The Complexity of Projections of Combinatorial Polytopes

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Joint work with student Michal Melamed
Motivation: Convex Multicriteria Optimization
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\max \{ f(Wx) : x \in S \}
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Where:

- **S**: Set of feasible points in \( \mathbb{Z}^n \)
- **W**: Integer \( d \times n \) matrix
- **f**: Convex function from \( \mathbb{R}^d \) to \( \mathbb{R} \)
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Interpreted as convex multicriteria optimization with objective

\[ f(Wx) = f(W_1 x, \ldots, W_d x) \] trading off \( d \) linear criteria \( W_i x \)
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We are particularly interested in one of the following situations:

- **S** is in \( \{0,1\}^n \) with some combinatorial structure
- **S** = \( \{x \in \mathbb{Z}^n : Ax \leq b\} \) is given by linear inequalities
Projections of Polytopes

The optimal solution of \( \max \{ f(Wx) : x \text{ in } S \} \)

is attained at a vertex of the projection integer polytope in \( \mathbb{R}^d \)

\[ P = \text{conv}(WS) = W \text{conv}(S) \]
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In this talk we assume that the entries of \( W \) are small or even 0-1.

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Example 1: Projections of Arbitrary 0-1 Sets

What is the maximum number $v(d,n)$ of vertices of \( \text{conv}(WS) \) when $S$ is any set in $\{0,1\}^n$ and $W$ is any 0-1 valued $d \times n$ matrix?
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We demonstrate it for $k=3$. Let $n=k+k^2=12$ and let

$$W = \begin{pmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{pmatrix}$$

$$S = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
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Then $WS=\{(i,i^2) : 0 \leq i \leq k \}$ has $\Omega(n^{0.5})$ points on the moment curve.
Now let $S$ in $\{0,1\}^n$ be a matroid of order $n$, that is, the set of indicating vectors of bases of a matroid with ground set $\{1,\ldots,n\}$. 

**Matroids**
Matroids

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**Theorem 1:** For any dimension $d$, the maximum number of vertices of $\text{conv}(WS)$ when $S$ is any matroid of any order $n$ and $W$ is any 0-1 $d \times n$ matrix is a constant $m(d)$ independent of the matroid and $n$. 
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In fact \( d2^d \leq m(d) \leq 2 \sum_{k=0}^{d-1} \left( \frac{1}{2} \binom{3^d - 3}{k} \right) = O(3^{d(d-1)}) \) and hence \( m(2) = 8 \).
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More generally, the maximum number of vertices of the projection of any matroid by any $\{0,\pm 1,\ldots, \pm p\}$ matrix $W$ is bounded by a constant $m(d,p)$. 

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Example 2: Planar Binary Projections of Matroids

The maximum number $m(2)$ of vertices of a planar projection $\text{conv}(WS)$ of any matroid $S$ of any order $n$ by any binary matrix $W$ is attained by the following matrix and uniform matroid of rank 3 and order 8,

$$W = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$

$$S = \text{U}(3,8) = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$
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where $\text{conv}(WS) =$
Now consider the set $S$ of $l \times m \times n$ tables with given line sums:

$$S = \{x \in \mathbb{Z}_+^{l \times m \times n} : \sum_i x_{i,j,k} = a_{j,k}, \sum_j x_{i,j,k} = b_{i,k}, \sum_k x_{i,j,k} = c_{i,j}\}$$
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**Theorem 2:** For any $d$, the maximum number of vertices of $\text{conv}(WS)$ for any set $S$ of $l \times m \times n$ tables with given line sums and any $\{0, \pm 1, \ldots, \pm p\}$ matrix $W$ is a constant $t(d,p;l,m)$ independent of $n$ and the line sums.
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More generally, a similar statement holds for $m_1 \times \cdots \times m_k \times n$ tables.
**The Main Theorem**

**Definition:** The edge complexity of $S$ is the smallest integer such that any edge direction of $\text{conv}(S)$ is parallel to some $v$ in $\mathbb{Z}^n$ with $|v|_1 \leq e(S)$. 

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set $E$ of all edge directions
The Main Theorem

**Definition:** The *edge complexity* of $S$ is the smallest integer such that any *edge direction* of $\text{conv}(S)$ is parallel to some $v$ in $\mathbb{Z}^n$ with $|v|_1 \leq e(S)$.

**Theorem 3:** For any $d$, the maximum number of vertices of $\text{conv}(W S)$ for any set $S$ of edge complexity $e(S)$ by any matrix $W$ is $O(e(S) |W|^{d(d-1)})$. 

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Lemma 1: If $E = \{e^1, \ldots, e^m\}$ covers all edge directions of a polytope $P$ then the zonotope $Z = [-1, 1] e^1 + \ldots + [-1, 1] e^m$ is a refinement of $P$. 
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Proof

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$Z$ $P$
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**Proof**

Lemma 2: In $\mathbb{R}^d$, the zonotope $Z = \text{zone}\{e^1, \ldots, e^m\}$ has $O(m^{d-1})$ vertices.

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Proof of Main Theorem 3

- Set $S$ in $\mathbb{Z}^n$ with edge complexity $e(S)$
- Weight $d \times n$ matrix $W$
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Each edge direction of $\text{conv}(WS)$ is parallel to $Wh$ for some edge direction $h$ of $\text{conv}(S)$.
Proof of Main Theorem 3

- Set $S$ in $\mathbb{Z}^n$ with edge complexity $e(S)$
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Each edge direction of $\text{conv}(WS)$ is parallel to $Wh$ for some edge direction $h$ of $\text{conv}(S)$.

Hence $E := \{0, \pm 1, \ldots, \pm e(S)|W|\}^d$ covers all edge directions of $\text{conv}(WS)$. 

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- Set $S$ in $\mathbb{Z}^n$ with edge complexity $e(S)$
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Each edge direction of $\text{conv}(WS)$ is parallel to $W h$ for some edge direction $h$ of $\text{conv}(S)$.

Hence $E := \{0, \pm 1, \ldots, \pm e(S) | W| \}^d$ covers all edge directions of $\text{conv}(WS)$.

So $\text{zone}(E)$ refines $\text{conv}(WS)$. Since $|E| = O(e(S) | W|^d)$ it follows that $\text{conv}(WS)$ has $O(e(S) | W|^{d(d-1)})$ vertices.
Proof of Theorem 1 from Theorem 3

**Theorem 1:** For any $d$, the maximum number of vertices of $\text{conv}(WS)$ when $S$ is any matroid of any order $n$ and $W$ is any $\{0, \pm 1, \ldots, \pm p\}$ valued $d \times n$ matrix is a constant $m(d,p)$ independent of the matroid and $n$. 

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Proof of Theorem 1 from Theorem 3

**Theorem 1:** For any \( d \), the maximum number of vertices of \( \text{conv}(WS) \) when \( S \) is any matroid of any order \( n \) and \( W \) is any \( \{0, \pm 1, \ldots, \pm p\} \) valued \( d \times n \) matrix is a constant \( m(d,p) \) independent of the matroid and \( n \).

**Proof:**

For any matroid \( S \) each edge direction of \( \text{conv}(S) \) is the difference \( 1_i - 1_j \) of two unit vectors. Hence the edge complexity of \( \text{conv}(S) \) for any matroid is constant \( e(S)=2 \).
Proof of Theorem 1 from Theorem 3

**Theorem 1:** For any $d$, the maximum number of vertices of $\text{conv}(WS)$ when $S$ is any matroid of any order $n$ and $W$ is any $\{0, \pm 1, \ldots, \pm p\}$ valued $d \times n$ matrix is a constant $m(d,p)$ independent of the matroid and $n$.

**Proof:**

For any matroid $S$ each edge direction of $\text{conv}(S)$ is the difference $1_i - 1_j$ of two unit vectors. Hence the edge complexity of $\text{conv}(S)$ for any matroid is constant $e(S)=2$.

For $d=2$ the refining zonotope is homothetic to $\text{conv}(WS)$ for the uniform matroid $S=U(3,8)$. 

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Proof of Theorem 2 from Theorem 3

**Theorem 2**: For any $d$, the maximum number of vertices of $\conv(WS)$ for any set $S$ of $l \times m \times n$ tables with given line sums and any $\{0, \pm 1, \ldots, \pm p\}$ matrix $W$ is a constant $t(d,p;l,m)$ independent of $n$ and the line sums.

**Proof:**

The **Graver basis** of the matrix which defines the line sum equations covers all edge directions of $\conv(S)$.

Our theory of **$n$-fold integer programming** implies that any element $v$ in that Graver basis satisfies $|v|_1 \leq e(l,m)$ and hence the edge complexity of $\conv(S)$ is constant $e(S)=e(l,m)$.
Some Open Problems

Determine or bound the maximum number of vertices of $\text{conv}(WS)$ for the following sets $S$ with $W$ any $\{0, \pm 1, \ldots, \pm p\}$ valued $d \times n$ matrix:

- $v(d,n)$ for arbitrary 0-1 sets $S$ and 0-1 matrix $W$, or even $v(2,n)$
- $m(d,p)$ for any matroid $S$
- $m(d)$ for any matroid $S$ and 0-1 matrix $W$, we only know $m(2)=8$
- $u(d)$ for any uniform matroid $S$ and 0-1 matrix $W$
- $t(d,p;l,m)$ for any set $S$ of $l \times m \times n$ tables with given line sums
- $t(2,1;l,m)$ for projections to the plane of tables by 0-1 matrix $W$
- $b(d,n)$ for planar projections of Birkhoff’s polytope by 0-1 matrix $W$

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Reference: Michal Melamed and Shmuel Onn,
Convex integer optimization by constantly many linear counterparts
Background and more info on convex and nonlinear multicriteria optimization, Graver bases, and n-fold integer programming, can be found in my monograph available electronically from my homepage (with kind permission of EMS).