which stay in \(N_K(u, w)\) [respectively, \(N_K(w, v)\)]. More precisely,
\[
\sum_{\lambda_1, n \to w} q_{\beta}^\theta(\lambda_1) \equiv \langle \sigma_{\beta} \sigma_{\beta}^* \rangle_{f}^\theta (1 + o(1)), \tag{B3}
\]

uniformly in all \((s, \epsilon)\)-compatible triples. Any such path \(\lambda_1 = (\lambda_1(0), \ldots, \lambda_1(n_1))\) could be decomposed as follows: Define
\[
n_w = \max\{k : \lambda_k \in N_K(u, w) \setminus N_K(w, v)\},
\]
and set \(\lambda_1^w = (\lambda_1(0), \ldots, \lambda_1(n_w))\), \(\lambda_1^y = (\lambda_1(n_w + 1), \ldots, \lambda_1(n_1))\); \(\lambda_1 = \lambda_1^w \cup \lambda_1^y\). The decomposition \(\lambda_1 = \lambda_1^w \cup \lambda_1^y\) is defined in a completely symmetric way. Notice that, by the construction, the paths \(\lambda_1^w\) and \(\lambda_1^y\) are disjoint and compatible, and, by (B2),
\[
\max(\|\lambda_1(n_w) - w\|_2, \|\lambda_1(n_w) - w\|_2) \leq c_2(\epsilon) \log s.
\]

The claim of the lemma follows now from (3.4.6) and (3.4.7). \(\square\)


50. S. Shlosman (private communication, 1999).
79. Y. Velenik, “Phase separation as a large deviations problem: a microscopic derivation of surface thermodynamics for some 2D spin systems,” These 1712 EPF-L, 1997; available electronically from the author.