N-Fold Integer Programming
and
Multicommodity Flows

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Based on several papers joint with several co-authors including
Berstein, De Loera, Hemmecke, Lee, Rothblum, Weismantel
Prologue:

Nonlinear Discrete Optimization
Setup for Nonlinear Discrete Optimization

The problem is: \[ \min/\max \{ f(Wx) : x \in S \} \]
Setup for Nonlinear Discrete Optimization

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The data is a triple:
Setup for Nonlinear Discrete Optimization

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The **data** is a triple:

- **S**: Set of feasible points in \( \mathbb{Z}^n \)
Setup for Nonlinear Discrete Optimization

The problem is: \[ \min/\max \{ f(Wx) : x \in S \} \]

The data is a triple:

- \( S \): Set of feasible points in \( \mathbb{Z}^n \)
- \( W \): Integer \( d \times n \) matrix
Setup for Nonlinear Discrete Optimization

The problem is:

\[ \min/\max \{ f(Wx) : x \text{ in } S \} \]

The data is a triple:

- **S**: Set of feasible points in \( \mathbb{Z}^n \)
- **W**: Integer \( d \times n \) matrix
- **f**: Function from \( \mathbb{Z}^d \) to \( \mathbb{R} \)
Setup for Nonlinear Discrete Optimization

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This setup enables:

- Determination of broad classes of triples \(S, W, f\) solvable efficiently (deterministically, randomly, or approximately)
**Setup for Nonlinear Discrete Optimization**

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This setup enables:

- **Determination of broad classes** of triples \( S, W, f \) solvable efficiently (deterministically, randomly, or approximately)

- **Interpretation as multi-objective optimization** with objective \[ f(Wx) = f(W_1x, \ldots, W_dx) \] balancing criteria \( W_i x \) of \( d \) players

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Setup for Nonlinear Discrete Optimization

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The presentation of \( S \) splits the theory into two branches:
Setup for Nonlinear Discrete Optimization

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The presentation of \( S \) splits the theory into two branches:

- **Combinatorial Optimization**:
  \( S \) in \( \{0,1\}^n \) given by oracle (membership, linear-optimization, etc.)
Setup for Nonlinear Discrete Optimization

The problem is: \[ \min/\max \{ f(Wx) : x \in S \} \]

The data is a triple:

- **S**: Set of feasible points in \( Z^n \)
- **W**: Integer \( d \times n \) matrix
- **f**: Function from \( Z^d \) to \( R \) given by comparison oracle

The presentation of \( S \) splits the theory into two branches:

- **Combinatorial Optimization**: \( S \) in \( \{0,1\}^n \) given by oracle (membership, linear-optimization, etc.)

- **Integer Programming**: \( S = \{ x \in Z^n : A(x) \leq 0 \} \) given by inequalities

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Example:

Convex Discrete Maximization
**Convex Discrete Maximization**

**Theorem 0:** For $S$ in $\mathbb{Z}^n$ given by linear-optimization oracle, $d \times n$ matrix $W$ with $d$ fixed, and convex function $f$, can solve in polynomial-time

$$\max \{ f(Wx) : x \text{ in } S \}$$
Convex Discrete Maximization

Theorem 0: For $S$ in $\mathbb{Z}^n$ given by linear-optimization oracle, $d \times n$ matrix $W$ with $d$ fixed, and convex function $f$, can solve in polynomial-time

$$\max \{ f(Wx) : x \in S \}$$

when $S$ is endowed with a set $E$ of all edge-directions of $\text{conv}(S)$

Reference: Convex combinatorial optimization (Onn, Rothblum)
Journal of Discrete and Computational Geometry
Convex Discrete Maximization

**Theorem 0:** For $S$ in $\mathbb{Z}^n$ given by linear-optimization oracle, $d \times n$ matrix $W$ with $d$ fixed, and convex function $f$, can solve in polynomial-time

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### Convex Discrete Maximization – Some Applications

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N-Fold Integer Programming
The *n*-fold product of an \((r+s) \times t\) bimatrix \(A\) is the following \((r+ns) \times nt\) matrix:

\[
A^{(n)} = \begin{pmatrix}
A_1 & A_1 & A_1 & \cdots & A_1 \\
A_2 & 0 & 0 & \cdots & 0 \\
0 & A_2 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & A_2
\end{pmatrix}.
\]
Theorem 1: Linear optimization in polynomial time:

\[ \max \{ wx : A^{(n)}x = b, \; l \leq x \leq u, \; x \text{ in } \mathbb{Z}^{n^t} \} \]
Five Theorems on N-Fold Integer Programming

Theorem 1: Linear optimization in polynomial time:

$$\max \{ wx : A^{(n)}x = b, \ l \leq x \leq u, \ x \text{ in } \mathbb{Z}^{nt} \}$$

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\end{pmatrix}$$

with $A$ fixed $(r+s) \times t$ bimatrix

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Five Theorems on N-Fold Integer Programming

**Theorem 1:** Linear optimization in polynomial time:

\[
\max \{ wx : A^{(n)}x = b, \quad l \leq x \leq u, \quad x \in \mathbb{Z}^{nt} \}
\]

**Reference:** N-fold integer programming (De Loera, Hemmecke, Onn, Weismantel)
Discrete Optimization (Volume in memory of George Dantzig)
Theorem 2: Separable convex minimization in polynomial time:

$$\min \{ \sum f_i(x_i) : A^{(n)}x = b, \ l \leq x \leq u, \ x \in \mathbb{Z}^n \}$$

Reference: A polynomial oracle-time algorithm for convex integer minimization (Hemmecke, Onn, Weismantel) Mathematical Programming
Theorem 3: Integer point $l_p$-nearest to $x$ in polynomial time:

$$\min \{ |x - x|^p : A^{(n)}x = b, \ l \leq x \leq u, \ x \in \mathbb{Z}^{nt} \}$$
Theorem 4: Weighted separable convex minimization in polytime:

\[ \min \{ f(W^{(n)}x) : A^{(n)}x = b, \ l \leq x \leq u, \ L \leq W^{(n)}x \leq U, \ x \in \mathbb{Z}^{nt} \} \]
Theorem 5: Weighted convex maximization in polynomial time:

\[
\max \{ f(Wx) : A^{(n)}x = b, \quad l \leq x \leq u, \quad x \in \mathbb{Z}^{nt} \}
\]

Multiway Tables
Multiway Tables

Consider optimization over $m_1 \times \cdots \times m_k \times n$ tables with given margins:
Multiway Tables

Consider optimization over $m_1 \times \cdots \times m_k \times n$ tables with given margins:

\[
\begin{bmatrix}
6 & 3 & 3 & 0 \\
3 & 2 & 0 & 1 \\
0 & 5 & 0 & 1 \\
0 & 1 & 0 & 9 \\
\end{bmatrix}
\]

Such tables form an $n$-fold program $\{ x : A^{(n)} x = b, \ x \geq 0, \ x \text{ integer} \}$ for suitable $A$ depending on $m_1, \ldots, m_k$ where $A_1$ controls the equations of margins involving summation over layers, whereas $A_2$ controls the equations of margins involving summation within a single layer at a time.
Multiway Tables

Consider optimization over $m_1 \times \cdots \times m_k \times n$ tables with given margins:

Corollary: Nonlinear optimization over $m_1 \times \cdots \times m_k \times n$ tables with given margins can be done in polynomial-time.

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Consider optimization over $m_1 \times \cdots \times m_k \times n$ tables with given margins:

Corollary: Nonlinear optimization over $m_1 \times \cdots \times m_k \times n$ tables with given margins can be done in polynomial-time

Universality Theorem for 3-way tables (De Loera, Onn, SIAM J. Optim.):
Every integer program is one over $3 \times m \times n$ tables with given line-sums
Multicommodity Flows
Many Commodity Transshipment

Find integer \textit{multicommodity transshipment} $x$ of \textit{minimum cost} satisfying \textit{vertex demands} $d$ and \textit{edge capacities} $u$ in \textit{digraph} $G$. 
Many Commodity Transshipment

Find integer multicommodity transshipment $x$ of minimum cost satisfying vertex demands $d$ and edge capacities $u$ in digraph $G$.

The cost of flow $x_e$ over edge $e$ can be either standard linear or more realistic nonlinear increasing function accounting for congestion over channel when subject to heavy traffic or communication load, such as

$$f_e(x_e) = c_e x_e^{a_e}$$
Small Example
Data:

Small Example

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Data:

digraph $G$
Small Example

Data:

digraph $G$

edge capacities $u_e$ unlimited

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Data:
digraph $G$
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edge costs $f_e(x_e) = x_e^2$
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vertex demands - two commodities:

$d^1 = (3\ -1\ -2)$
$d^2 = (-3\ 2\ 1)$
Data:

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$d^1 = (3 \quad -1 \quad -2)$
$d^2 = (-3 \quad 2 \quad 1)$

Solution:

$X^1 = (3 \quad 2 \quad 0)$
$X^2 = (0 \quad 2 \quad 3)$

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**Small Example**

**Data:**
- digraph $G$
- edge capacities $u_e$ unlimited
- edge costs $f_e(x_e) = x_e^2$
- vertex demands - two commodities:
  - $d^1 = (3, -1, -2)$
  - $d^2 = (-3, 2, 1)$

**Solution:**
- $X^1 = (3, 2, 0)$
- $X^2 = (0, 2, 3)$

**Cost:**
- $f(x) = (3+0)^2 + (2+2)^2 + (0+3)^2 = 34$
Many Commodity Transshipment

Find integer multicommodity transshipment $x$ of minimum $f$ cost satisfying vertex demands $d$ and edge capacities $u$ in digraph $G$

Let $D$ be the $s \times t$ vertex-edge incidence matrix of $G$
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Then the multicommodity transshipment problem can be written as

$$\min \{ f(u - x^0) : \sum x^k = u, \ D x^k = d^k, \ x \geq 0, \ x \in \mathbb{Z}^{(n+1)t} \}$$
Many Commodity Transshipment

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This is an $(n+1)$-fold integer program over the $(t+s) \times t$ bimatrix $A$ with first block $A_1$ the $t \times t$ identity and second block $A_2 = D$
Many Commodity Transshipment

Find integer \textit{multicommodity transshipment} \( x \) of minimum \( f \) cost satisfying \( \text{vertex demands} \ d \) and \( \text{edge capacities} \ u \) in \( \text{digraph} \ G \)

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\textbf{Corollary:} For fixed digraph \( G \) and variable number \( n \) of commodities can solve the \textit{n-commodity transshipment problem} in polynomial time

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Many Commodity Transshipment

Find integer multicommodity transshipment $x$ of minimum $f$ cost satisfying vertex demands $d$ and edge capacities $u$ in digraph $G$

Let $D$ be the $s \times t$ vertex-edge incidence matrix of $G$

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Then the multicommodity transshipment problem can be written as

$$\min \{ f(u - x^0) : \sum x^k = u, \ D x^k = d^k, \ x \geq 0, \ x \in \mathbb{Z}^{(n+1)t} \}$$

This is an $(n+1)$-fold integer program over the $(t+s) \times t$ bimatrix $A$ with first block $A_1$ the $t \times t$ identity and second block $A_2 = D$

**Corollary:** For fixed $s$ and variable $n$ can solve the $n$-commodity transshipment problem over any $s$-digraph in polynomial time
Multicommodity Transportation

Find integer $k$-commodity transportation $x$ of minimum $f$ cost from $m$ suppliers to $n$ consumers in the bipartite digraph $K_{m,n}$.
Multicommodity Transportation

Find integer $k$-commodity transportation $x$ of minimum cost from $m$ suppliers to $n$ consumers in the bipartite digraph $K_{m,n}$.

Also given are supply and consumption vectors $s^i$ and $c^j$ in $\mathbb{Z}^k$, edge capacities $u_v$, and volume $v_i$ per unit commodity $i$. 

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**Multicommodity Transportation**

Find integer \( k \)-commodity transportation \( x \) of minimum \( f \) cost from \( m \) suppliers to \( n \) consumers in the bipartite digraph \( K_{m,n} \).

Also given are supply and consumption vectors \( s^i \) and \( c^j \) in \( \mathbb{Z}^k \), edge capacities \( u_e \), and volume \( v_i \) per unit commodity \( i \).

For suitable \( (km+k) \times km \) bimatrix \( A \) and suitable \( (0+m) \times km \) bimatrix \( W \) derived from the \( v_i \) the problem becomes the \( n \)-fold integer program

\[
\min \left\{ f(W^{(n)}x) : A^{(n)}x = (s^i, c^j), \ x \geq 0, \ W^{(n)}x \leq u, \ x \in \mathbb{Z}^{nm} \right\}
\]
**Multicommodity Transportation**

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For suitable $(km+k) \times km$ bimatrix $A$ and suitable $(0+m) \times km$ bimatrix $W$ derived from the $v_i$, the problem becomes the $n$-fold integer program

$$\min \{ f(W^{(n)}x) : A^{(n)}x = (s^i, c^j), \ x \geq 0, \ W^{(n)}x \leq u, \ x \in \mathbb{Z}^{nk^m} \}$$

**Corollary:** For fixed $k$ commodities and $m$ suppliers, can find optimal multicommodity transportation for $n$ consumers in polynomial time.

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Universality
Universality of N-Fold Integer Programming

Consider the following special form of the \( n \)-fold product operator,

\[
A^{[n]} = \begin{pmatrix}
I & I & I & \cdots & I \\
A & 0 & 0 & \cdots & 0 \\
0 & A & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & A \\
\end{pmatrix}_{n}.
\]
Universality of N-Fold Integer Programming

Consider the following special form of the n-fold product operator,

\[ A^{[n]} = \begin{pmatrix} I & I & I & \cdots & I \\ A & 0 & 0 & \cdots & 0 \\ 0 & A & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & A \end{pmatrix} \]

Consider such m-fold products of the 1 x 3 matrix \([1 \ 1 \ 1]\). For example,

\[ [1 \ 1 \ 1]^{[3]} = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix} \]
Universality of N-Fold Integer Programming

$$A^{[n]} = \begin{pmatrix}
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A & 0 & 0 & \cdots & 0 \\
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\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & A
\end{pmatrix}$$

**Universality Theorem**: Any bounded set \{ y integer : By = b, y \geq 0 \} is in polynomial-time-computable coordinate-embedding-bijection with some

\[
\{ x \text{ integer} : [1 \ 1 \ 1]^{[m][n]} x = a, \ x \geq 0 \}
\]

**Reference**: All linear and integer programs are slim 3-way programs

(De Loera, Onn) SIAM Journal on Optimization

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Universality of N-Fold Integer Programming

\[ A^{[n]} = \begin{pmatrix} I & I & I & \cdots & I \\ A & 0 & 0 & \cdots & 0 \\ 0 & A & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & A \end{pmatrix} . \]

**Universality Theorem:** Any bounded set \( \{ y \text{ integer} : By = b, y \geq 0 \} \) is in polynomial-time-computable coordinate-embedding-bijection with some \( \{ x \text{ integer} : [1 \ 1 \ 1]^{[m][n]}x = a, \ x \geq 0 \} \)

**Scheme for Nonlinear Integer Programming:**

any integer program \( \max \{ f(Wx) : By = b, \ y \geq 0, \ y \text{ integer} \} \)

can be lifted to

an n-fold program: \( \max \{ f(W'x) : [1 \ 1 \ 1]^{[m][n]}x = a, \ x \geq 0, \ x \text{ integer} \} \)
Proofs
Graver Bases

The Graver basis of an integer matrix $A$ is the finite set $G(A)$ of conformal-minimal nonzero integer vectors $x$ satisfying $Ax = 0$. 
The **Graver basis** of an integer matrix $A$ is the finite set $G(A)$ of conformal-minimal nonzero integer vectors $x$ satisfying $Ax = 0$.

**Lemma**: For fixed $A$, can compute in polytime the Graver basis $G(A^{(n)})$ of

$$A^{(n)} = \begin{pmatrix} A_1 & A_1 & A_1 & \cdots & A_1 \\
A_2 & 0 & 0 & \cdots & 0 \\
0 & A_2 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
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Graver Bases

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The proof uses finiteness results of Santos-Sturmfels & Hosten-Sullivant

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Proof of Theorem 2
(convex n-fold minimization)
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(convex n-fold minimization)

- Set $S = \{ x \in \mathbb{Z}^{nt} : A^{(n)}x = b, \; l \leq x \leq u \}$
Proof of Theorem 2
(convex n-fold minimization)

- Set \( S = \{ x \in \mathbb{Z}^n : A^{(n)}x = b, \; l \leq x \leq u \} \)

Construct the Graver basis \( G(A^{(n)}) \)
**Proof of Theorem 2**
*(convex n-fold minimization)*

Let $S = \{ x \in \mathbb{Z}^{nt} : A^{(n)}x = b, \; l \leq x \leq u \}$

*Construct the Graver basis $G(A^{(n)})$*

*Find initial point by auxiliary n-fold program*
Proof of Theorem 2
(convex n-fold minimization)

- Set \( S = \{ x \in \mathbb{Z}^{n^t} : A^{(n)}x = b, \ l \leq x \leq u \} \)

Construct the Graver basis \( G(A^{(n)}) \)

Find initial point by auxiliary n-fold program

- Separable convex function \( f \) to be minimized
Proof of Theorem 2
(convex n-fold minimization)

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Construct the Graver basis $G(A^{(n)})$

Find initial point by auxiliary n-fold program

- Separable convex function $f$ to be minimized

Iteratively greedily augment initial point to optimal one using elements from $G(A^{(n)})$
Proof of Theorem 2
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Integer Caratheodory Theorem assures polynomial convergence
Proof of Theorem 2
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Iteratively greedily augment initial point to optimal one using elements from \( G(A^{(n)}) \)

Integer Caratheodory Theorem assures polynomial convergence

Theorems 1, 3 (linear optimization, minimal distance) follow from Thm. 2

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Proof of Theorem 2
(convex n-fold minimization)

- Set $S = \{ x \in \mathbb{Z}^{nt} : A^{(n)}x = b, \ l \leq x \leq u \}$

Construct the Graver basis $G(A^{(n)})$

Find initial point by auxiliary n-fold program

- Separable convex function $f$ to be minimized

Iteratively greedily augment initial point to optimal one using elements from $G(A^{(n)})$

Integer Caratheodory Theorem assures polynomial convergence

Theorems 1, 3 (linear optimization, minimal distance) follow from Thm. 2

Theorem 4 (weighted convex minimization) reduces to unweighted Thm. 2

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Proof of Theorem 5
(convex n-fold maximization)
Proof of Theorem 5
(convex n-fold maximization)

- Set \( S = \{ x \in \mathbb{Z}^{nt} : A^{(n)}x = b, \ l \leq x \leq u \} \)
Proof of Theorem 5
(convex n-fold maximization)

Set $S = \{ x \in \mathbb{Z}^{nt} : A^{(n)}x = b, \ 1 \leq x \leq u \}$

Construct the Graver basis $G(A^{(n)})$
Proof of Theorem 5
(convex \(n\)-fold maximization)

Set \( S = \{ x \in \mathbb{Z}^{nt} : A^{(n)} x = b, \ l \leq x \leq u \} \)

Construct the Graver basis \( G(A^{(n)}) \)

Simulate linear-optimization oracle over \( S \) using Theorem 1
Proof of Theorem 5
(convex n-fold maximization)

- Set $S = \{x \in \mathbb{Z}^{nt} : A^{(n)}x = b, \; l \leq x \leq u\}$

Construct the Graver basis $G(A^{(n)})$

Simulate linear-optimization oracle over $S$ using Theorem 1

The Graver basis covers all edge-directions of $\text{conv}(S)$
Proof of Theorem 5
(convex n-fold maximization)

- Set $S = \{ x \in \mathbb{Z}^{nt} : A^{(n)}x = b, \ l \leq x \leq u \}$

Construct the Graver basis $G(A^{(n)})$

Simulate linear-optimization oracle over $S$ using Theorem 1

The Graver basis covers all edge-directions of conv($S$)

Apply Theorem 0 on convex discrete maximization

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Epilogue:
Nonlinear Combinatorial Optimization
Nonlinear Combinatorial Optimization

\[ \min/\max \{ f(Wx) : x \in S \} \]

\[ S \text{ in } \{0,1\}^n \]

\[ W \text{ unary } d \times n, \text{ fixed } d \]

\[ f \text{ arbitrary from } \mathbb{Z}^d \text{ to } \mathbb{R} \]
Nonlinear Combinatorial Optimization

\[
\min/\max \{ f(Wx) : x \in S \}
\]

\[
\begin{align*}
S & \text{ in } \{0,1\}^n \\
W & \text{ unary } d \times n \text{, fixed } d \\
f & \text{ arbitrary from } \mathbb{Z}^d \text{ to } \mathbb{R}
\end{align*}
\]

**Theorem A:** For \( S \) bipartite matching in randomized polynomial time.

*Shmuel Onn*
Nonlinear Combinatorial Optimization

\[ \min/\max \{ f(Wx) : x \in S \} \]

where \( W \) is unary \( d \times n \), fixed \( d \), \( f \) is arbitrary from \( \mathbb{Z}^d \) to \( \mathbb{R} \), and \( S \) is in \( \{0,1\}^n \).

**Theorem A:** For \( S \) bipartite matching in randomized polynomial time.

Berstein, Onn, Discrete Optimization

Shmuel Onn
Nonlinear Combinatorial Optimization

\[
\min/\max \{ f(Wx) : x \in S \}
\]

\[
\begin{align*}
S & \text{ in } \{0,1\}^n \\
W & \text{ unary } d \times n, \text{ fixed } d \\
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\end{align*}
\]

**Theorem A:** For \( S \) bipartite matching in randomized polynomial time.

*Berstein, Onn, Discrete Optimization*
Nonlinear Combinatorial Optimization

\[ \min/\max \{ f(Wx) : x \in S \} \]

\[ S \text{ in } \{0,1\}^n \]
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\[ f \text{ arbitrary from } \mathbb{Z}^d \text{ to } \mathbb{R} \]

**Theorem A:** For \( S \) bipartite matching in randomized polynomial time.

Berstein, Onn, Discrete Optimization

**Theorem B:** For \( S \) matroid (e.g. spanning tree) in polynomial time.

Nonlinear Combinatorial Optimization

\[ \min/\max \{ f(Wx) : x \in S \} \]

\( S \) in \( \{0,1\}^n \)

\( W \) unary \( d \times n \), fixed \( d \)

\( f \) arbitrary from \( Z^d \) to \( R \)

**Theorem A**: For \( S \) bipartite matching in randomized polynomial time.

Berstein, Onn, Discrete Optimization

**Theorem B**: For \( S \) matroid (e.g. spanning tree) in polynomial time.


**Theorem C**: For \( S \) matroid intersection in randomized polynomial time.

Berstein, Lee, Onn, Weismantel, Mathematical Programming?
**Independence Systems**

$$\min/\max \{ f(wx) : x \in S \}$$

- $S$ in $\{0,1\}^n$ independence system given by linear optimization oracle
- $w$ in $\{a_1, \ldots, a_p\}^n$ (d=1)
- $f$ arbitrary from $\mathbb{Z}$ to $\mathbb{R}$

**Theorem D:** Can find an $r(a_1, \ldots, a_p)$-best solution in polynomial time.

For $p=2$ weight values $r(a_1, a_2) = F(a_1, a_2)$ is the Frobenius number.

So for $w$ in $\{2,3\}^n$ can efficiently find a 1-best solution.

Amazingly, this is best possible:

**Theorem E:** For $w$ in $\{2,3\}^n$ finding 0-best solution takes exponential time.

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Comprehensive up-to-date development of the general theory is available in my

Nachdiplom Lectures
on
Nonlinear Discrete Optimization

ETH Zurich, Spring 2009

http://www.fim.math.ethz.ch/activities/eth_lectures