This Course

deals with super-rational agents:
1. Every agent is rational.
2. Every agent knows that every agent is rational.
3. Every agent knows that every agent knows that every agent is rational.
   
   n. Every agent knows that Statement n-1.
   
   This course is intended for students who are not afraid of mathematics.
Simple Decision Problems.
The numbers are in monetary units.

\[ x_3 \text{ is optimal} \]

\[ x_2 \text{ and } x_4 \text{ are optimal} \]
And What Would You Do Now?

$x_1$ guarantees at least 1
$x_2$ guarantees at least 0

$x_1$ may yield 4
$x_2$ may yield 5

$x_1$ is optimal
$x_2$ is optimal
## Adding Probabilities

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<th>x₁</th>
<th>1 or 4 with equal probabilities</th>
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<tr>
<td>x₂</td>
<td>5 or 0 with probabilities 0.75 and 0.25</td>
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x₁ gives an expected payoff of \( 0.5 \times 1 + 0.5 \times 4 = 2.5 \)

x₂ gives an expected payoff of \( 0.75 \times 5 + 0.25 \times 0 = 3.75 \)

An agent who always chooses the lottery with the highest expected payoff is called a **risk-neutral** agent. Such an agent prefers x₂ to x₁, and x₁ to a sure payoff of 2.
In This Course:

We Assume That agents are risk-neutral, unless we say otherwise. Moreover, we use this assumption in cases, in which the payoffs are not in units of money!
Games in Strategic Form

A game in strategic form is defined by a 2n-tuple: 
\( G = (X_1, X_2, \ldots, X_n, u_1, u_2, \ldots, u_n) \), 
where the players are 1, 2, \ldots, n, the strategy set of i is the nonempty set \( X_i \), and 
\( u_i: X \rightarrow \mathbb{R} \) is the payoff function of i, where 
\( X = X_1 \times X_2 \times \ldots \times X_n \) 
is the set of strategy profiles.

The game has one stage. In this stage the players are making simultaneous choices, \( x_1, \ldots, x_n \), and each player i receives the payoff \( u_i(x_1, \ldots, x_i, \ldots, x_n) \).

A Player, when making her choice does not know the other player’s choices.
Some terminology:

For every player $i$ we denote by $X_i$ the set of all strategy profiles of all other players. That is:

$$X_i = X_1 \times X_2 \times \ldots \times X_{i-1} \times X_{i+1} \times \ldots \times X_n.$$ 

E.g., if $n=3$, $X_2 = X_1 \times X_3$

A generic element of $X_i$ is denoted by $x_i$.

More over if $x = (x_1, x_2, \ldots, x_n)$ in $X$, then $x_i = (x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$

A game in strategic form is **finite** if all strategy sets $X_i$, are finite sets.
III

Example:

Player 1 chooses a row, Player 2 chooses a column, and Player 3 chooses a matrix.

\[
\begin{array}{c|c}
\text{matrix A} & \text{matrix B} \\
\hline
2 & 2 \\
-2 & 9 \\
1 & 0 \\
\hline
0 & 1 \\
6 & -2 \\
-3 & -1 \\
\end{array}
\begin{array}{c|c}
2 & 3 \\
-2 & 7 \\
1 & 5 \\
\hline
0 & 3 \\
6 & -2 \\
-2 & 1 \\
\end{array}
\]

For example,

\(u_1(1,2,B) = 3, \quad u_3(1,2,B) = 5\)
Bimatrix Games

Two-person finite games are sometimes described by a bimatrix \((A,B)\), where \(A,B\) are \(m \times n\) matrices; Player 1 chooses a row \(i\), \(1 \leq i \leq m\), Player 2 chooses a column \(j\), \(1 \leq j \leq n\), and the players receive \(a_{i,j}\), \(b_{i,j}\), respectively. It is sometime useful to describe The pair of matrices as a matrix, each of its cells is a vector of the form \((a_{i,j},b_{i,j})\). That is,

\[
\begin{array}{|c|c|}
\hline
i & j \\
\hline
\hline
a_{1,1} & a_{1,2} \\
\hline
b_{1,1} & b_{1,2} \\
\hline
\end{array}
\]

Player 1 chooses a row \(i\), Player 2 chooses a column \(j\). Player 1 receives \(a_{i,j}\), and Player 2 receives \(b_{i,j}\).
Dominated Strategies

Let $x_i$ and $z_i$ be two strategies of Player $i$. We say that $x_i$ dominates $z_i$, or that $z_i$ is dominated by $x_i$ if $x_i$ yields a better payoff than $z_i$ for every choice of the other players. That is,

\[ u_i(x_i, x_{-i}) > u_i(z_i, x_{-i}), \quad \text{for every } x_{-i} \text{ in } X_{-i}. \]
Player 1 and 2 are being held by the police at separate cells. The police knows that the two together robbed a bank, but lack evidence to convict. The police offers each of them the following deal; each is asked to implicate her partner. If neither does so, then each gets no time in jail. If one implicate the other but is not implicated, the first gets off, while the one implicated go to jail for 10 years. If each implicate the other, then each goes to jail for 2 years.

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d is dominated by c. That is, each player implicates the other, and both will get two years in jail.
## Sequential Elimination

**Example:**

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2: $2 > 1$

1: $2 > 3$

1: $2 > 4$
Example:

Player 1 chooses 1, and Player 2 chooses 2
Let $G=(X_1,\ldots,X_n, u)$ be a game in strategic form, where $u=(u_1,\ldots,u_n)$. For every player $i$, let $Z_i$ be a nonempty subset of $X_i$. We say that the game $H=(Z_1,\ldots,Z_n, u)$ is obtained by a one-stage elimination from $G$ if at least for one player $i$, $Z_i$ is a strict subset of $X_i$, and for every $i$, every $x_i$ in $D_i=X_i/Z_i$ is dominated in $G$ by some strategy in $X_i$.

A sequence $G_0, G_1,\ldots,G_k,\ldots$ is an elimination sequence of games, if $G_s$ is obtained by a one-stage elimination from $G_{s-1}$ for every $s>0$.

Note that if $G$ is a finite game, every elimination sequence starting with $G$ is a finite sequence. In order to deal with infinite elimination sequences we introduce the following terminology:

We denote by $G_m^i$, the strategy set of $i$ at the game $G_m$.

Let $G^i_\infty$ be the intersection of all $G_m^i$, $m\geq 0$. Hence, $G^i_\infty$ is a well-defined game.
in strategic form if $G_i^\infty$ is a nonempty set for every $i$. An elimination sequence terminates at the game $H$, if it is finite and the last game in the sequence is $H$, or it is infinite and $G^\infty=H$. If an elimination sequence terminates at the game $H$, we say that $H$ is obtained from $G$ by a **sequential elimination**. An elimination sequence is maximal, if it terminates at a game that does not have dominated strategies.
Recall:

The process of elimination is a virtual process run in the players’ minds.

The game has only one stage !!!!

**Definition**: Let $G=(X_1,\ldots,X_n, u)$ be a game in strategic form, and let $x$ be a strategy profile in $X$. We say that $x=(x_1,\ldots,x_n)$ is a solution obtained by a sequential elimination of dominated strategies, if there exists an elimination sequence of games beginning with $G$ and terminating with $H=({x_1},\ldots,{x_n},u)$. If such a solution $x$ exists, we say that $G$ is solvable by a sequential elimination of dominated strategies.

**Note**: Every elimination sequence of games that terminates in $H$ must be a maximal elimination sequence, because $H$ does not possess dominated strategies.
Order of Elimination

**Theorem 1:**
Let \( G \) be a finite game. All maximal elimination sequences that begin with \( G \) terminate at the same game. That is, if \( G_0, \ldots, G_s, \) and \( H_0, \ldots, H_t \) are maximal elimination sequences with \( G_0 = H_0 = G \), then \( G_s = H_t \).

**Proof:**
Denote the strategy set of Player \( i \) at the games \( G_k, H_p \), by \( G_k^i \), \( H_p^i \), respectively. Denote the sets of strategy profiles of all other players in the games \( G_k, H_p \), by \( G_k^{-i} \), \( H_p^{-i} \), respectively.

We need the following definitions and lemmas.

**Definition:** We say that \( z_i \) is eliminated at the \( k^{th} \) stage, if \( z_i \) belongs to the strategy set of \( i \) at \( G_k \), \( G_k^i \), and \( z_i \) does not belong to the strategy set of \( i \) at \( G_{k+1} \).
II

**Definition:** Let $x_i$, $z_i$, be strategies of $i$, and let $Y_{-i}$ be a subset of $X_{-i}$. We say that $z_i$ dominates $x_i$ w.r.t. $Y_{-i}$, if $u_i(z_i, y_{-i}) > u_i(x_i, y_{-i})$ for every $y_{-i}$ in $Y_{-i}$.

**Lemma 1:** If $x_i$ is dominated by $z_i$ w.r.t. $Z_{-i}$, and $Y_{-i}$ is a subset of $Z_{-i}$, then $x_i$ is dominated by $z_i$ w.r.t. $Y_{-i}$.

The proof of Lemma 1 is obvious, and therefore it is omitted.

**Lemma 2:** If $x_i$ was eliminated at the $k^{th}$ stage, then there exists $z_i$ that dominates $x_i$ at $G_k$, and $z_i$ belongs to the strategy set of $i$ at $G_{k+1}$. 


III

Proof of Lemma 2: As $x_i$ was eliminated at the $k^{th}$ stage, there exists $z_i(1)$ that dominates $x_i$ at $G_k$. If $z_i(1)$ was not eliminated at the $k^{th}$ stage, the Lemma is proved. Otherwise there exists $z_i(2)$ that dominates $z_i(1)$ at $G_k$. If $z_i(2)$ was not eliminated at the $k^{th}$ stage, then the lemma is proved (choose $z_i = z_i(2)$, and note that $z_i(2)$ dominates $x_i$). Continuing recursively we can find a sequence $z_i(1), z_i(2), z_i(3), \ldots$ of distinct strategies of $i$, such that $z_i(m+1)$ dominates $z_i(m)$ at $G_k$. As $G_k^i$ is a finite set, for some $m$, $z_i = z_i(m)$ is not eliminated at the $k^{th}$ stage. Hence $z_i$ dominates $x_i$ at $G_k$, and $z_i$ belongs to $G_k^i$. QED

Lemma 3: If $x_i$ belongs to $G_s^i$, then $x_i$ is not dominated w.r.t. $G_s^{-i}$. 

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IV

**Proof of Lemma 3**: Suppose in negation that $x_i$ is dominated by some $z_i$ w.r.t. $G_{s}^{-i}$. This $z_i$ belongs to $G_k^i$, for some $k$. Let $k$ be the largest integer for which there exists $z_i$ in $G_k^i$, such that $z_i$ dominates $x_i$ w.r.t. $G_{s}^{-i}$, and let $z_i$ be associated with this $k$. If $k=s$, $x_i$ is dominated in the game $G_s$, which contradicts the maximality of the elimination sequence. Hence, $k<s$. Since $z_i$ does not belong to $G_{k+1}^i$, we can apply Lemma 2 to find $y_i$ in $G_{k+1}^i$ that dominates $z_i$ w.r.t. $G_k^{-i}$. By Lemma 1, $y_i$ dominates $z_i$ w.r.t. $G_{s}^{-i}$. Therefore, $y_i$ dominates $x_i$ w.r.t. $G_{s}^{-i}$, contradicting the maximality of $k$. Hence, $x_i$ is not dominated w.r.t. $G_{s}^{-i}$. QED

We continue with the proof of Theorem 1.
Assume in negation that $G_s \neq H_t$. Hence, there exists $l$, such that $G_s^l \neq H_t^l$. Therefore, one of these sets is not a subset of the other one. Without loss of generality, assume that $G_s^l$ is not a subset of $H_t^l$. Therefore there exists a $k$, $0 \leq k < t$, such that $G_s^j$ is a subset of $H_k^j$ for every player $j$, and for some $i$ $G_s^i$ is not a subset of $H_{k+1}^i$. Let $x_i$ be an element of $G_s^i$, which does not belong to $H_{k+1}^i$. Therefore, there exists $z_i$ in $H_{k+1}^i$, that dominates $x_i$ in the game $H_k$. By Lemma 1, because $G_s^{-i}$ is a subset of $H_k^{-i}$, $z_i$ dominates $x_i$ w.r.t. $G_s^{-i}$. this contradicts Lemma 3. Hence, $G_s = H_t$. QED

Note that by Theorem 1 every finite game has at most one solution obtained by a sequential elimination of dominated strategies.
Example

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Process 1:
First stage: eliminate Row 2 and Row 3.
Second stage: eliminate column 1.
Process 2:
First stage: eliminate Row 3.
Second stage: eliminate column 1 and Row 2.
### Process 3:
First stage: eliminate Row 3.
Second stage: eliminate Column 1.
Third stage: eliminate Row 2.
Infinite Games

We will be mainly interested in a special type of infinite games. **Definition:** A game in strategic form is a regular game, if for every player $i$, $X_i$ is a compact metric space, and the payoff functions are continuous on $X$.

Non-regular infinite games may possess solutions that violate our intuition. For example, consider the two-person game, in which $X_1 = X_2 = (0,1)$ – the open interval of all real numbers $r$, $0 < r < 1$, and $u_1(x_1, x_2) = x_1$, $u_2(x_1, x_2) = x_2$. In this (non-regular) game every pair $(x_1, x_2)$ is a solution obtained by a sequential elimination of dominated strategies. In particular Theorem 1 is false in this example.
For regular games we have:

**Theorem 2:**
Let $G$ be a regular game, and let $G_0, G_1, \ldots, G_m, \ldots$ be an elimination sequence, then the sequence terminates at a game. If the elimination sequence is maximal, the sequence terminates at a regular game.

**Theorem 3:** Let $G$ be a regular game. Any two maximal elimination sequences terminate at the same game.

The proofs of theorems 2 and 3 appear in the Appendix.

A famous example for an infinite regular game, which is solvable by a sequential elimination of dominated strategies is the linear Cournot games described below:
Linear Cournot Games

Two similar firms are producing a divisible good, say wine. The cost of producing one unit of wine is \( c \geq 0 \). The market demand function for wine is \( d(p) = \text{Max}(b-ap,0) \), \( a>0, b>0 \), where \( p \) is the price per unit, and \( d(p) \) is the demand for wine when the market price is \( p \). Let \( P(x) \) be the inverse demand function. That is, \( P(x) \) is the minimal price in which it is possible to sell \( x \) units of wine. That is,

\[
P(x) = \begin{cases} 
  a^{-1}(b-x), & \text{if } x \leq b \\
  0, & \text{if } x \geq b.
\end{cases}
\]
Linear Cournot Games, an Example

In the following example $c=10$, and:

\[ P(x) = \begin{cases} 
  100 - x, & \text{if } x \leq 100 \\
  0, & \text{if } x \geq 100.
\end{cases} \]

How much should firm 1 produce? Note that if a firm produces more than 90 units, the market price will be less than 10, and the firm cannot possibly make a non-negative profit. Hence, we assume that a firm does not produce more than 90 units.
The profit of each firm depends also on the other firm’s decision. Hence we have a two-person game $G=(X_1,X_2,u_1,u_2)$, where for $i=1,2$, $X_i$ is the set of all feasible levels of production for Firm $i$. That is, $X_1=X_2=[0,90]$, and
\[ u_1(x_1,x_2) = x_1P(x_1+x_2) - 10x_1, \]
\[ u_2(x_1,x_2) = x_2P(x_1+x_2) - 10x_2. \]
Note that $G$ is an infinite regular game.

We show that for player 1, every $z_1>45$ is dominated by 45. That is
\[ u_1(45,x_2) > u_1(z_1,x_2), \text{ for every } x_2 \text{ in } X_2. \]
Proof: Fixed $z_1$, $45 < z_1 \leq 90$. Fixed $x_2$, $0 \leq x_2 \leq 90$.
We plot the graph of $u_1(x_1, x_2)$ for $0 \leq x_1 \leq 90$, where

$$u_1(x_1, x_2) = \begin{cases} 
  x_1(90 - x_2 - x_1), & \text{if } x_1 \leq 100-x_2, \\
  -10x_1, & \text{if } 100-x_2 < x_1 \leq 90.
\end{cases}$$
From the graph, $u_1(x_1,x_2)$ is decreasing when $45 \leq x_1 \leq 90$. As $45 < z_1$, $u_1(45,x_2) > u_1(z_1,x_2)$.

QED
Hence, the interval \((45,90]\) is eliminated for Firm 1 at the first stage of our sequential elimination process. Similarly we eliminate \((45,90]\) for the second firm. Hence, after the first stage of elimination \(X_1(1) = X_2(1) = [0,45]\). In the second stage of elimination we eliminate the interval \([0,22.5)\) for every firm. That is, we show (for Firm 1) that for every fixed \(z_1, 0 \leq z_1 < 22.5,\)

\[ u_1(22.5,x_2) > u_1(z_1,x_2), \text{ for every } x_2 \text{ in } [0,45], \]

and we show the analogous inequalities for Firm 2. Actually, you show it.

Just draw the graph of \(u_1(x_1,x_2)\) for a fixed \(x_2\) in \([0,45]\), note that \(0.5(90-x_2) \geq 22.5\), and deduce that \(u_1(x_1,x_2)\) is increasing at the interval \(0 \leq x_1 \leq 22.5.\)
VI

Hence, the interval \([0,22.5)\) is eliminated for each firm at the second stage of our sequential elimination process. Hence, after the second stage of elimination,

\[ X_1(2) = X_2(2) = [22.5,45].\]

You can guess that

\[ X_1(3) = X_2(3) = [22.5, 33.75].\]

We continue inductively. If after the nth round of elimination, \(n \geq 0\), \(X_1(n) = X_2(n) = [a_n, b_n]\) (\(a_0=0\) and \(b_0=90\)), we define:

\[
\begin{align*}
    b_{n+1} &= (90-a_n)/2, \text{ if } n \text{ is an even number, and } b_{n+1} = b_n, \text{ if } n \text{ is an odd number}, \\
    a_{n+1} &= (90-b_n)/2, \text{ if } n \text{ is an odd number, and } a_{n+1} = a_n, \text{ if } n \text{ is an even number,}
\end{align*}
\]

and

\[ X_1(n+1) = X_2(n+1) = [a_{n+1}, b_{n+1}]. \]
It can be easily seen that when $n$ converges to infinity, $a_n \to 30$, and $b_n \to 30$, and therefore $x=(30,30)$ is the solution of the Cournot game obtained by a sequential elimination of dominated strategies.
Weak Domination

Let $x_i, z_i$ be strategies of Player $i$.

$x_i$ weakly dominates $z_i$ if the following two conditions hold:

- $u_i(x_i, x_{-i}) \geq u_i(z_i, x_{-i})$, for every $x_{-i}$ in $X_{-i}$.
- There exists $x_{-i}$ in $X_{-i}$ such that $u_i(x_i, x_{-i}) > u_i(z_i, x_{-i})$.

Example

Row 1 weakly dominates Row 2

Sequential elimination of weakly dominated strategies yields the solution (Row 1, Column 1)
In a process of sequential elimination of weakly dominated strategies, we may eliminate the reason for elimination in previous steps.

Row 1 weakly dominates Row 2, because of the possibility that Column 1 will be played.
However, at the second round of elimination, column 1 is eliminated!!
Even for a finite game, in the process of sequential elimination of weakly dominated strategies, the order of elimination may make a difference. That is two distinct maximal elimination sequences may yield two different solutions.
Nash Equilibrium

Definition: \( x_i \) is a best response to \( x_{-i} \) if

\[
u_i(x_i, x_{-i}) \geq u_i(z_i, x_{-i}), \text{ for all } z_i \text{ in } X_i
\]

Row 3 is a best response of Player 1 to Column 2

Column 1 is a best response of Player 2 to Row 3
Definition:
A profile of strategies $x = (x_1, x_2, \ldots, x_n)$ is in equilibrium if $x_i$ is a best response versus $x_{-i}$ for every player $i$.

Example: In the following 3-person game, (Row 2, Column 2, Matrix A) is in equilibrium:
**III**

**Definition**: Let $x$ be in equilibrium in the game $G$. The equilibrium payoff of Player $i$ at $x$ is $u_i(x)$.

Unlike previous solution concepts, an equilibrium profile is a joint recommendation. A player should follow the recommendation if he believes that all other players follow their recommendations.

Another example: $(1,1)$ is in equilibrium.

Player 1 should play Row 1 if she believes that Player 2 will play Column 1.

Player 2 should play column 1 if he believes that Player 1 will play Row 1.
If the players can talk before the game starts, and can agree on a joint play, they must agree on an equilibrium profile, because these are the only self-enforcing agreements.

If the players cannot communicate, they still play in equilibrium, because they conclude that this is what they would have done if they could communicate.....

Equilibrium Assumption in Economics:

Economic Agents play in equilibrium.
Classical Games

(c,c) is the unique equilibrium in this Prisoner’s Dilemma

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There are two equilibrium profiles. How would you play this game??
Battle of the Sexes

Battle of the sexes

Woman’s choices

Football

Movie

Football

Movie

4 1
2 0
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1 4

man’s choices
No Equilibrium

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Equilibrium and Domination

**Theorem 4:** Let G be a regular game. Let x in X be a solution obtained by a sequential elimination of dominated strategies, or a solution obtained by a sequential elimination of weakly dominated strategies. Then x is in equilibrium.

**Proof:** Note that x is in equilibrium if and only if, for every player i, xi is not dominated w.r.t \(\{x_{-i}\}\) if and only if for every player i, xi is not weakly dominated w.r.t \(\{x_{-i}\}\).

**Case 1:** G is a finite game.
If x is the solution obtained by a sequential elimination of dominated strategies the result follows from Lemma 3 in the proof of Theorem 1. The proof of Lemma 3 uses Lemma 1,
which is not true for the weak dominance relation. However, it can be easily verified that Lemma 3 holds for the weak dominance relation. Therefore, the result for the weak dominance relation holds.

**Case 2: G is an infinite game.**
This case is proved in the Appendix. QED
The Traveler’s Dilemma

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The traveller's dilemma

Two travelers returning home from a remote island discover that the identical antiques they bought have been smashed in transit. The airline manager proposes the following scheme to elicit the value of the articles.

The two travelers independently submit compensation claims between $2 and $100. The airline will reimburse each traveler at the minimum of the two claims. In addition, if the claims differ, a reward of $2 will be paid to the person making the smaller claim and a penalty of $2 deducted from the reimbursement of the larger claimant.

The outcome (2,2) is the unique Nash equilibrium and the only outcome obtained by a sequential elimination of weakly dominated strategies. Yet, it seems very unlikely that any two individuals, no matter how rational they are and how certain they are about each other's rationality, each other's knowledge of each other's rationality, and so on, will play (2,2). How would you play this game?

The traveller's dilemma can be formally specified as follows:

<table>
<thead>
<tr>
<th>Strategy space</th>
<th>{2, 3, 4, . . . , 100}</th>
</tr>
</thead>
<tbody>
<tr>
<td>Payoff function</td>
<td>\Pi_i(\hat{x}, \hat{y}) = \begin{cases} \min{\hat{x}, \hat{y}} - 2, &amp; \hat{x} &gt; \hat{y} \ \min{\hat{x}, \hat{y}}, &amp; \hat{x} = \hat{y} \ \min{\hat{x}, \hat{y}} + 2, &amp; \hat{x} &lt; \hat{y} \end{cases}</td>
</tr>
</tbody>
</table>

or in matrix form

<table>
<thead>
<tr>
<th></th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>98</th>
<th>99</th>
<th>100</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>(2,2)</td>
<td>(4,0)</td>
<td>(4,0)</td>
<td>(4,0)</td>
<td>(4,0)</td>
<td>(4,0)</td>
</tr>
<tr>
<td>3</td>
<td>(0,4)</td>
<td>(3,3)</td>
<td>(5,1)</td>
<td>(5,1)</td>
<td>(5,1)</td>
<td>(5,1)</td>
</tr>
<tr>
<td>4</td>
<td>(0,4)</td>
<td>(1,5)</td>
<td>(4,4)</td>
<td>(6,2)</td>
<td>(6,2)</td>
<td>(6,2)</td>
</tr>
<tr>
<td>98</td>
<td></td>
<td></td>
<td></td>
<td>(1,5)</td>
<td>(2,6)</td>
<td>(98,98)</td>
</tr>
<tr>
<td>99</td>
<td>(0,4)</td>
<td>(1,5)</td>
<td>(2,6)</td>
<td>(96,100)</td>
<td>(99,99)</td>
<td>(101,97)</td>
</tr>
<tr>
<td>100</td>
<td>(0,4)</td>
<td>(1,5)</td>
<td>(2,6)</td>
<td>(96,100)</td>
<td>(97,101)</td>
<td>(100,100)</td>
</tr>
</tbody>
</table>

The following theorem shows that by eliminating dominated strategies we do not lose equilibrium profiles:

**Theorem 5:** Let $G$ be a game in strategic form, and let $x$ be an equilibrium profile in $G$. Let $G_0, G_1, ..., G_m, ...$ be an elimination sequence in $G$, then $x$ belongs to $G_m$ for every $m \geq 0$.

**Proof:** Assume in negation that $x$ does not belong to $G_m$. Let $k$ be the smallest integer for which there exists a player $i$ such that $x_i$ does not belong to $G_k^i$. Obviously, $0 < k \leq m$, and $x$ belongs to $G_{k-1}$. Let $i$ be a player for which $x_i$ was eliminated at the $(k-1)^{th}$ stage. Hence, there exists $z_i$, such that $z_i$ dominates $x_i$ w.r.t. $G_{k-1}^i$. In particular $u_i(z_i, x_{-i}) > u_i(x_i, x_{-i})$, contradicting $x_i$ being a best response to $x_{-i}$. Therefore, $x$ belongs to $G_m$. QED
IV

The following simple example shows that equilibrium profiles may be eliminated in the process of elimination of weakly dominated strategies:

```
  2  3  2
  6  2
  0  3  2
```

By eliminating Row 2 at the first stage of elimination, we eliminate the equilibrium profile (2,2).
Moreover, even if the game has equilibrium profiles, the elimination process that uses the weak relation may eliminate all equilibrium profiles. Moreover, a maximal elimination sequence may terminate in a game that does not have equilibrium profiles.

**Example:**

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>3</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>7</td>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Note that (2,3) is the unique equilibrium profile in this game. However, Row 2 is weakly dominated by Row 1. Please check that the elimination process yields a game that does not have equilibrium profiles.
Cournot (again!)

Let us find a best response of firm 1 versus $x_2$, $0 \leq x_2 \leq 90$.

We wish to find a level of production $x_1$ that maximizes $u_1(x_1,x_2) = x_1P(x_1+x_2)-10x_1$, over $x_1$ in $X_1=[0,90]$. The graph reveals that the optimal level of production is $x_1=0.5(90-x_2)$.

Similarly, for each $x_1$ in $[0,90]$ the best response of Firm 2 is to produce $0.5(90-x_1)$. Thus,
Cournot Equilibrium

if firm 2 produces $x_2$, and firm 1 knows it, it should produce $b_1(x_2) = 0.5(90-x_2)$, $0 \leq x_2 \leq 90$.

If firm 1 produces $x_1$, and firm 2 knows it, it should produce $b_2(x_1) = 0.5(90-x_1)$, $0 \leq x_1 \leq 90$.

And

Recall that $(z_1, z_2)$ is in equilibrium if

• $z_1$ is a best response versus $z_2$, that is $z_1 = b_1(z_2)$, and
• $z_2$ is a best response versus $z_1$, that is $z_2 = b_2(z_1)$.

Solving these two linear equations with two unknowns yield $z_1 = 30 = z_2$.

Are you surprised?
This Graph May Help

The graph of the best response function, $b_1(x_2)$ of firm 1.

Equilibrium $z_1=30=z_2$

The graph of the best response function, $b_1(x_2)$ of firm 1.
In General

We may have:

3 equilibrium points

No equilibrium
Safety Level

Player i can guarantee the payoff w if it has a strategy \( x_i \) such that \( u_i(x_i, x_{-i}) \geq w \), for all \( x_{-i} \) in \( X_{-i} \).

**Definition:** Let \( G \) be a regular game.
the safety level of i at the game \( G \) is a number \( L_i = L_i(G) \) that satisfies:
1. i can guarantee \( L_i \).
2. For every \( e > 0 \), i cannot guarantee \( L_i + e \).

Obviously, every player i in a regular game \( G \) has a safety level. Moreover,
\[ L_i = \text{Max}\{w: \text{player i can guarantee w}\} \]
that is, the safety level of Player i is the maximal payoff that i can guarantee.
Let $G$ be a regular game.
For a strategy $x_i$ of $i$, let $L_i(x_i)$ be the payoff that $i$ guarantees by playing $x_i$. That is,

\[ L_i(x_i) = \min \{ u_i(x_i, x_{-i}) : x_{-i} \text{ in } X_{-i} \}. \]

\textbf{Theorem 6:} For regular games, $L_i = \max \{ L_i(x_i) : x_i \text{ in } X_i \}$.

The proof of Theorem 6 is obvious, and therefore it is omitted.

Note that by Theorem 6,

$L_i = \max \{ \min \{ u_i(x_i, x_{-i}) : x_{-i} \} : x_i \}.$

Therefore, the safety level of $i$ is also called the \textbf{MaxMin value} of $i$. 
x_i is a **safety level strategy** (or a **MaxMin strategy**) for Player i if x_i guarantees the safety level L_i. That is, x_i is a safety level strategy for i if L_i = L_i(x_i).

**Example:**

The safety level of Player 3 is L_3 = -2, and a maxmin strategy is x_3 = B

\[
\begin{array}{ccc|ccc}
2 & 2 & 2 \\
-2 & 9 & 1 \\
0 & 0 & 0 \\
\end{array}
\]

L_1 = 2  
\[x_1 = 1\]

L_2 = -2  
\[x_2 = 1 \text{ or } 2\]
**Example:**

The safety level strategy of Player 1:

\[
\begin{array}{ccc}
3 & 4 & 4 \\
4 & 2 & 0 \\
2 & 0 & 1 \\
0 & -1 & 4 \\
1 & -1 & 2 \\
4 & 2 & 2 \\
\end{array}
\]

\[L_2 = 0 \]
\[x_2 = 1 \text{ or } 3\]

Player 1 can guarantee

\[L_1 = \text{maxmin} = 3\]
Let $G$ be a regular game. For $x_i$ in $X_i$, define

$$H_i(x_i) = \max\{u_i(x_i, x_i): x_i \text{ in } X_i\}.$$  

By jointly using $x_i$, all other players can guarantee that $i$ does not receive more than $H_i(x_i)$. Define:

$$H_i = \min\{H_i(x_i): x_i \text{ in } X_i\} = \min\{\max\{u_i(x_i, x_i): x_i \text{ in } X_i\}: x_i \text{ in } X_i\}.$$  

$H_i$ is called the **punishment level** of $i$, or the MinMax value of $i$. Obviously,

$$H_i = \min\{w: \text{ all other players can guarantee that } i \text{ will not get more than } w\}.$$  

$x_i$ is a **punishing-i profile of strategies** if it guarantees that $i$ will not receive more than $H_i$, that is, if $H_i = H_i(x_i)$. 
Example:

\[
\begin{array}{c|ccc}
& y_1 & & \\
\hline
3 & 4 & 4 & 4 \\
4 & 2 & 0 & 4 \\
2 & 0 & -1 & 1 \\
0 & 1 & 4 & 2 \\
1 & -1 & 2 & 2 \\
4 & 2 & 2 & 4 \\
\end{array}
\]

Value of punishment: 3, 4, 4

H₁ = 3  
H₂ = 4

punishing strategy = y₁ 

punishing strategy = x₁ or x=2 or x₃
Theorem 7: For every regular game $G$, and for every player $i$ 
$L_i \leq H_i$. 
That is, 

$$\max\{\min\{u_i(x_i, x_{-i}): x_{-i}\}: x_i\} \leq \min\{\max\{u_i(x_i, x_{-i}): x_i\}: x_{-i}\}.$$ 

Proof: There exists a strategy $x_i$ that guarantees $L_i$, and there 
exists $x_{-i}$ that guarantees that $i$ will not get more 
than $H_i$. Hence, $H_i \geq u_i(x_i, x_{-i}) \geq L_i$. 

Because $x_{-i}$ guarantees that $i$ will not 
receive more than $H_i$ for every choice 
of player $i$ 

Because $x_i$ guarantees $L_i$ versus every choice of 
the other players. QED
MaxMin, MinMax, and Equilibrium

Theorem 8: For regular games, for every equilibrium profile $x$, $u_i(x) \geq H_i$.

Proof: Let $z_i$ be a punishing strategy of all other players. Therefore, $\text{Max}\{u_i(z_i,z_{-i}) : z_i\} = H_i$, and for every other joint strategy, and in particular for $x_{-i}$, $\text{Max}\{u_i(z_i,x_{-i}) : z_i\} \geq H_i$. As $x_i$ is a best response to $x_{-i}$, $u_i(x) = u_i(x_i,x_{-i}) = \text{Max}\{u_i(z_i,x_{-i}) : z_i\}$. Hence, $u_i(x) \geq H_i$. QED

Because $H_i \geq L_i$, Player $i$’s payoff in equilibrium is at least her safety level payoff.
Mixed Strategies

Example:

<table>
<thead>
<tr>
<th></th>
<th>6</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>*</td>
<td>*</td>
</tr>
</tbody>
</table>

The safety level of Player 1 is $L_1=2$.

Player 1 can draw a fair coin:

- Row 1
- Row 2

If the coin falls on the green side she will choose Row 1, and otherwise she will choose Row 2.

Whatever player 2 chooses, the outcome of the game is not deterministic for Player 1 (who makes her calculation before she draws the coin).
Let us denote the mixed strategy of drawing the coin by \( p_1 = (0.5, 0.5) \). From the point of view of player 1, her payoff matrix is:

\[
\begin{array}{c|cc}
 & 1 & 2 \\
\hline
1 & 6 & 2 \\
2 & 0 & 6 \\
p_1 & 3 & 4 \\
\end{array}
\]

where the numbers in the third row represent expected payoffs. However, as a risk-neutral player, Player 1 relates to expected payoffs as payoffs. Hence, without using mixed strategies player 1 can guarantee a payoff of 2. By using the coin \( p_1 \), she can guarantee 3.

Hence, using mixed strategies may be a good idea.
Formal Definitions

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Formal Definitions

For a finite set \( Z \) we denote by \( \Delta(Z) \) the set of all probability distributions over \( Z \). That is, every \( q \in \Delta(Z) \) is a function \( q : Z \rightarrow [0,1] \), such that \( \sum_{z \in Z} q(z) = 1 \). Let \( G \) be a finite game. Every \( p_i \in \Delta(X_i) \) is called a mixed strategy of Player \( i \). The set of all mixed strategies of Player \( i \), \( \Delta(X_i) \) is denoted by \( M_i \).

• When Player \( i \) uses the mixed strategy \( p_i \), she conducts a lottery over \( X_i \), in which the probability of the outcome \( x_i \in X_i \) is \( p_i(x_i) \). She then uses the strategy, which is the outcome of this lottery. The set of all mixed strategy profiles \( p = (p_1, \ldots, p_n) \) is denoted by \( M \). That is, \( M = M_1 \times \cdots \times M_n \).

• If the players use \( p \in M \), that is, each player \( i \) uses \( p_i \), independently of the other players, the probability that the outcome \( x = (x_1, \cdots, x_n) \in X \) is chosen is

\[
p(x) = p_1(x_1)p_2(x_2)\cdots p_n(x_n).
\]

When we deal with bimatrix games \((A, B)\) with \( m \) rows and \( n \) columns, it is convenient to denote the probability that 1 uses Row \( s \), \( p_1(s) \), by \( p_{1s} \), and to use a similar terminology for Player 2. Hence, a mixed strategy of Player 1 is a vector \( p_1 = (p_{11}, \ldots, p_{1m}) \), a mixed strategy of Player 2 is a vector \( p_2 = (p_{21}, \ldots, p_{2n}) \), and for every Row \( i \) and Column \( j \), \( p(i,j) = p_1p_{2j} \) is the probability that the cell \((i, j)\) is chosen.
Example:

<p>| | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
</table>

Assume $p_1 = (z_1, 1-z_1)$, $p_2 = (w_1, 1-w_1)$, $p=(p_1,p_2)$. Hence,

- $p(1,1) = z_1w_1$,
- $p(1,2) = z_1(1-w_1)$,
- $p(2,1) = (1-z_1)w_1$,
- $p(2,2) = z_2w_2$. 
Example:

If Player 1 uses $p_1=(0.6,0.4)$, Player 2 uses $p_2=(0.5,0.5)$, and Player 3 uses $p_3=(0.8,0.2)$, the probabilities of the cells are shown above.
Still Mixing

Let $U_i(p)$ be the expected payoff of $i$ when all players, use the profile of mixed strategies $p$. That is,

$$U_i(p) = \sum_{x \in X} u_i(x)p(x)$$

**Example:**

$$p_1 = (z, 1-z)$$
$$p_2 = (w, 1-w)$$

$$U_1(p_1, p_2) = 4zw + 3z(1-w) + 3(1-z)w + 1(1-z)(1-w)$$

$$U_2(p_1, p_2) = 2zw - 1z(1-w) + 2(1-z)w + 4(1-z)(1-w)$$
Pure Strategies

Assume player i has 3 strategies a, b, and d. What is the meaning of the mixed strategy $p_i(a)=0, p_i(b)=1, p_i(d)=0$ ($p_i=(0,1,0)$)?

The player conducts a lottery whose outcome is b with probability 1. This is obviously equivalent to the usage of the pure strategy b.

For $x_i$ in $X_i$ we denote by $e_{x_i}$ in $M_i$, the mixed strategy that assigns probability 1 to $x_i$. $e_{x_i}$ is called a pure strategy of i.

From the practical point of view, using the pure strategy $e_{x_i}$ and using the strategy $x_i$ are equivalent.

Note that: $U_i(e_x)=u_i(x)$ for every $x$ in $X$, where $e_x=(e_{x1},...,e_{xn})$. 
The mixed extension

Let $G = (X_1, X_2, \ldots, X_n, u_1, u_2, \ldots, u_n)$ be a finite game in strategic form. We define the **mixed extension** of $G$ as the game $G^m$, in which the strategy set of player $i$ is $M_i = M(X_i)$, and the payoff function of $i$ is $U_i$. That is,

$$G^m = (M_1, M_2, \ldots, M_n, U_1, U_2, \ldots, U_n).$$

**Note:** $G^m$ is an infinite game.
Important Mathematical Facts

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In particular, $G^m$ is a regular infinite game
Important Mathematical Facts

Denote by $s_i$ the number of strategies of $i$. Not that every mixed strategy of $i$ is a vector in $R^{s_i}$, $p_i = (p_i(x_i))_{x_i \in X_i}$. Hence, the sum, $p_i + q_i$, and the multiplication by a real number $c$, $cp_i$ are well-defined mathematical objects, although they are not necessarily mixed strategies. However, a convex combination of mixed strategies of $i$ is a mixed strategy. That is, for every $p_i^1, p_i^2, ..., p_i^m$ in $M_i$, and for every non-negative real number $c_1, c_2, ..., c_m$ with $\sum_{s=1}^{m} c_s = 1$, $\sum_{s=1}^{m} c_s p_i^s$ is a mixed strategy in $M_i$. A subset of an Euclidean space that includes every convex combination of its members is called a convex set. Thus we have,

- The set of mixed strategies of $i$, $M_i$, is a convex set for every Player $i$.

It is important to note that:

- Every mixed strategy is a convex combination of pure strategies.

Indeed, for every $p_i \in M_i$,

$$p_i = \sum_{x_i \in X_i} p_i(x_i)x_i.$$ 

As the Cartesian product of convex sets is a convex set,

- The set of mixed strategy profiles $M$ is a convex set.

That is, if $p^1, ..., p^k \in M$, where $p^s = (p_1^s, p_2^s, ..., p_n^s)$ for every $s$, $1 \leq s \leq k$, and $c_1, ..., c_k$ are convex coefficients, $p = \sum_{s=1}^{k} c_s p^s \in M$. Note that $p = (p_1, p_2, ..., p_n)$, where

$$p_i = \sum_{s=1}^{k} c_s p_i^s \quad \text{for every Player } i.$$ 

Recall that $U_i$ is defined over $M$ as follows:

$$U_i(p) = U_i(p_1, ..., p_n) = \sum_{x \in X} p(x)u_i(x) = \sum_{(x_1, ..., x_n) \in X} p_1(x_1)p_2(x_2) \cdots p_n(x_n)u_i(x_1, x_2, ..., x_n).$$

Therefore,

$$U_i(p) = \sum_{x_j \in X_j} p_j(x_j)(\sum_{x_{-j} \in X_{-j}} p_{-j}(x_{-j})u_i(x_j, x_{-j})).$$ \hspace{1cm} (1)
where \( p_{-j} = (p_s)_{s \neq j} \), and \( p_{-j}(x_{-j}) = \prod_{s \neq j} p_s(x_s) \). Therefore, \( U_i \) is linear in the \( j^{th} \) variable. Hence,

- \( U_i \) is a linear function in each variable separately, for every Player \( i \).

More precisely, let \( i \) be a player. For every Player \( j \), for every \( p_{-j} \in M_{-j} \), for every \( p_j^1, p_j^2, ..., p_j^m \) in \( M_j \), and for every convex coefficients \( c_1, c_2, ..., c_m \),

\[
U_i\left( \sum_{s=1}^{m} c_s p_j^s, p_{-j} \right) = \sum_{s=1}^{m} c_s U_i(p_j^s, p_{-j}).
\]

Note that the distance between two mixed strategies of \( i \), \( p_i, q_i \) is defined by the usual Euclidean norm:

\[
||p_i - q_i||^2 = \sum_{x_i \in X_i} (p_i(x_i) - q_i(x_i))^2.
\]

Under this (or any other Euclidean norm), \( M_i \) is a metric space. It is well known that:

- For every \( i \), \( M_i \) is a compact metric space.

Similarly, for \( p, q \in M \), the distance between \( p \) and \( q \) is:

\[
||p - q||^2 = \sum_{i=1}^{n} ||p_i - q_i||^2.
\]

With this metric,

- \( M \) is a compact metric space.

In particular,

- \( G^m \) is an infinite regular game.
Safety Level and Punishment Level

Definitions:

The safety level of Player $i$ in the game $G^m$, $L_i(G^m)$, is called the safety level of $i$ in mixed strategies in $G$, and it is denoted by $L_i^m = L_i^m(G)$. That is,

$$L_i^m = \max_{p_i} \min_{p_{-i}} U_i(p_i, p_{-i}).$$

Similarly,

$$H_i^m = \min_{p_{-i}} \max_{p_i} U_i(p_i, p_{-i}),$$

is the punishment level of $i$ in mixed strategies.
Important Properties

**Theorem 8:**

If $U_i(p_i, e_{x_i}) \geq c$ for every $x_i$ in $X_i$, then $U_i(p_i, p_i) \geq c$ for every $p_i$ in $M_i$.

Or, in other words: If a player has a mixed strategy that guarantees $c$ versus all pure strategies of all other players, then this mixed strategy guarantees $c$ versus all mixed strategies of all other players.

**Proof of Theorem 8:**
Proof of Theorem 8

By (1) in the document "Important mathematical facts",

\[ U_i(p_i, p_{-i}) = \sum_{x_i \in X_i} p_i(x_i) \left( \sum_{x_{-i} \in X_{-i}} p_{-i}(x_{-i}) u_i(x_i, x_{-i}) \right). \]

Changing the order of summation yields

\[ U_i(p_i, p_{-i}) = \sum_{x_{-i} \in X_{-i}} p_{-i}(x_{-i}) \left( \sum_{x_i \in X_i} p_i(x_i) u_i(x_i, x_{-i}) \right). \]

By the equalities \( u_i(x_i, x_{-i}) = U_i(e_{x_i}, e_{x_{-i}}) \), and \( p_i = \sum_{x_i \in X_i} p_i(x_i) e_{x_i} \),

\[ U_i(p_i, p_{-i}) = \sum_{x_{-i} \in X_{-i}} p_{-i}(x_{-i}) U_i(p_i, e_{x_{-i}}). \]

As \( U_i(p_i, e_{x_{-i}}) \geq c \) for every \( x_{-i} \),

\[ U_i(p_i, p_{-i}) \geq \sum_{x_{-i} \in X_{-i}} p_{-i}(x_{-i}) c = c \sum_{x_{-i} \in X_{-i}} p_{-i}(x_{-i}) = c. \]
Theorem 9: If $U_i(e_{xi}, p_{-i}) \leq c$ for every $x_i$ in $X_i$, then $U_i(p_i, p_{-i}) \leq c$, for every $p_i$ in $M_i$.

Or, in words, if all other players have a strategy $p_{-i}$, which guarantees that $i$ cannot receive more than $c$ by using a pure strategy, then this $p_{-i}$ guarantees that $i$ cannot receive more than $c$, whatever mixed strategy $i$ uses.

The proof of Theorem 9 is similar to the proof of Theorem 8, and therefore it is omitted.
II

Corollary 10:
1. For every $p_i$, $\min_{p_i} U_i(p_i, p_i) = \min_{x_i} U_i(p_i, e_{x_i})$, and therefore
   $L_i^m = \max_{p_i} \min_{p_i} U_i(p_i, p_i) = \max_{p_i} \min_{x_i} U_i(p_i, e_{x_i})$.
2. For every $p_i$, $\max_{p_i} U_i(p_i, p_i) = \max_{x_i} U_i(e_{x_i}, p_i)$, and therefore
   $H_i^m = \min_{p_i} \max_{p_i} U_i(p_i, p_i) = \min_{p_i} \max_{x_i} U_i(e_{x_i}, p_i)$.

Proof:
The proof of (1) follows from Theorem 8. The proof of (2) follows from Theorem 9. QED

Corollary 11:
If $u_i(x_i, x_i) \geq c$ for every $x_i$ in $X_i$, then
$U_i(e_{x_i}, p_i) \geq c$ for every $p_i$ in $M_i$.

Proof: Since $u_i(x_i, x_i) = U_i(e_{x_i}, e_{x_i})$, the result follows from Theorem 8 with $p_i = e_{x_i}$. QED
\[ H_i \geq H_i^m \geq L_i^m \geq L_i. \]

Theorem 12: \[ H_i \geq H_i^m \geq L_i^m \geq L_i. \]

Proof: We know the inequality \( H_i^m \geq L_i^m \), because it was proved for every regular game, and in particular for the game \( G^m \).

We prove the right-hand-side inequality: Let \( z_i \) be a safety level strategy for \( i \) in \( G \). That is: \( u_i(z_i,x_{-i}) \geq L_i \), for every \( x_{-i} \) in \( X_{-i} \). By Corollary 11, \( U_i(e_{z_i},p_{-i}) \geq L_i \) for every \( p_{-i} \) in \( M_{-i} \). Hence, we have a mixed strategy of \( i \) that guarantees \( L_i \) versus all mixed strategy profiles of all other players. Therefore, the safety level of \( i \) in \( G^m \), is at least \( L_i \). That is, \( L_i^m \geq L_i \). The proof that \( H_i \geq H_i^m \) is similar, hence it is omitted. QED
Example:

<table>
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<tr>
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<tbody>
<tr>
<td>6</td>
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<tr>
<td>0</td>
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</table>

In this game $L_1 = 0$, $H_1 = 6$. Let us find $L_1^m$ and $H_1^m$.

Let $p_1 = (0.5, 0.5)$. $U_1(p_1, e_1) = 3 \geq 3$, and $U_1(p_1, e_2) = 3 \geq 3$.
Hence $L_1^m \geq 3$.

Let $p_2 = (0.5, 0.5)$. $U_1(e_1, p_2) = 3 \leq 3$, and $U_1(e_2, p_2) = 3 \leq 3$.
Hence $H_1^m \leq 3$. Therefore:

$$3 \leq L_1^m \leq H_1^m \leq 3,$$
that is
$$L_1^m = 3 = H_1^m.$$

Note that $p_1$ is a safety level strategy in mixed strategies, and $p_2$ is a punishing-of-1 strategy (to be used by 2) in mixed strategies.

Is the equality between the safety level and punishment level in mixed strategies is a coincidence? **No and Yes**
The MinMax Theorem:

**Theorem 13 (The MinMax Theorem):**
For finite two-person games, $L_i^m = H_i^m$, for every player $i=1,2$.

**Proof:** This theorem can be easily derived from the Duality Theorem in linear programming proved in the course "Deterministic models". A detailed description of the proof is given in the Appendix.
It does not work with more players

The following is a 3-person game in which, $L_3^m \leq 1$, and $H_3^m > 1$.

Indeed, let $p_3 = (z, 1-z)$ be a safety level strategy of Player 3 in mixed strategies. The expected payoff of 3 is given in the next slide, as a function of the pure strategies of players 1 and 2 respectively.
\[ U_3^m(e_i^1, e_j^2, (z, 1-z)) = \]

Here \( i = 1, j = 1 \).

\begin{align*}
\begin{array}{|c|c|}
\hline
2z+ & 2 \\
0(1-z) & \\
\hline
2 & 0z+ \\
2(1-z) & \\
\hline
\end{array}
\end{align*}

Hence, \( 2z \geq L_3^m, 2 \geq L_3^m, 2 \geq L_3^m, 2(1-z) \geq L_3^m \).

Adding the first and the forth inequalities yields: \( 1 \geq L_3^m \).
If \(((z_1,1-z_1),(y_1,1-y_1))\) is a punishing joint mixed strategy of players 1 an 2, then by using it they guarantee that 3 will not get more than \(H_1^m\).

If Player 3 uses Matrix A, she receives \(2z_1+2(1-z_1)y_1\).

Hence, \(2z_1+2(1-z_1)y_1\leq H_1^m\).

If Player 3 uses Matrix B, she receives \(2(1-z_1)+2z_1(1-y_1)\).

Hence, \(2(1-z_1)+2z_1(1-y_1)\leq H_1^m\).

If, \(H_1^m\leq1\), then adding the two inequalities yield: \(1+(1-z_1)y_1 + z_1(1-y_1)\leq1\).

Therefore \(y_1=0\) and \(z_1=0\) OR \(z_1=1\) and \(y_1=1\).

\[2 \leq 1\] \[2 \leq 1\]

Hence, \(H_1^m\) must be greater than 1. QED
A Useful Corollary

**Theorem 14**: Consider a two-person finite game. If for every mixed strategy $p_2$, of Player 2, Player 1 has a strategy $x_1$, such that $U_1(e_{x_1},p_2) \geq c$, then Player 1 has a mixed strategy $p_1$, such that $U_1(p_1,p_2) \geq c$, for every mixed strategy $p_2$ of Player 2.

**Proof**: Note that the statement “for every mixed strategy $p_2$, of Player 2, Player 1 has a strategy $x_1$, such that $U_1(e_{x_1},p_2) \geq c$” is equivalent to “$\max_{x_1} U_1(e_{x_1},p_2) \geq c$, for every $p_2$”, which is equivalent to the statement “$\min_{p_2} \max_{x_1} U_1(e_{x_1},p_2) \geq c$”, which, by Corollary 10 is equivalent to “$H_{1,m} \geq c$”. Similarly, the statement “Player 1 has a mixed strategy $p_1$, such that $U_1(p_1,p_2) \geq c$, for every mixed strategy $p_2$ of Player 2” is equivalent to “$L_{1,m} \geq c$”. Hence, Theorem 14 can be restated as follows:
II

For every $c$, $H_1^m \geq c$ implies $L_1^m \geq c$.

Since by Theorem 13, $H_1^m = L_1^m$, the proof is completed. QED
Mixed Strategy Equilibrium

**Definition**: Let \( p=(p_1,p_2,\ldots,p_n) \) be a mixed strategy profile in \( M \). \( p \) is a mixed strategy equilibrium in the game \( G \), if \( p \) is in equilibrium in the game \( G^m \).

Thus, \( p \) is a mixed strategy equilibrium if for every \( i \), \( p_i \) is a best response to \( p_{-i} \) in the game \( G^m \). We proceed to explore properties of best-response strategies in \( G^m \):

**Theorem 15**: Let \( p_i \) be a best response to \( q_{-i} \). Let \( z_i \) be a strategy of \( i \) such that the associated pure strategy \( e_{z_i} \) is not a best response to \( q_{-i} \). Then \( p_i(z_i)=0 \).

**Theorem 16**: Let \( q_{-i} \) belongs to \( M_{-i} \). Let \( p_i \) belongs to \( M_i \). \( p_i \) is a best response to \( q_{-i} \) if the following condition holds: For every \( z_i \) in \( X_i \), \( p_i(z_i)>0 \) implies \( zee \) is a best response to \( q_{-i} \).

The proofs of theorems 15, 16 are given here:
Proofs of theorems 15 and 16.

Remark: Recall that every \( p_i \in M_i \) is a convex combination of the pure strategies as follows:

\[
p_i = \sum_{x_i \in X_i} p_i(x_i) e_{x_i}.
\]

It is important to note that the set of vectors \( \{e_{x_i} : x_i \in X_i\} \) is linearly independent, and therefore the coefficients in the above formula are uniquely determined by \( p_i \). That is, if

\[
p_i = \sum_{x_i \in X_i} \alpha_{x_i} e_{x_i},
\]

then \( \alpha_{x_i} = p_i(x_i) \) for every \( x_i \in X_i \). The next theorem shows that if \( p_i \) is a best response to \( q_{-i} \), then \( p_i(x_i) = 0 \) for every \( x_i \), such that \( e_{x_i} \) is not a best response to \( q_{-i} \).

Proof of Theorem 15: Let \( r = U_i(p_i, q_{-i}) \). By the linearity of \( U_i \) in the \( i^{th} \) variable, for every \( q_i \in M_i \)

\[
U_i(q_i, q_{-i}) = \sum_{x_i \in X_i} q_i(x_i) U_i(e_{x_i}, q_{-i}).
\]

In particular,

\[
r = U_i(p_i, q_{-i}) = \sum_{x_i \in X_i} p_i(x_i) U_i(e_{x_i}, q_{-i}).
\]

Note that \( e_{x_i} \) is a best response to \( q_{-i} \) iff \( U_i(e_{x_i}, q_{-i}) = r \). By Corollary 9 there exists \( y_i \) such that \( U_i(e_{y_i}, q_{-i}) = r \). On the other hand, \( U_i(e_{z_i}, q_{-i}) < r \). Supposed in negation that \( p_i(z_i) > 0 \). Define a new mixed strategy \( \tilde{p}_i \) as follows: \( \tilde{p}_i(z_i) = 0 \), \( \tilde{p}_i(y_i) = p_i(y_i) + p_i(z_i) \), and \( \tilde{p}_i(x_i) = p_i(x_i) \), for every \( x_i, x_i \neq y_i, z_i \). Obviously,

\[
U_i(\tilde{p}_i, q_{-i}) - U_i(p_i, q_{-i}) = p_i(z_i) (U_i(e_{y_i}, q_{-i}) - U_i(e_{z_i}, q_{-i})) > 0,
\]

contradicting the fact that \( p_i \) is a best response to \( q_{-i} \). Hence \( p_i(z_i) = 0 \). \( \square \)

The proof of Theorem 16 is very similar to the proof of Theorem 15, and therefore it is omitted.
Characterizations of Best Response Strategies:

By theorems 15, 16, and by the remark preceding the proof of Theorem 15,

**Theorem 17**: $p_i$ is a best response to $q_{-i}$ if and only if $p_i$ assigns positive probabilities only to pure strategies that are best response to $q_{-i}$.

**Corollary 18**: Let $q_{-i}$ be in $M_{-i}$. Let $r=\max_{x_i} U_i(e_{x_i}, q_{-i})$. Then, $p_i$ is a best response to $q_{-i}$ if and only if $p_i(x_i)=0$ for every $x_i$, for which $U_i(e_{x_i}, q_{-i})<r$. Consequently, $e_{x_i}$ is a best response to $q_{-i}$ if and only if $U_i(e_{x_i}, q_{-i})=r$. 
Exercises:

**Exercise 1**: Reply with a “True” or a “False”

Consider a two-person 3x19 game, in which \( p=(0.7,0,0.3) \) is a best response to \( q \), and \( U_1(p,q)=6 \). Then:

1. \( U_1(e_1,q) \geq 6 \).
2. \( U_1(e_2,q) < 6 \).
3. \( U_1(e_3,q) = 6 \).
4. \( U_1((0.1,0.4,0.5),q) < 6 \).

\[ \begin{array}{ccc}
T & F & T \\
1. U_1(e_1,q) \geq 6 & 2. U_1(e_2,q) < 6 & 3. U_1(e_3,q) = 6 \\
4. U_1((0.1,0.4,0.5),q) < 6 & F \\
\end{array} \]
II

**Exercise 2:**

Consider a game $G$ with 250 players in which Player 1 has four strategies, $1,2,3,4$.

Let $q$ be a profile of mixed strategies of all other players.

Suppose the following holds:

1. $U_1(e_1,q)=6$,
2. $U_1(e_2,q)=6$
3. $U_1(e_3,q)=6$
4. $U_1(e_4,q)=5$

**AND** $U_1(p,q)\leq 6$ for every mixed strategy $p$ of player 1

**True or False:**

$p=(0.25,0.25,0.25,0.25)$ is a best response to $q$. \(\text{F}\)

$p=(0.25,0,0.75,0)$ is a best response to $q$. \(\text{T}\)
Exercise 3:

Consider a game $G$ with 19 players in which Player 1 has four strategies, 1, 2, 3, 4.

Let $q$ be a profile of mixed strategies of all other players.

Suppose the following holds:

1. $U_1(1,q)=6$,
2. $U_1(2,q)=6$
3. $U_1(3,q)=6$
4. $U_1(4,q)=5$

True or False:

- $p=(0.25,0.25,0.25,0.25)$ is a best response to $q$.  
  - F
- $p=(0.25,0,0.75,0)$ is a best response to $q$.  
  - T
A Characterization of Mixed Strategy Equilibrium

**Theorem 19**: \((p_1, p_2, \ldots, p_n)\) is a mixed-strategy equilibrium if and only if the following holds for every player \(i\):
For every pure strategy \(x_i\),
either
\(x_i\) is a best response to \(p_{-i}\)
or
\(p_i(x_i) = 0\).
**Proof**: The proof is a simple consequence of Theorem 17. QED

A simple corollary of Theorem 19 is that by extending the game \(G\) to the game \(G^m\), we do not loose equilibrium profiles:

**Corollary 20**: Let \(G\) be a finite game.\(e_x\) is a mixed strategy equilibrium if and only if \(x\) is an equilibrium strategy.
**Example:**

In this example you learn how to prove that a certain profile of strategies is a mixed-strategy equilibrium.

In the following three-person game, player $i$ wishes to be with player $i+1$ and without player $i+2$.

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Matrix 1

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Matrix 2
We show that \((p_1, p_2, p_3)\) is in equilibrium, where \(p_i = (0.5, 0.5)\) for every \(i = 1, 2, 3\).

We have to show that \(p_i\) is a best response to \(p_{\neg i}\) for \(i = 1, 2, 3\). However, because the game is symmetric it suffices to show that \(p_3\) is a best response to \(p_{\neg 3}\).

We show that:
\[
U_3(p_1, p_2, e_{\text{matrix 1}}) = U_3(p_1, p_2, e_{\text{matrix 2}}).
\]

Therefore, by Corollary 17, both \(e_{\text{matrix 1}}\) and \(e_{\text{matrix 2}}\) are best response to \((p_1, p_2)\).

Hence, \(p_3\) assigns a positive probability only to pure strategies, which are best responses to \((p_1, p_2)\), and thus, \(p_3\) is a best response to \((p_1, p_2)\).
Exercise 4: True or False?

Is \((1,0),(0,1)\) is a mixed-strategy equilibrium in \(G\)?

Solution:

True;
As (Row 1, Column 2) is in equilibrium in \(G\), then by Corollary 17, \((e_{Row1},e_{Column2})\) is in equilibrium in \(G^m\).
Existence of Mixed Strategy Equilibrium

**Theorem 21** (Nash, 1951): Every finite game has a mixed-strategy equilibrium.

**Proof:** Can be found in the Appendix.

Methods for computing mixed-strategy equilibrium will be given at the TA class.
We will not deal with a complete theory of domination with mixed strategies. Just note the following example:

\[
\begin{array}{ccc}
4 & 1 & 4 \\
* & * & * \\
1 & 3 & 0 \\
* & * & * \\
0 & 6 & 1 \\
* & * & *
\end{array}
\]

In G, player 1 does not have dominated strategies. However, with mixed strategies: (0.5,0,0.5) dominates Row 2.

Hence, the elimination process of strategies in the game G may yield a smaller game, when mixed strategies are taken into account!
Two-Person Zero-Sum Games

A two-person game \( G=(X_1,X_2,u_1,u_2) \) is zero-sum, if 
\[ u_2(x_1,x_2) = -u_1(x_1,x_2), \text{ for every } x=(x_1,x_2) \text{ in } X. \]

A finite zero-sum game can be described by a simple matrix, in which we write the payoffs of Player 1.

Thus, the following zero sum game has the two equivalent forms:

\[
\begin{array}{cc}
1 & -2 \\
-1 & 2 \\
0 & -3 \\
0 & 3 \\
\end{array}
\quad \text{OR} \quad
\begin{array}{cc}
1 & -2 \\
0 & -3 \\
\end{array}
\]
Please note:
A zero-sum game is first of all a (two-person) game, and all previous theorems hold.!!!!

**Theorem 22**: Let $G=(X_1,X_2,u_1,u_2)$ be a regular 2-person zero-sum game. Then, $L_2=-H_1$, and $H_2=-L_1$.

**Proof**: Let $x_2$ be a punishing-1 strategy. That is, $u_1(x_1,x_2) \leq H_1$, for every $x_1$ in $X_1$. Hence, $-u_1(x_1,x_2) \geq -H_1$ for every $x_1$. That is, $u_2(x_1,x_2) \geq -H_1$ for every $x_1$. Therefore, $L_2 \geq -H_1$.

Let $z_2$ be a safety level strategy of Player 2. Therefore, $u_2(x_1,z_2) \geq L_2$ for all $x_1$, yielding $u_1(x_1,z_2) \leq -L_2$. Therefore $H_1 \leq -L_2$, or $L_2 \leq -H_1$. Hence, $L_2=-H_1$. The proof of the other equality is similar, and therefore it is omitted. QED
Corollary 23: Let G be a regular two-person zero-sum game. \( x_i \) is a safety level strategy for i iff \( x_i \) is a punishment-of-[-i] strategy, \( i=1,2 \).

**Example:**

\[
\begin{array}{c|c|c}
 & 3 & -3 \\
-1 & 1 & \\
2 & 1.5 & -1.5 \\
-2 & \\
\end{array}
\]

\( L_1=1.5, H_2=-1.5 \) and Row 2 is both, a safety level strategy for 1, and a punishing-player 2 strategy.

Similarly,

\( L_2=-2, H_1=2 \), and Column 1 is a safety level strategy for 2, which is also a punishing-Player 1 strategy.
IV

**Definition:** A 2-person regular zero-sum game has a value, if $L_i = H_i$, $i = 1, 2$. The value of $G$ denoted by $v(G)$ is defined to be $L_1$.

Note that by Theorem 22, it suffices to check only one of the equalities, that is the game has a value iff $L_1 = H_1$ iff $L_2 = H_2$.

**Definition:** in a 2-person zero sum game with a value, a safety level strategy for $i$ is called an optimal strategy for $i$.

**Definition:** A finite 2-person zero-sum game $G$ has a value in mixed strategies, if $G^m$ has a value, that is, if $L_1^m = H_1^m$.

**Theorem 24:** Every finite 2-person zero sum game has a value in mixed strategies.

**Proof:** By the Minmax theorem (Theorem 13), $L_1^m = H_1^m$. QED
Examples

L₁ = 1.5 < 2 = H₁.
Hence, the game does not have a value.

L₁ = 3 = H₁.
Hence, the game has a value, v = 3. Row 1 is optimal for 1, and Column 2 is optimal for 2.
Value and Equilibrium

**Theorem 25:**

Let $G$ be a regular 2-person zero-sum game in strategic form. Then,

1. $G$ has a value iff it has an equilibrium.
2. If $G$ has a value, then $(x_1, x_2)$ is in equilibrium iff $x_i$ is optimal for $i$, for every $i = 1, 2$.
3. If $G$ has a value, and $(x_1, x_2)$ is in equilibrium, then $u_i(x_1, x_2) = L_i$ for every $i = 1, 2$, and in particular $u_1(x_1, x_2) = v(G)$.

**Proof:** Suppose $G$ has an equilibrium $(z_1, z_2)$. By Theorem 8, $u_1(z_1, z_2) \geq H_1$. Also, by Theorem 8, $u_2(z_1, z_2) \geq H_2$. That is, $u_1(z_1, z_2) \leq -H_2 = L_1$, by Theorem 22. Hence, $L_1 = H_1$, that is the game has a value.
Suppose the game has a value. Let $x_i$ be an optimal strategy for $i$, $i=1,2$. We first show that $u_1(x_1,x_2)=L_1$. Indeed, because $x_1$ is optimal for 1, $u_1(x_1,x_2) \geq L_1$, and because $x_2$ is optimal for 2, $u_1(x_1,x_2) \leq L_1$. Similarly we can show that $u_2(x_1,x_2)=L_2$.

Next we show that $(x_1,x_2)$ is in equilibrium. We have to show that:

- $u_1(z_1,x_2) \leq L_1$ for every $z_1$, and that $u_2(x_1,z_2) \leq L_2$ for every $z_2$.

The inequalities follow from the optimality of $x_2$ and $x_1$, respectively. Please verify that we have proved all three parts. 

QED
A Few Exercises

**Question:** Let $G$ be a finite 2-person zero-sum game, and let $(p,q)$ and $(p^*,q^*)$ be mixed strategy equilibrium profiles. Then, $(p,q^*)$ is a mixed strategy equilibrium profile. True or False?

**True**

**Question:** Consider the following game, $G=$

\[
\begin{array}{cc}
4 & 3 \\
-4 & -3 \\
2 & 1.5 \\
-2 & -1.5 \\
\end{array}
\]

If $(p,q)$ is a mixed strategy equilibrium in $G$, then $U_2(p,q)=-3$. True or False?

**True**