Multiway Polytopes: Universality and Convex Optimization

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Based on several papers joint with various subsets of
{De Loera, Hemmecke, Rothblum, Weismantel}

Supported in part by ISF - Israel Science Foundation
Multiway Tables and Margins

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\begin{array}{ccc}
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\end{array}
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\begin{array}{ccc}
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2 & 2 & 0 & 4 \\
2 & 3 & 2 & \\
\end{array}
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![3-way table diagram]
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Two main contrasting results:

**Universality Theorem:** Any rational polytope is an $r \times c \times 3$ line-sum polytope.

**Optimization Theorem:** Convex Integer Programming over $m_1 \times \cdots \times m_k \times n$ polytopes is solvable in polynomial time.

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Universality Theorem for Short 3-Way Polytopes

**Theorem:** Any rational polytope $P = \{ y \in \mathbb{R}^m : Ay = b \}$ is polytime representable as an $r \times c \times 3$ line-sum polytope

$$T = \left\{ x \in \mathbb{R}_+^{r \times c \times 3} : \sum_i x_{i,j,k} = w_{j,k}, \sum_j x_{i,j,k} = v_{i,k}, \sum_k x_{i,j,k} = u_{i,j} \right\}$$

(there is a coordinate-erasing projection from $\mathbb{R}^{r \times c \times 3}$ to $\mathbb{R}^m$ giving a bijection between $T$ and $P$ and between their integer points).

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→ Optimization over $r \times c \times 3$ tables is NP-hard.
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\[\rightarrow\] Any linear/integer program is polytime representable as an \( r \times c \times 3 \) multiway program.

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\[\rightarrow\] Implications on the rational version of Hilbert’s 10\textsuperscript{th} problem on the decidability of the realization problem for polytopes?
Table Security (confidential data disclosure)

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Question: how does the set of values that can occur in a specific entry in all tables with the released margins look like?

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Fact: for $k$-way tables with fixed hyperplane-sums, the set of values in an entry is always an interval.

Example: the values 0, 2 occur in an entry:

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\begin{array}{ccc}
0 & 1 & 2 \\
2 & 2 & 0 \\
2 & 3 & 2 \\
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Fact: for k-way tables with fixed hyperplane-sums, the set of values in an entry is always an interval.

Example: the values 0, 2 occur in an entry:

Therefore, also the value 1 occurs in that entry:
In contrast we have the following universality:

**Theorem:** For every finite set $S$ of nonnegative integers, there are $r$, $c$ and line-sums for $r \times c \times 3$ tables such that the set of values occurring in a fixed entry in all possible tables with these line-sums is precisely $S$. 
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**Proof:** Given $S = \{s_1, \ldots, s_m\}$, let

$$P := \{y \in \mathbb{R}_+^{m+1} : y_0 - \sum_{i=1}^m s_i y_i = 0, \sum_{i=1}^m y_i = 1\}.$$ 

Lift $P$ using the universality theorem to $r \times c \times 3$ line-sum polytope $T$. 

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Consider the designated entry:
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Consider the following line-sums for $6 \times 4 \times 3$ tables:

The only values occurring in that entry in all possible tables with these line-sums are 0, 2

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More Consequences

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Universality Theorem for Toric Ideals: Every toric ideal is embeddable in a toric ideal of $r \times c \times 3$ tables with fixed line-sums.
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**Solution of the Vlach Problems:** Many problems of the cornerstone paper by M. Vlach on transportation polytopes resolved.
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Solution of the Vlach Problems: Many problems of the corner stone paper by M. Vlach on transportation polytopes resolved.

Universality Theorem for Bitransportation Polytopes:

Theorem: Any rational polytope $P = \{y \in \mathbb{R}_+^m : Ay = b\}$ is polytime representable as an $n \times n$ bitransportation polytope

$$B = \left\{ (x^1, x^2) \in \oplus_2 \mathbb{R}_+^{n \times n} : \sum_j x^k_{i,j} = r^k_i, \sum_i x^k_{i,j} = c^k_j, x^1_{i,j} + x^2_{i,j} \leq u_{i,j} \right\}$$

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Convex Integer Optimization
over Long Multiway Polytopes

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Consider more generally the following convex integer programming problem

\[ \text{max} \{ c(w_1x, \ldots, w_dx) : x \geq 0, \ Ax = b, \ x \text{ integer} \} \]

where \( w_1, \ldots, w_d \) are linear forms and \( c \) is a convex functional on \( \mathbb{R}^d \).
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The problem can be interpreted as balancing $d$ given linear criteria. It is generally intractable even for fixed $d=1$, since standard linear integer programming is the special case with $c$ the identity on $\mathbb{R}$. 

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Nonetheless, we can show the following (and more):

**Theorem:** Fix \( d, m_1, \ldots, m_k \). Then convex integer programming over any \( m_1 \times \cdots \times m_k \times n \) multiway polytope is solvable in polynomial oracle-time for any margins, \( w_1, \ldots, w_d \), and convex \( c \) presented by comparison oracle.

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Proof Ingredient 1: Edge-Directions
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Exploit edge symmetry of the integer hull

\[ P = \text{conv}\{x : x \geq 0, \ Ax = b, \ x \text{ integer}\} \subseteq \mathbb{R}^n \]
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**Lemma 1:** Fix \( d \). Then, given a set \( E \) covering all edge-directions of \( P \), the convex integer programming problem over \( P \) is reducible to solving polynomially many linear integer programming counterparts over \( P \).
Zonotope Refinement and Construction

Prop. 1: If \( E = \{e^1, \ldots, e^m\} \) covers all edge-directions of a polytope \( P \) then the zonotope \( Z = [-1, 1] e^1 + \ldots + [-1, 1] e^m \) is a refinement of \( P \).

(Minkowsky, Grunbaum, \ldots, )
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Zonotope Refinement and Construction

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**Prop. 2:** In \( \mathbb{R}^d \), the zonotope \( Z \) can be constructed from \( E = \{ e^1, \ldots, e^m \} \) along with a vector \( a_i \) in the cone of every vertex in \( O(m^{d-1}) \) operations.

(Edelsbrunner, Gritzmann, Orouk, Seidel, Sharir, Sturmfels, …)
The Algorithm Establishing Lemma 1

Input: Polytope $P$ in $\mathbb{R}^n$ given via $A, b$, set $E$ covering its edge-directions, $d \times n$ matrix $w$, and convex functional $c$ on $\mathbb{R}^d$ given by comparison oracle.
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2. Lift each \( a_i \) in \( \mathbb{R}^d \) to \( b_i = w^T \cdot a_i \) in \( \mathbb{R}^n \) and solve linear integer programming with objective \( b_i \) over \( P \)

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4. Output any $v_i$ attaining maximum value $c(w \cdot v_i)$ using comparison oracle.
Proof ingredient 2: Graver Bases

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A vector $u$ is conformal to vector $v$ if $|u_i| \leq |v_i|$ and $u_iv_i \geq 0$ for all $i$. 
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A vector $u$ is conformal to vector $v$ if $|u_i| \leq |v_i|$ and $u_i v_i \geq 0$ for all $i$.

The Graver basis of an integer matrix $A$ is the set of conformal-minimal nonzero integer dependencies on $A$, i.e. vectors with $Av = 0$. For instance, the Graver basis of $A = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix}$ is $\pm \{ [2 -1 0], [0 -1 2], [1 0 -1], [1 -1 1] \}$.
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Lemma 2: The Graver basis of $A$ allows to augment in polynomial time any feasible solution to an optimal solution of any linear integer program

$$\max \{ wx : x \geq 0, \ Ax = b, \ x \text{ integer} \}$$
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Lemma 3: The Graver basis of $A$ covers all edge-directions of any fiber $P = \text{conv}\{ x : x \geq 0, \ A x = b, \ x \text{ integer} \}$
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Lemma 3: The Graver basis of $A$ covers all edge-directions of any fiber

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Lemma 4: The Graver basis of the matrix $A$ defining the margin equations for any $m_1 \times \cdots \times m_k \times n$ polytope is polytime computable.

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Combining Lemmas 1 – 4 plus some additional components, we obtain the optimization theorem for long k-way polytopes:

**Theorem:** Fix $d, m_1, \ldots, m_k$. Then convex integer programming over any $m_1 \times \cdots \times m_k \times n$ multiway polytope is solvable in polynomial oracle-time for any margins, $w_1, \ldots, w_d$, and convex $c$ presented by comparison oracle.

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In contrast, short 3-way polytopes are universal:

**Theorem:** Any rational polytope is an $r \times c \times 3$ line-sum 3-way polytope.

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