N-Fold Integer Programming
and
Multicommodity Flows

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Based on several papers joint with several co-authors including
Berstein, De Loera, Hemmecke, Lee, Rothblum, Weismantel

Prologue:

Nonlinear Discrete Optimization

Setup for Nonlinear Discrete Optimization

The **problem** is: $\min/\max\{ f(Wx) : x \text{ in } S \}$

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This setup enables:

- **Determination of broad classes** of triples **S**, **W**, **f** solvable efficiently (deterministically, randomly, or approximately)

- **Interpretation as multi-objective optimization** with objective $f(Wx) = f(W_1x, \ldots, W_dx)$ balancing criteria W_ix of d players

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The presentation of **S** splits the theory into two branches:

- **Combinatorial Optimization**:
  S in $\{0,1\}^n$ given by oracle (membership, linear-optimization, etc.)

- **Integer Programming**:
  S $= \{x \text{ in } Z^n : A(x) \le 0\}$ given by inequalities

Example:

Convex Discrete Maximization

Convex Discrete Maximization

Theorem 0: For S in Z^n given by linear-optimization oracle, $d \times n$ matrix W with d fixed, and convex function f, can solve in polynomial-time

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when S is endowed with a set **E** of all edge-directions of conv(S)

Reference: Convex combinatorial optimization (Onn, Rothblum)
Journal of Discrete and Computational Geometry

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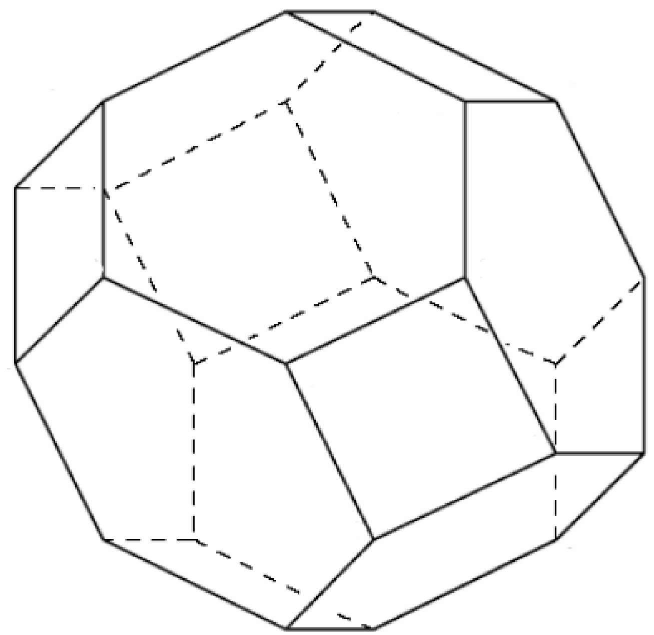

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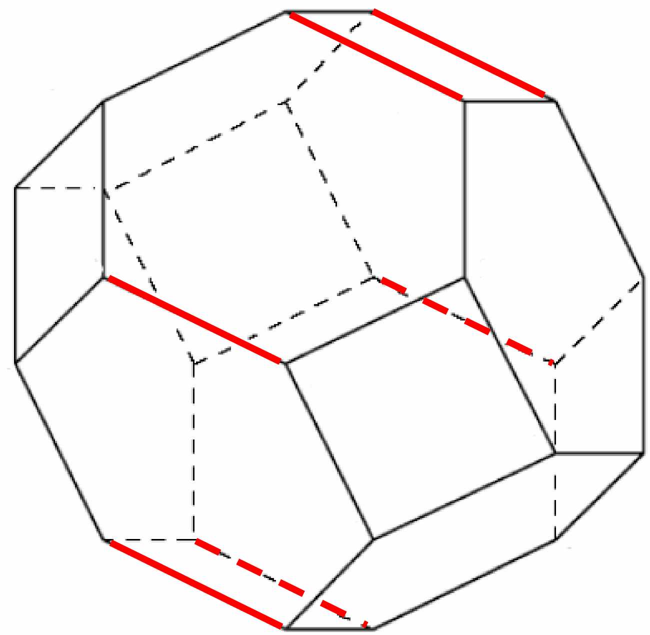

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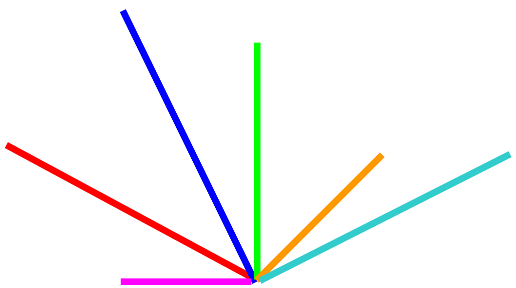

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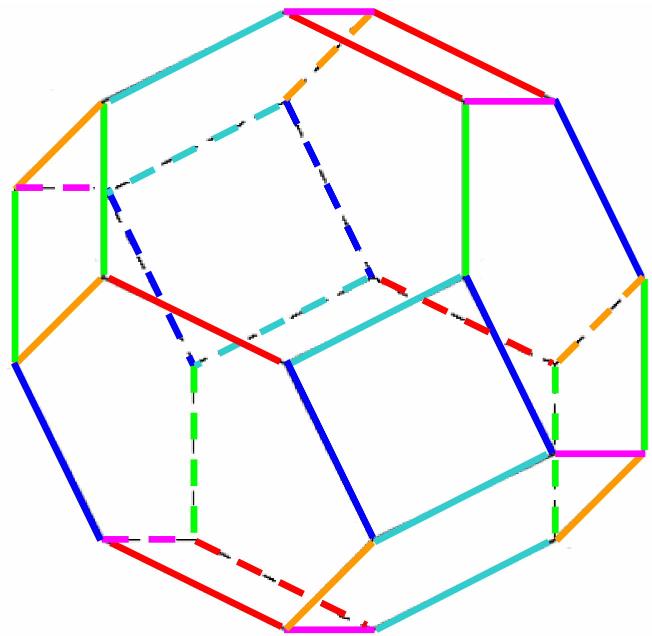

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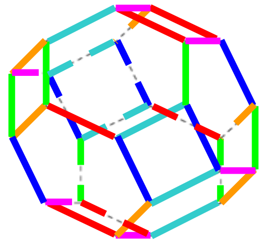
Convex Discrete Maximization – Some Applications



Some edge-behaved polytopes	And their applications
Matroid polytopes: pairs $1_i - 1_j$ Also poly-matroids	e.g. spanning trees, experimental design
High dimensional Transportation polytopes:	e.g. vector partitioning, privacy in data bases
Integer-hulls of N-fold integer programs:	**Many, some to be discussed below**

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N-Fold Integer Programming

N-Fold Products

The n-fold product of an $(r+s) \times t$ bimatrix A is the following $(r+ns) \times nt$ matrix:

$$A^{(n)} = \underbrace{\begin{pmatrix} A_1 & A_1 & A_1 & \cdots & A_1 \\ A_2 & 0 & 0 & \cdots & 0 \\ 0 & A_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & A_2 \end{pmatrix}}_{n} .$$

Five Theorems on N-Fold Integer Programming

Theorem 1: **Linear optimization** in **polynomial time**:

$$\max \{ wx \ : \ A^{(n)}x = b, \ \ l \le x \le u, \ \ x \text{ in } Z^{nt} \}$$

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with A fixed $(r+s) \times t$ bimatrix

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Reference: N-fold integer programming (De Loera, Hemmecke, Onn, Weismantel)
Discrete Optimization (Volume in memory of George Dantzig)

Five Theorems on N-Fold Integer Programming

Theorem 2: **Separable convex minimization** in **polynomial time**:

$$\min \{ \sum f_i(x_i) : A^{(n)}x = b, \quad l \le x \le u, \quad x \text{ in } Z^{nt} \}$$

Reference: *A polynomial oracle-time algorithm for convex integer minimization* (Hemmecke, Onn, Weismantel) Mathematical Programming

Five Theorems on N-Fold Integer Programming

Theorem 3: Integer point l_p-nearest to x in polynomial time:

$$\min \{ |x - x|_p : A^{(n)} x = b, \ l \le x \le u, \ x \in Z^{nt} \}$$

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Five Theorems on N-Fold Integer Programming

Theorem 4: **Weighted separable convex minimization** in **polytime**:

$$\min \{ f(W^{(n)}x) : A^{(n)}x = b, \ \ l \le x \le u, \ \ L \le W^{(n)}x \le U, \ \ x \text{ in } Z^{nt} \}$$

Five Theorems on N-Fold Integer Programming

Theorem 5: **Weighted convex maximization** in **polynomial time**:

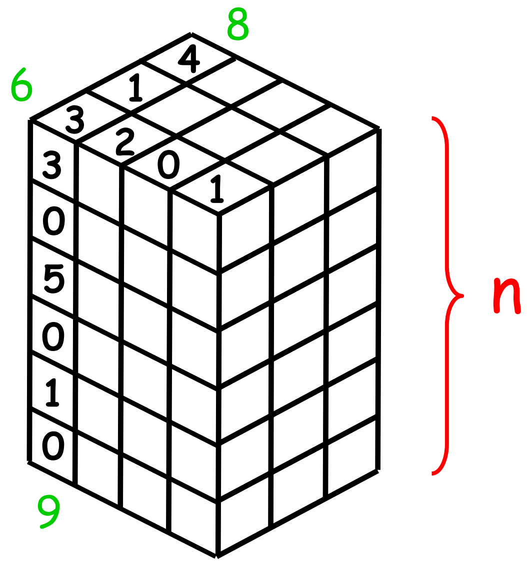
$$\max \{ f(Wx) : \; A^{(n)}x = b, \;\; l \le x \le u, \;\; x \in Z^{nt} \}$$

Reference: Convex integer maximization via Graver bases (De Loera, Hemmecke, Onn, Rothblum, Weismantel) Journal of Pure and Applied Algebra

Multiway Tables

Multiway Tables

Consider optimization over $m_1 \times \cdots \times m_k \times n$ tables with given margins:



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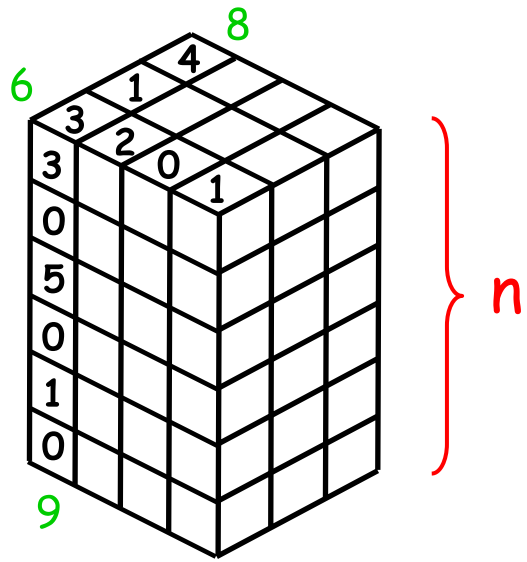

$$A^{(n)} = \underbrace{\begin{pmatrix} A_1 & A_1 & A_1 & \cdots & A_1 \\ A_2 & 0 & 0 & \cdots & 0 \\ 0 & A_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & A_2 \end{pmatrix}}_{n}$$

Such tables form an n-fold program $\{x \ : \ A^{(n)}x = b, \ x \geq 0, \ x \text{ integer}\}$ for suitable A depending on $m_1, \ldots, m_k$ where A_1 controls the equations of margins involving summation over layers, whereas A_2 controls the equations of margins involving summation within a single layer at a time.

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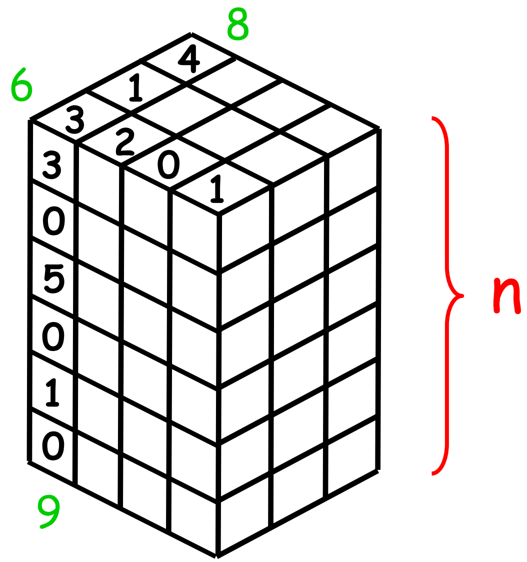

Corollary: Nonlinear optimization over $m_1 \times \cdots \times m_k \times n$ tables with given margins can be done in polynomial-time

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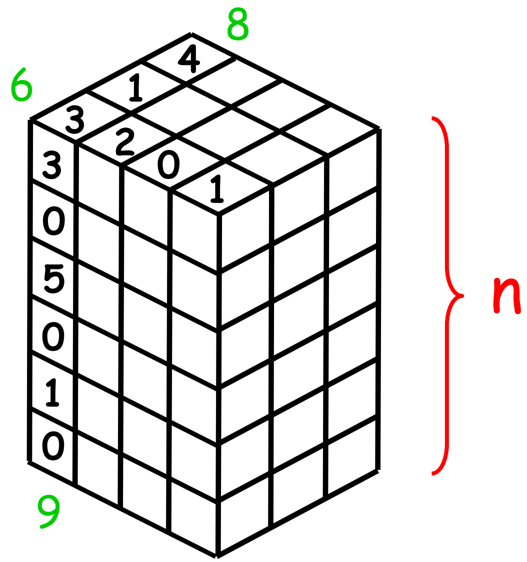

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Universality Theorem for 3-way tables (De Loera, Onn, SIAM J. Optim.)**:**
Every integer program is one over $3 \times m \times n$ tables with given line-sums

Multicommodity Flows

Many Commodity Transshipment

Find integer multicommodity transshipment x of minimum f cost satisfying vertex demands d and edge capacities u in digraph G

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The **cost** of flow x_e over edge e can be either standard linear or more realistic nonlinear increasing function accounting for **congestion** over channel when subject to heavy traffic or communication load, such as

$$f_e(x_e) = c_e \, x_e^{a_e}$$

Small Example

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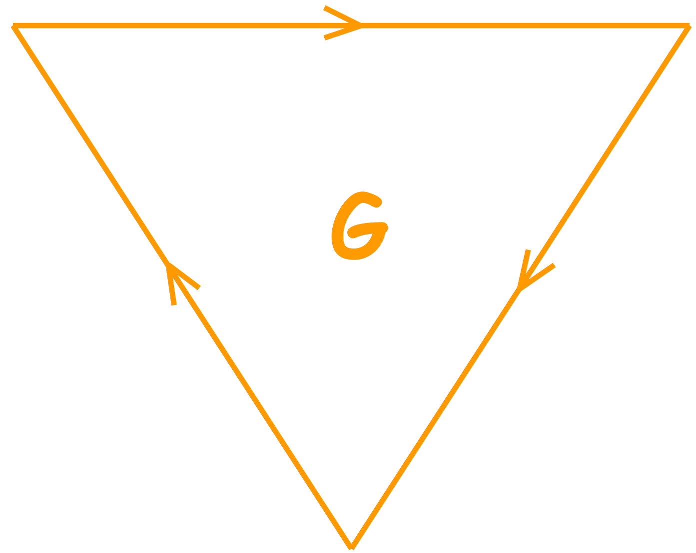
digraph G



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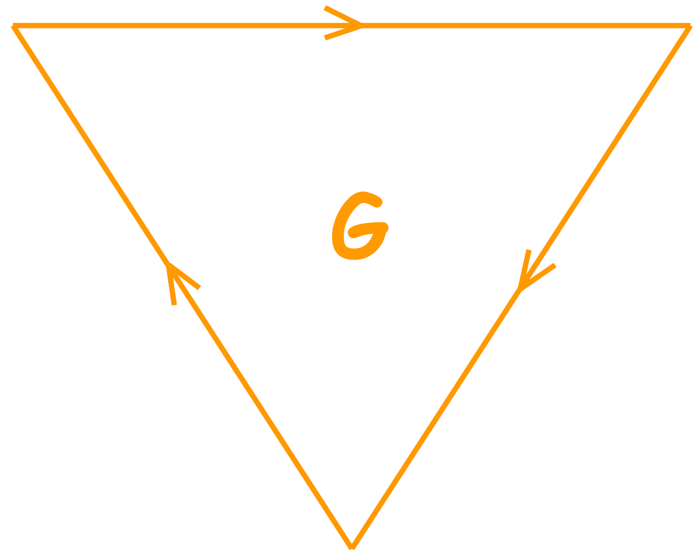

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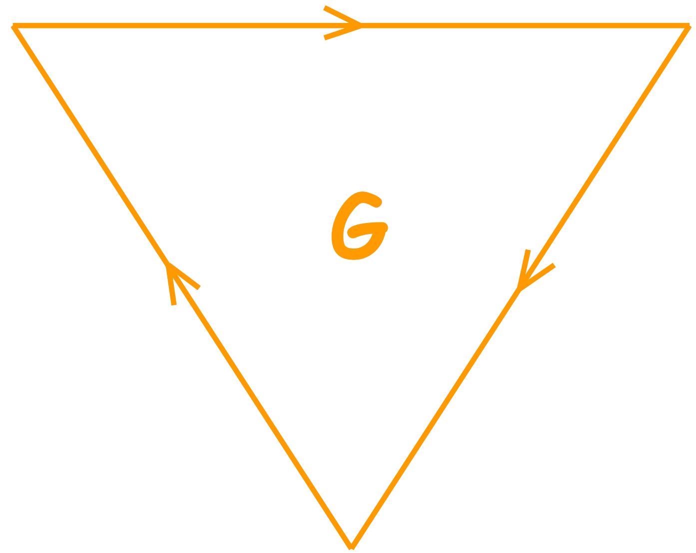
edge costs $f_e(x_e) = x_e^2$



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$d^1 = (3 \;\; -1 \;\; -2)$

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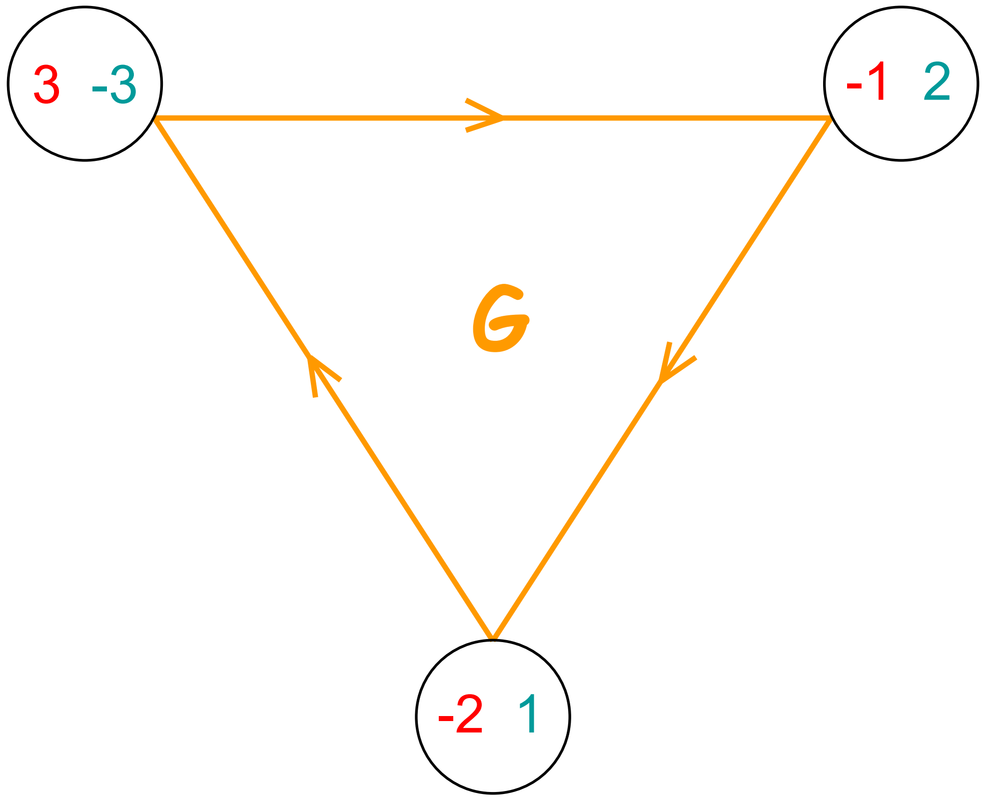

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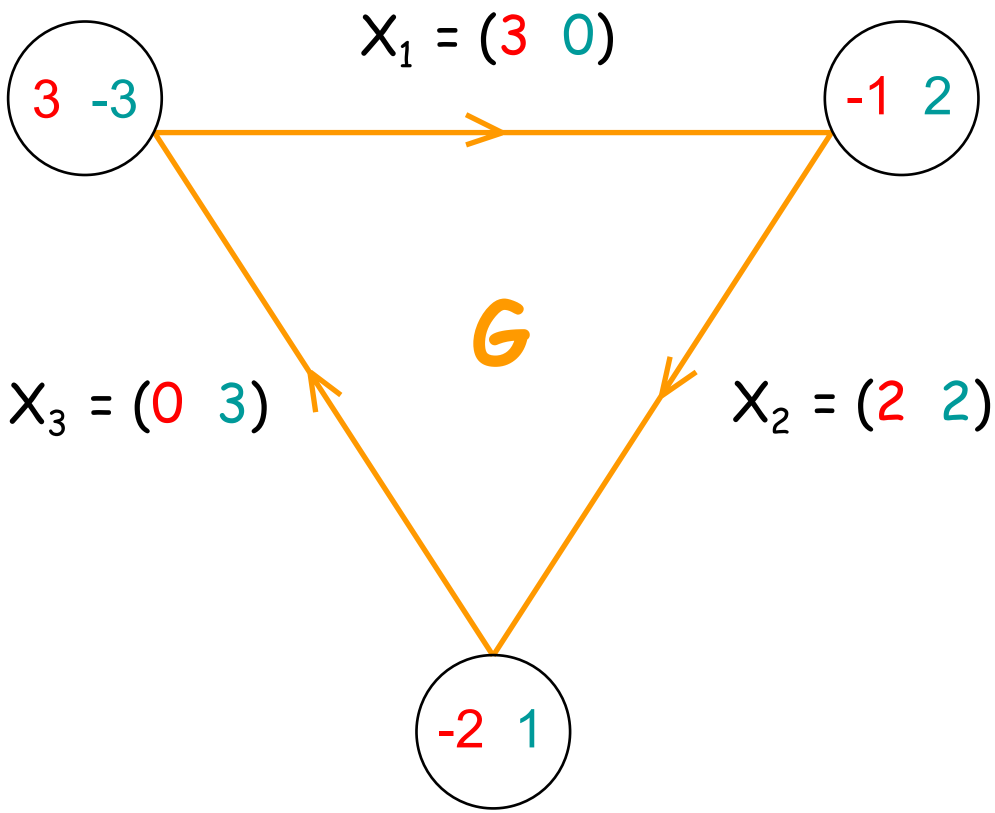
Solution:

$X^1 = (3\ \ 2\ \ 0)$

$X^2 = (0\ \ 2\ \ 3)$



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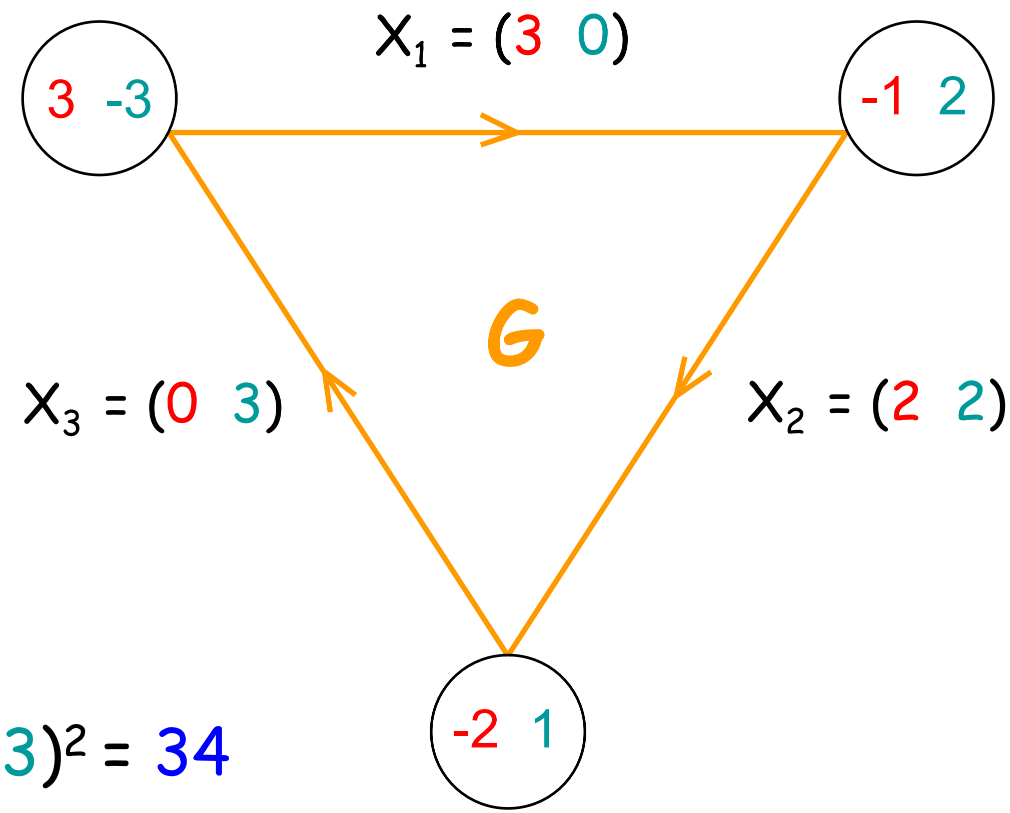

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Cost:

$f(x) = (3+0)^2+(2+2)^2+(0+3)^2 = 34$

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Let D be the $s \times t$ vertex-edge incidence matrix of G

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$$\min\{ f(u - x^0) : \sum x^k = u,\ Dx^k = d^k,\ x \geq 0,\ x \in Z^{(n+1)t} \}$$

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This is an (n+1)-fold integer program over the (t+s) x t bimatrix A with first block A_1 the t x t identity and second block $A_2 = D$

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Corollary: For fixed digraph G and variable number n of commodities can solve the n-commodity transshipment problem in polynomial time

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Corollary: For fixed s and variable n can solve the n-commodity transshipment problem over any s-digraph in polynomial time

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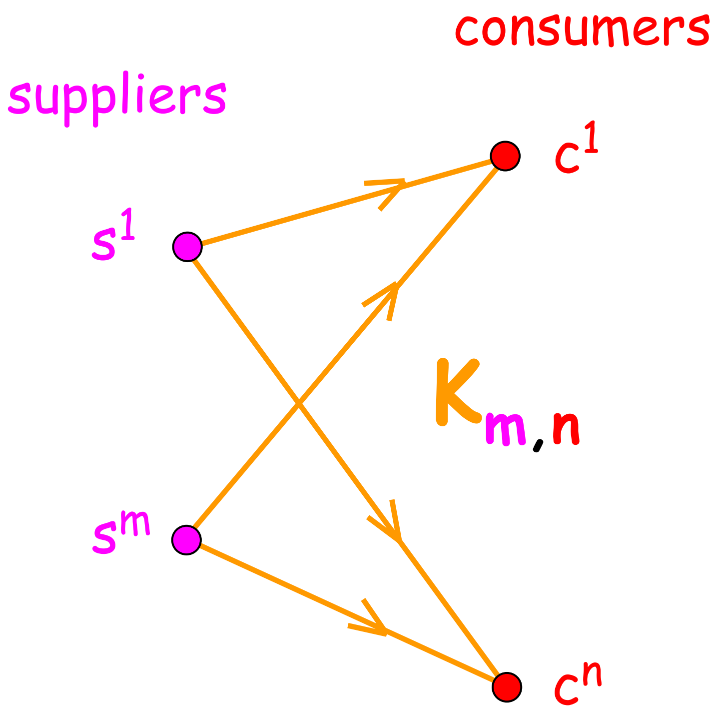
Multicommodity Transportation

Find integer k-commodity transportation x of minimum f cost from m suppliers to n consumers in the bipartite digraph $K_{m,n}$



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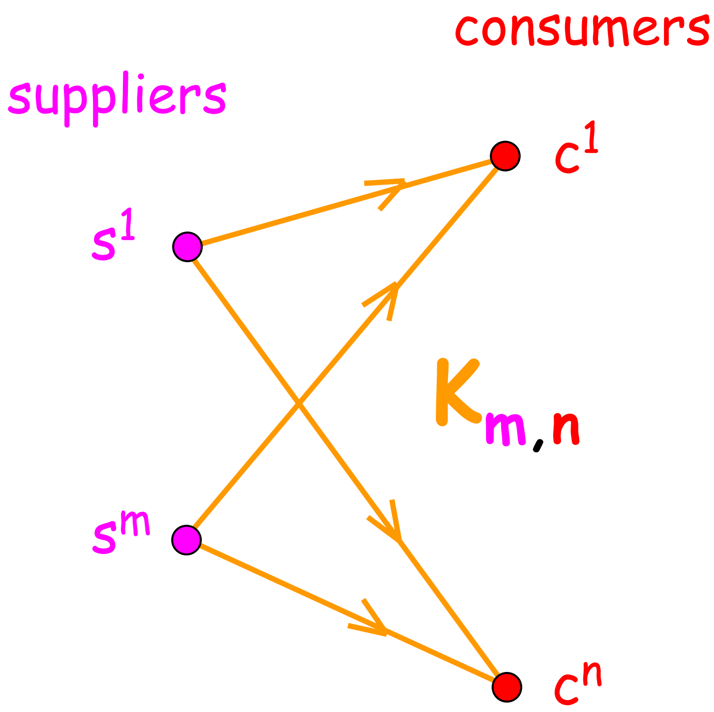

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For suitable (km+k) x km bimatrix A and suitable (0+m) x km bimatrix W derived from the v_i the problem becomes the n-fold integer program

$$\min \{ f(W^{(n)}x) \;:\; A^{(n)}x = (s^i, c^j),\quad x \geq 0,\quad W^{(n)}x \leq u,\quad x \text{ in } Z^{nkm} \}$$

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Corollary: For fixed k commodities and m suppliers, can find optimal multicommodity transportation for n consumers in polynomial time

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Universality

Universality of N-Fold Integer Programming

Consider the following special form of the n-fold product operator,

$$A^{[n]} = \underbrace{\begin{pmatrix} I & I & I & \cdots & I \\ A & 0 & 0 & \cdots & 0 \\ 0 & A & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & A \end{pmatrix}}_{n} \quad .$$

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Consider such m-fold products of the 1×3 matrix [1 1 1]. For example,

$$[1\ \ 1\ \ 1]^{[3]} = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix} .$$

Universality of N-Fold Integer Programming

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Universality Theorem: Any bounded set $\{ y \text{ integer} : By = b,\ y \geq 0 \}$ is in polynomial-time-computable coordinate-embedding-bijection with some

$$\{ x \text{ integer} : [1\ \ 1\ \ 1]^{[m][n]} x = a,\ \ x \geq 0 \}$$

Reference: *All linear and integer programs are slim 3-way programs* (De Loera, Onn) SIAM Journal on Optimization

Universality of N-Fold Integer Programming

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Scheme for Nonlinear Integer Programming:

any integer program $\max \{ f(Wx) \ : \ By = b,\ y \geq 0,\ y \text{ integer} \}$

can be lifted to

an n-fold program: $\max \{ f(W'x) \ : \ [1\ 1\ 1]^{[m][n]} x = a,\ x \geq 0,\ x \text{ integer} \}$

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Proofs

Graver Bases

The Graver basis of an integer matrix A is the finite set $G(A)$ of conformal-minimal nonzero integer vectors x satisfying $Ax = 0$.

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Lemma: For fixed A, can compute in polytime the Graver basis $G(A^{(n)})$ of

$$A^{(n)} = \begin{pmatrix} A_1 & A_1 & A_1 & \cdots & A_1 \\ A_2 & 0 & 0 & \cdots & 0 \\ 0 & A_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & A_2 \end{pmatrix} .$$

Graver Bases

The Graver basis of an integer matrix A is the finite set $G(A)$ of conformal-minimal nonzero integer vectors x satisfying $Ax = 0$.

Lemma: For fixed A, can compute in polytime the Graver basis $G(A^{(n)})$ of

$$A^{(n)} = \begin{pmatrix} A_1 & A_1 & A_1 & \cdots & A_1 \\ A_2 & 0 & 0 & \cdots & 0 \\ 0 & A_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & A_2 \end{pmatrix} .$$

The proof uses finiteness results of Santos-Sturmfels & Hosten-Sullivant

Proof of Theorem 2
(convex n-fold minimization)

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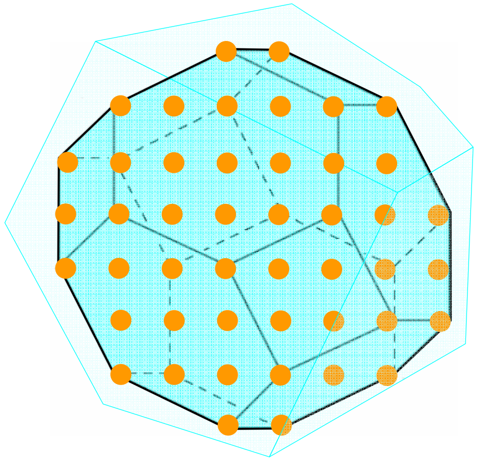
R^{nt}



- Set $S = \{ x \text{ in } Z^{nt} : A^{(n)} x = b,\ l \le x \le u \}$

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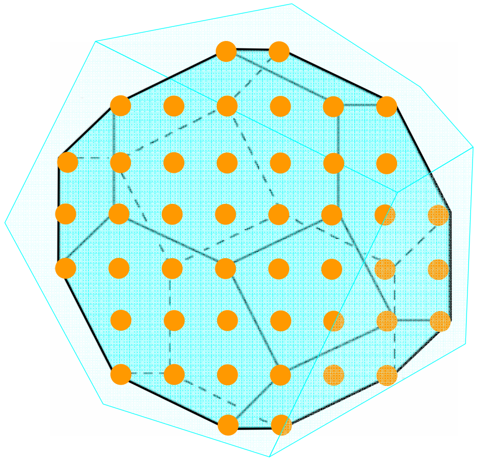

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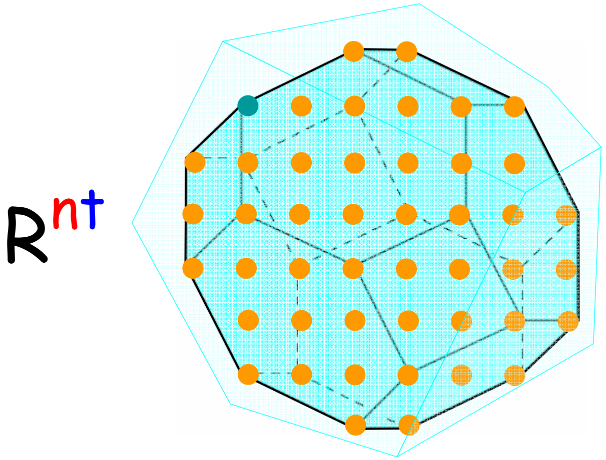

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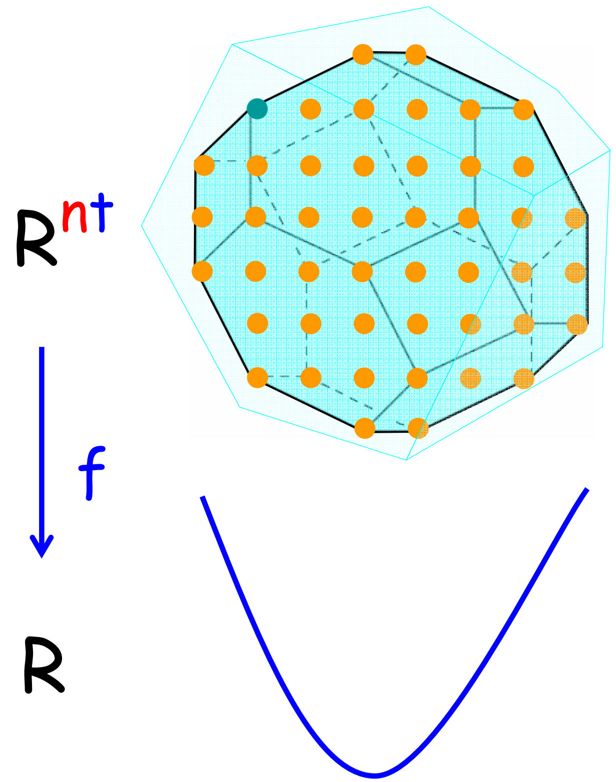

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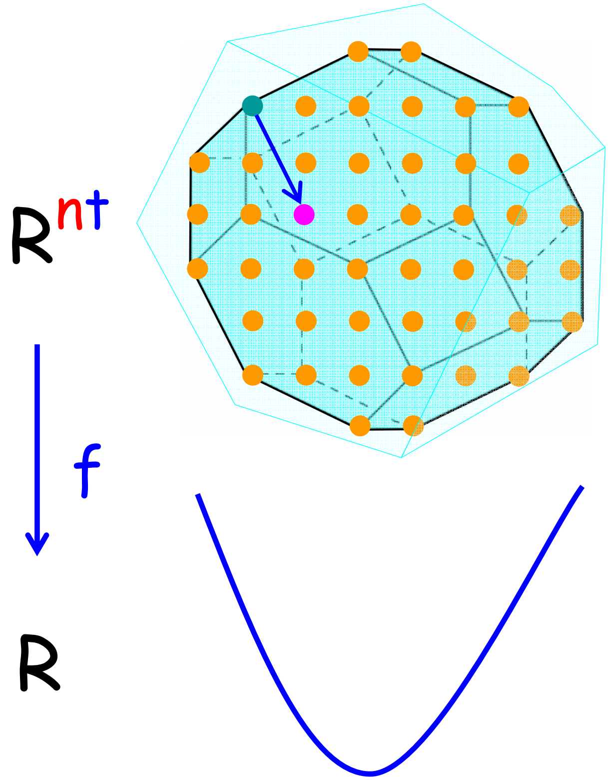

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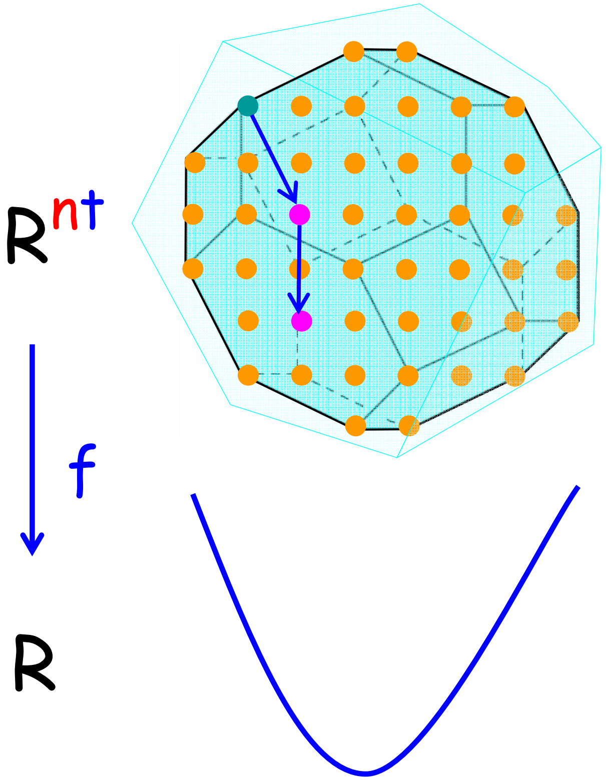

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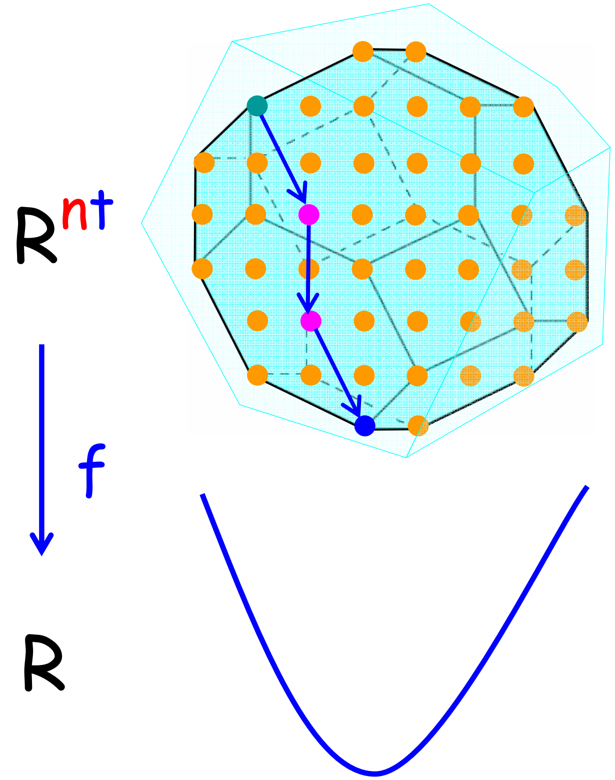

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Integer Caratheodory Theorem assures polynomial convergence

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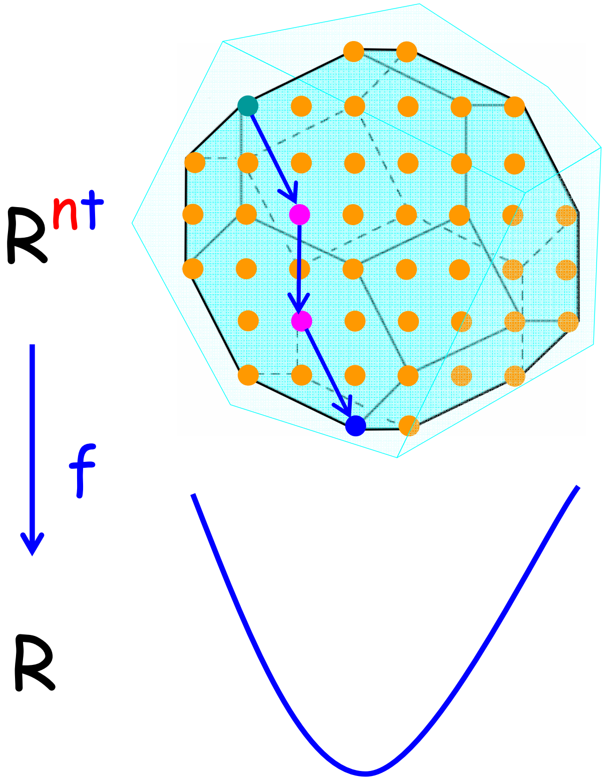

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Theorems 1, 3 (linear optimization, minimal distance) follow from Thm. 2

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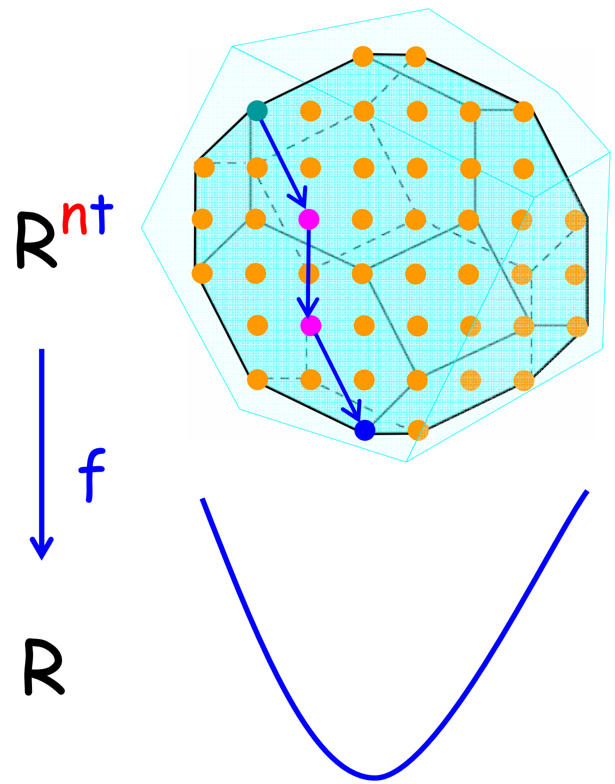

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Integer Caratheodory Theorem assures polynomial convergence

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Theorem 4 (weighted convex minimization) reduces to unweighted Thm. 2

Shmuel Onn

Proof of Theorem 5
(convex n-fold maximization)

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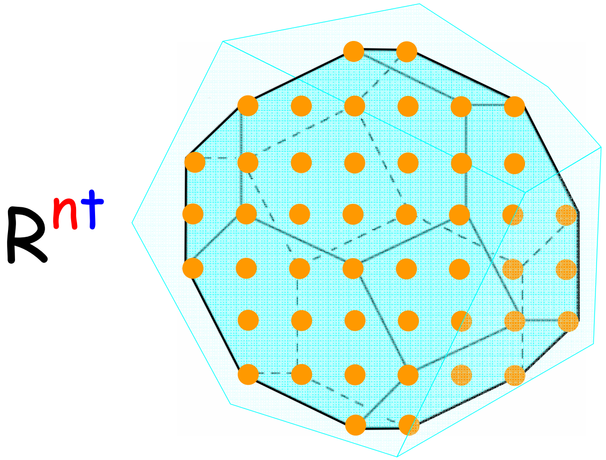

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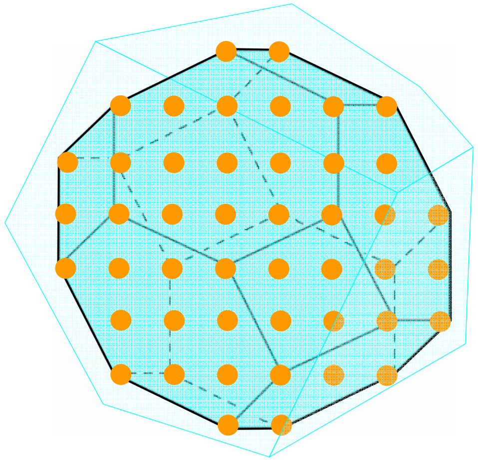

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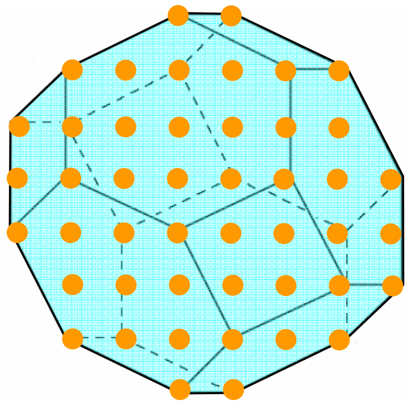

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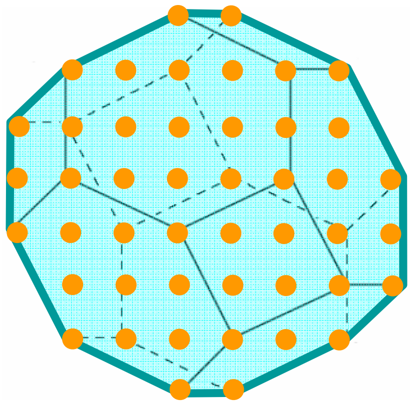

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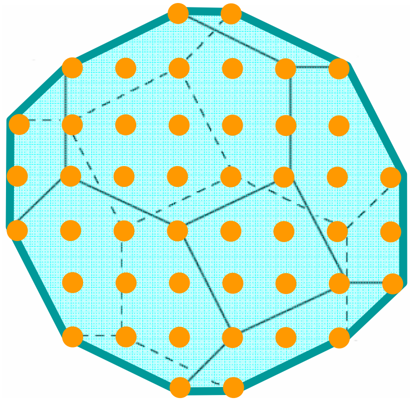

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Apply Theorem 0 on convex discrete maximization

Epilogue:

Nonlinear Combinatorial Optimization

Nonlinear Combinatorial Optimization

min/max { f(Wx) : x in S }

- **S** in $\{0,1\}^n$
- **W** unary $d \times n$, fixed d
- **f** arbitrary from Z^d to R

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Berstein, Onn, Discrete Optimization

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Berstein, Lee, Maruri-Aguilar, Onn, Riccomagno, Weismantel, Wynn, SIAM J. Disc. Math.

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Berstein, Lee, Maruri-Aguilar, Onn, Riccomagno, Weismantel, Wynn, SIAM J. Disc. Math.

Theorem C: For S **matroid intersection** in **randomized polynomial time**.

Berstein, Lee, Onn, Weismantel, Mathematical Programming ?

Independence Systems

$\min/\max \{ f(wx) : x \text{ in } S \}$

- **S** in $\{0,1\}^n$ **independence system** given by **linear optimization oracle**
- w in $\{a_1, \ldots, a_p\}^n$ (d=1)
- **f** arbitrary from Z to R

Theorem D: Can find an $\mathbf{r}(a_1, \ldots, a_p)$**-best** solution in **polynomial time**.

For p=2 weight values $r(a_1, a_2) = F(a_1, a_2)$ is the Frobenius number.

So for w in $\{2,3\}^n$ can efficiently find a 1-best solution.

Amazingly, this is best possible:

Theorem E: For w in $\{2,3\}^n$ finding 0-best solution takes **exponential time**.

Lee, Onn, Weismantel, SIAM Journal on Discrete Mathematics ?

Bibliography (mostly available at http://ie.technion.ac.il/~onn)

Onn: Convex Discrete Optimization (SMS Lecture Notes, CRM Montréal)

Onn: **Nonlinear Discrete Optimization** (Nachdiplom Lectures, ETH Zurich)

Lee, Onn, Weismantel: Nonlinear Discrete Optimization (book in preparation)

- Partition problems with convex objectives (Math. OR)
- Convex matroid optimization (SIAM Disc. Math.)
- The complexity of 3-way tables (SIAM Comp.)
- Convex combinatorial optimization (Disc. Comp. Geom.)
- Markov bases of 3-way tables (J. Symb. Comp.)
- **All linear and integer programs are slim 3-way programs** (SIAM Opt.)
- Entry Uniqueness in margined tables (Lect. Notes Comp. Sci.)
- Graver complexity of integer programming (Annals Combin.)
- Nonlinear bipartite matching (Disc. Opt.)
- **N-fold integer programming** (Disc. Opt. in memory of Dantzig)
- **Convex integer maximization via Graver bases** (J. Pure App. Algebra)
- **Polynomial oracle-time convex integer minimization** (Math. Prog.)
- Nonlinear matroid optimization and experimental design (SIAM Disc. Math.)
- Nonlinear optimization for matroid intersection and extensions (Math. Prog. ?)
- Nonlinear optimization over a weighted independence system (SIAM Disc. Math. ?)

Comprehensive up-to-date development
of the general theory is available in my

Nachdiplom Lectures
on
Nonlinear Discrete Optimization

ETH Zurich, Spring 2009

http://www.fim.math.ethz.ch/activities/eth_lectures