

# INFINITE RATE MUTUALLY CATALYTIC BRANCHING IN INFINITELY MANY COLONIES: THE LONGTIME BEHAVIOUR

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Consider the infinite rate mutually catalytic branching process (IMUB) constructed in [11] and [12]. For finite initial conditions, we show that only one type survives in the long run if the interaction kernel is recurrent. On the other hand, under a slightly stronger condition than transience, we show that both types can coexist.

## 1. Introduction and Main Results.

1.1. *Background and Motivation.* As a model for mutually catalytic branching, Dawson and Perkins [5] considered the following system of coupled stochastic differential equations:

$$(1.1) \quad \begin{aligned} Y_{i,t}(k) &= Y_{i,0}(k) + \int_0^t \sum_{l \in S} \mathcal{A}(k, l) Y_{i,s}(l) ds \\ &\quad + \int_0^t (\gamma Y_{1,s}(k) Y_{2,s}(k))^{1/2} dW_{i,s}(k) \quad \text{for } t \geq 0, k \in S, i = 1, 2. \end{aligned}$$

Here  $S$  is a countable set that is thought of as the site space and  $\gamma > 0$  is a parameter. (In fact, Dawson and Perkins made the explicit choice  $S = \mathbb{Z}^d$ .) The matrix  $\mathcal{A}$  is defined by

$$(1.2) \quad \mathcal{A}(k, l) = \mathbf{A}(k, l) - \mathbb{1}_{\{k=l\}},$$

where  $\mathbf{A}$  is the transpose of a stochastic matrix  $\mathbf{A}^T$  on  $S$  such that  $\sup_{k \in S} \sum_{l \in S} \mathbf{A}(k, l) < \infty$ . Note that  $\mathbf{A}^T$  is the  $q$ -matrix of the continuous time Markov chain on  $S$  with jump kernel  $\mathbf{A}^T$ . (In fact, Dawson and Perkins assumed that  $\mathbf{A}$  be symmetric but this is not substantial.) Finally,  $(W_i(k), k \in S, i = 1, 2)$  is an independent family of one-dimensional Brownian motions.

This is a spatial model for the evolution of two populations  $i = 1, 2$ .  $Y_{i,t}(k)$  is the size of the population of type  $i$  at site  $k \in S$  at time  $t$ . The individuals migrate on the site space  $S$  according to the discrete space heat flow induced by  $\mathbf{A}^T$ . Furthermore, at each given site each type of the population undergoes a random dynamic that can be interpreted as continuous state Feller's branching with a branching rate proportional to the local size of the respective other type. Both types undergo (independently) the same branching dynamics and influence each other in a symmetric way – hence the name mutually catalytic branching process.

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Dawson and Perkins studied the longtime behaviour of this model with summable initial conditions (and symmetric  $\mathbf{A}$ ) and established a dichotomy of coexistence versus non-coexistence of types depending on transience and recurrence of the Markov chain associated with  $\mathbf{A}$ . Via the self-duality of the mutually catalytic branching process, its total mass behaviour for summable initial conditions provides information about the local behaviour if the initial condition is infinite and sufficiently homogeneous. For  $x \in [0, \infty)^2$ , let  $\underline{x}$  denote the state in  $([0, \infty)^2)^S$  with  $\underline{x}_i(k) = x_i$  for all  $k \in S$ ,  $i = 1, 2$ . Assume that  $Y_0 = \underline{x}$ . In [5, Theorem 1.4], it is shown that  $Y_t$  converges in distribution to some random field  $Y_\infty$  as  $t \rightarrow \infty$ . Furthermore (under some mild regularity assumptions on  $\mathbf{A}^T$ ), we have

$$\mathbf{P}_{\underline{x}}[Y_{1,\infty}(0)Y_{2,\infty}(0) > 0] > 0 \iff \mathbf{A}^T \text{ is transient.}$$

Hence, in the recurrent case, for constant initial conditions, the two types segregate locally and form clusters. The assumption that the initial point is constant can be weakened to an ergodic random initial condition (see [3]).

The starting point for this work was the wish to get a quantitative description of the cluster growth in the recurrent case. We only briefly give the heuristics. Dawson and Perkins also constructed a version of their process in continuous space  $\mathbb{R}$  instead of  $S$ . This process is defined as the solution of the stochastic partial differential equation

$$(1.3) \quad \frac{dY_{i,t}(r)}{dt} = \Delta Y_{i,t}(r) + \sqrt{\gamma Y_{1,t}(r)Y_{2,t}(r)} \dot{W}_i(t, r) \quad \text{for } r \in \mathbb{R}, i = 1, 2,$$

where  $\dot{W}_1$  and  $\dot{W}_2$  are independent space time white noises and  $\Delta$  is the Laplace operator. As the Laplace operator generates the semigroup of Brownian motion, and since Brownian motion on the real line is recurrent, here also the types segregate. Now, due to Brownian scaling, if we denote by  $Y^\gamma$  the solution of (1.3) with that given value of  $\gamma$ , then we obtain

$$(1.4) \quad \mathbf{P}_{\underline{x}}[(Y_T^\gamma(r\sqrt{T}))_{r \in \mathbb{R}} \in \cdot] = \mathbf{P}_{\underline{x}}[(Y_1^{\gamma T}(r))_{r \in \mathbb{R}} \in \cdot].$$

Equation (1.4) shows that clusters of  $Y_{1,T}$  grow like  $\sqrt{T}$  and that a better understanding of the precise cluster formation can be obtained by letting  $\gamma \rightarrow \infty$  for fixed time. The process  $X$  that is the limit (in distribution) of  $Y^\gamma$  as  $\gamma \rightarrow \infty$  is called the *infinite rate mutually catalytic branching process (IMUB)*.

In [10], the infinite rate mutually catalytic branching process  $X$  was constructed for  $S$  a singleton. It was shown that  $Y^\gamma$  converges to  $X$  as  $\gamma \rightarrow \infty$  and  $X$  was characterised in terms of a martingale problem and in terms of its generator. Since the two types cannot coexist in the limit  $\gamma \rightarrow \infty$ , the proper state space for the one-colony IMUB is

$$E := [0, \infty)^2 \setminus (0, \infty)^2.$$

In [11], the IMUB process  $X$  was constructed for countable  $S$  via approximate solutions to a related Poisson noise stochastic partial differential equation. Here  $X$  is a strong Markov process that takes values in a suitable subspace of  $E^S$  that fulfills some growth condition (Liggett-Spitzer space). More precisely, as shown in [11, discussion after (1.32)], for  $k \in S$ ,  $i = 1, 2$ , the process

$$(1.5) \quad \mathbb{M}_{i,t}(k) := X_{i,t}(k) - X_{i,0}(k) - \int_0^t \mathcal{A}X_{i,s}(k) ds, \quad t \geq 0,$$

is an  $L^p$ -martingale for every  $p \in [1, 2)$  (but not for  $p = 2$ ) that could be represented as the stochastic integral with respect to a Poisson point process.

Furthermore, the IMUB process  $X$  was characterised as the solution to a certain martingale problem and it was shown that  $Y^\gamma$  converges to  $X$  (in distribution). In order to formulate the martingale problem, we need the notation

$$x \diamond y := -(x_1 + x_2)(y_1 + y_2) + i(x_1 - x_2)(y_1 - y_2) \quad \text{for } x, y \in E$$

(with  $i = \sqrt{-1}$ ) and

$$\langle\langle x, y \rangle\rangle = \sum_{k \in S} x \diamond y$$

for  $x, y \in E^S$  such that the sum is well-defined. Then the  $E^S$  valued Markov process  $X$  with initial value  $X_0 = x$  is characterised by the requirement that

$$(1.6) \quad e^{\langle\langle X_t, y \rangle\rangle} - e^{\langle\langle x, y \rangle\rangle} - \int_0^t \langle\langle \mathcal{A}X_s, y \rangle\rangle e^{\langle\langle X_s, y \rangle\rangle} ds, \quad t \geq 0,$$

be a martingale for all suitable  $y \in E^S$ .

In [12], a construction of  $X$  is performed via a Trotter type approximation scheme, see also [15]. Loosely speaking, for given  $\varepsilon > 0$ , consider a process  $X^\varepsilon$  that solves (1.1) with  $\gamma = 0$  in each interval  $[n\varepsilon, (n+1)\varepsilon)$ ,  $n \in \mathbb{N}_0$ . At the times  $n\varepsilon$  the state  $X_{n\varepsilon-}^\varepsilon$  is replaced by the limit (as  $t \rightarrow \infty$ ) of a solution  $Y$  of (1.1) with  $\mathcal{A} = 0$  and  $\gamma > 0$  with initial state  $Y_0 := X_{n\varepsilon-}^\varepsilon$ . That is, the value  $X_{n\varepsilon-}^\varepsilon(k)$  at each colony  $k \in S$  is replaced (independently) by a point in  $E$  chosen randomly according to the exit distribution of planar Brownian motion in  $[0, \infty)^2$  started at  $X_{n\varepsilon-}^\varepsilon(k)$ . (See Section 2.2 for a more detailed description.) It was shown in [12] that  $X^\varepsilon$  converges as  $\varepsilon \rightarrow 0$  to a process that solves the martingale problem (1.6).

In this paper, we aim at understanding the longtime behaviour of  $X$  for summable initial states  $x$  (and thus, in (1.6) we could take any bounded  $y \in E^S$ ). Let  $\tau$  be the amount of time that two independent Markov chains with  $q$ -matrix  $\mathcal{A}^T$  spend together. We show that if  $\tau$  has infinite expectation (and  $\mathcal{A}^T$  fulfils some modest regularity condition), then the types cannot coexist in the long run. On the other hand, if  $\mathcal{A}$  fulfils a condition that is somewhat stronger than  $\mathbf{E}[\tau] < \infty$ , then the types can coexist in the long run. As  $X$  is a process with an infinite variance random dynamics, the meta theorem that relates stability of the longtime behaviour to transience of the migration dynamics does not apply here. It remains open to check if there are cases where  $\mathbf{E}[\tau] < \infty$  but coexistence of types is impossible. In particular, it would be interesting to know if this could happen for certain (transient) random walk kernels  $\mathcal{A}$ .

**1.2. Results.** Before we present our result, we give a more detailed description the longtime behaviour for the case of finite  $\gamma$  and *finite* initial conditions as stated in [5]. While segregation of types for constant initial conditions can be rephrased as “two types cannot be present at the same site” in the longrun, for finite initial conditions, this does not make sense, as the population dissipates in space anyway. Here, the notion of (local) segregation of types is replaced by the notion of (global) non-coexistence of types.

Let

$$(1.7) \quad M_{i,t}^Y := \langle Y_{i,t}, 1 \rangle := \sum_{k \in S} Y_{i,t}(k), \quad t \geq 0,$$

be the total mass process of type  $i = 1, 2$  and assume that  $M_{1,0}^Y, M_{2,0}^Y \in (0, \infty)$ . Since  $M_1^Y$  and  $M_2^Y$  are (orthogonal) nonnegative martingales, they converge almost surely and, in fact, also in  $L^1$ . Denote by

$M_{i,\infty}^Y$  the limit variables. For the case where  $\mathbf{A}$  is a random walk kernel, Theorem 1.2 of [5] takes the concise form

$$(1.8) \quad \mathbf{P}[M_{1,\infty}^Y M_{2,\infty}^Y > 0] > 0 \iff \mathbf{A} \text{ is transient.}$$

In order to formulate the result for the more general case, we have to be a bit more careful. Let  $a_t$  be the continuous time kernel; that is

$$(1.9) \quad a_t = \exp(\mathcal{A}t) := \sum_{n=0}^{\infty} \frac{t^n \mathcal{A}^n}{n!} = \sum_{n=0}^{\infty} e^{-t} \frac{t^n \mathbf{A}^n}{n!},$$

where  $\mathbf{A}^n$  and  $\mathcal{A}^n$  denote the matrix powers. Note that  $a_t^T$  is then the continuous time Markov kernel related to  $\mathbf{A}^T$ . Furthermore, for  $t \geq 0$ , define the Green kernels

$$(1.10) \quad G_t(k, l) := \int_0^t a_s(k, l) ds \quad \text{and} \quad G(k, l) := \int_0^{\infty} a_s(k, l) ds.$$

Finally, let

$$(1.11) \quad G_t^* := \sup_{k \in S} G_t(k, k).$$

We say that coexistence of types is possible if  $\mathbf{P}[M_{1,\infty}^Y M_{2,\infty}^Y > 0] > 0$  for all initial states  $Y_0$  with  $M_{1,0}^Y, M_{2,0}^Y \in (0, \infty)$ . We say that coexistence of types is impossible if  $\mathbf{P}[M_{1,\infty}^Y M_{2,\infty}^Y > 0] = 0$  for all initial states  $Y_0$  with  $M_{1,0}^Y, M_{2,0}^Y \in [0, \infty)$ .

Now, Theorem 1.2 of [5] states the following:

**Theorem 0 (Theorem 1.2 of [5])** *Assume that  $\mathbf{A}$  is symmetric.*

- (i) *If  $\sup_{k \in S} G(k, k) < \infty$ , then coexistence of types is possible.*
- (ii) *If*

$$(1.12) \quad \inf_{k \in S} \liminf_{t \rightarrow \infty} \frac{G_t(k, k)}{G_t^*} > 0$$

*and if  $\mathbf{A}$  is recurrent and irreducible, then coexistence of types is impossible.*

The theorem describes a dichotomy between a stable behaviour (coexistence of types) and an instable or clustering behaviour (segregation of types) depending on properties of the Green function of the underlying migration dynamics. A similar dichotomy along the same line of transience and recurrence (in the case of migration of random walk type) was observed before for many interacting models with finite variance dynamics such as the voter model (see [1, 8]), interacting diffusions on a compact interval ([14, 17, 2]), branching random walk ([9]), the generalized smoothing and potlatch process ([7]), and the so-called linear systems ([13, Chapter IX]). If the local dynamics has moments only up to order  $1 + \beta$  for some  $\beta \in (0, 1)$ , then the critical line between the two regimes may shift to the point where higher powers of the Green operator are finite (see [4]). In the model we study in this paper, all moments less than the second are finite and thus a dichotomy could be expected close to the transience/recurrence line but a little shifted to the transient side.

In order to prepare for the formulation of our theorem and since we want to get rid of the symmetry assumption that Dawson and Perkins imposed on  $\mathbf{A}$ , we have to introduce some more notation first. Recall that  $\mathbf{A}^T(k, l) = \mathbf{A}(l, k)$  is the transpose of  $\mathbf{A}$  and define  $a_t^T, G_t^T$  and so on similarly as for  $\mathbf{A}$ . Furthermore, define the symmetrised kernels

$$(1.13) \quad \begin{aligned} \bar{a}_t(k, l) &= a_{t/2}^T a_{t/2}(k, l) = \sum_{m \in S} a_{t/2}(m, k) a_{t/2}(m, l), \\ \bar{\mathcal{A}} &= \frac{1}{2}(\mathcal{A} + \mathcal{A}^T) \quad \text{and} \quad \bar{\mathbf{A}} = \frac{1}{2}(\mathbf{A} + \mathbf{A}^T). \end{aligned}$$

Note that  $(\bar{a}_t)_{t \geq 0}$  is a semigroup (and is generated by  $\bar{\mathcal{A}}$ ) if  $\mathcal{A}\mathcal{A}^T = \mathcal{A}^T\mathcal{A}$ . In particular, if  $S$  is an Abelian group and  $\mathbf{A}$  is a random walk kernel, then  $(\bar{a}_t)_{t \geq 0}$  is the semigroup of the difference of two (rate 1/2) random walks and its one step transition matrix is  $\bar{a}$ .

Define the expected amount of time  $\tau$  two independent Markov chains with  $q$ -matrix  $\mathcal{A}^T$  spend together:

$$(1.14) \quad \bar{G}(k, l) := \lim_{t \rightarrow \infty} \bar{G}_t(k, l), \quad \text{where} \quad \bar{G}_t(k, l) := \int_0^t \bar{a}_s(k, l) ds.$$

As mentioned above, for many models with migration and local finite variance random fluctuations, finiteness of  $\bar{G}$  is equivalent to stability. We will show for the IMUB model here that  $\bar{G}(0, 0) = \infty$  (plus some mild regularity conditions) is sufficient for non-coexistence of types. In order to formulate the regularity conditions properly, we will also need

$$(1.15) \quad \bar{G}_t^* := \sup_{k \in S} \bar{G}_t(k, k).$$

In order to show coexistence of types we need more refined quantities. Define

$$(1.16) \quad p_s(k, l) := (a_s(\mathbf{A} + \mathbf{A}^T)a_s^T)(k, l) = \sum_{m, n \in S} a_s(k, m)(\mathbf{A}(m, n) + \mathbf{A}(n, m))a_s(l, n).$$

Let us present a very rough heuristic for the appearance of this object. If we start with a unit mass of type 1 at  $k_1$  and a unit mass of type 2 at  $k_2$ , then, as we will show, the expected mass of type  $i$  at time  $s$  at site  $m$  is  $a_s(m, k_i)$ . Recall that the types exclude each other at any given site. If type  $i$  is absent at  $m$  at time  $s$ , then the infinitesimal impact of type  $i$  at site  $m$  is governed by the immigration at rate  $\mathbf{A}(m, l)$  from the other sites  $l \in S$ ; that is, it is of order  $\mathbf{A}a_s(m, k_i)$ . Summing over all sites, we see that the expected total ‘‘activity’’ is of order  $p_s(k_1, k_2)$ . Since the interaction of types has infinite variance, it is not  $p$  itself that is the crucial quantity, but rather we will see that we will need a logarithmic correction term. In order to quantify the total amount of interaction in the ‘‘transient’’ case, we define

$$(1.17) \quad G_{p, \log}(k, l) := \int_0^\infty p_s(k, l)(1 + |\log(p_s(k, l))|) ds.$$

It is easy to check that

$$G_{p, \log}(k, l) \geq \int_0^\infty p_s(k, l) ds = \bar{G}(k, l) - \mathbb{1}_{\{k=l\}}.$$

Hence,  $G_{p, \log}$  is infinite if  $\bar{G}$  is infinite.

Define the total mass process

$$(1.18) \quad M_{i,t} := \langle X_{i,t}, \mathbf{1} \rangle$$

and assume that  $M_0 \in [0, \infty)^2$ . Recall the martingales  $\mathbb{M}_i(k)$  from (1.5) and note that  $M_{i,t} = \sum_{k \in S} \mathbb{M}_{i,t}(k)$  since  $\sum_{k \in S} \mathcal{A}(k, l) = 0$  for all  $l \in S$ . Hence,  $M_1$  and  $M_2$  are is a nonnegative martingales and therefore the almost sure limits

$$Z_i := \lim_{t \rightarrow \infty} M_{i,t}$$

exist,  $i = 1, 2$ .

Recall that  $\mathbf{A}^T$  is a stochastic matrix and that  $\mathcal{A}(k, l) = \mathbf{A}(k, l) - \mathbb{1}_{\{k=l\}}$ .

**Theorem 1** *Let  $X$  be the infinite rate mutually catalytic branching process with kernel  $\mathbf{A}$ . Assume that the initial state is given by*

$$X_{1,0} = \mathbb{1}_{\{l_1\}} \quad \text{and} \quad X_{2,0} = \mathbb{1}_{\{l_2\}}$$

for some  $l_1, l_2 \in S$ ,  $l_1 \neq l_2$ .

If  $l_1$  and  $l_2$  are such that  $G_{p, \log}(l_1, l_2)$  is sufficiently small, then there is coexistence of types in the longtime limit; that is,  $\mathbf{P}[Z_1 > 0, Z_2 > 0] > 0$ .

**Theorem 2** *Assume that  $\mathbf{A}$  fulfills*

$$(1.19) \quad \sup_{k \in S} \mathbf{A}(k, k) < 1$$

and

$$(1.20) \quad c := \inf_{k, l \in S} \liminf_{t \rightarrow \infty} \frac{\bar{G}_t(k, l)}{\bar{G}_t^*} > 0.$$

Then coexistence of types is impossible; that is, if  $M_0 \in [0, \infty)^2$ , then  $Z_1 Z_2 = 0$  almost surely.

**Remark 3** (i) Note that in the case where  $\mathbf{A} = \mathbf{A}^T$ , condition (1.20) implies irreducibility of  $\mathbf{A}$ . Furthermore, if  $\mathbf{A} = \mathbf{A}^T$  is recurrent and irreducible, due to the strong Markov property, (1.20) is equivalent to (1.12).

(ii) Assume that  $S$  is an Abelian group and  $\mathbf{A}$  is a random walk kernel. In this case, (1.20) is equivalent to  $\bar{a}$  being irreducible and recurrent. In particular, (1.20) holds if  $\mathbf{A}$  is irreducible and recurrent.

(iii) For symmetric simple random walk on the  $d$ -dimensional integer lattice  $\mathbb{Z}^d$ ,  $d \geq 3$ , it is simple to show (using the CLT), that  $G_{p, \log}(l_1, l_2) \approx c_d \|l_1 - l_2\|^{2-d}$  as  $\|l_1 - l_2\| \rightarrow \infty$ , for some constant  $c_d \in (0, \infty)$ . Hence, the assumptions of Theorem 1 are fulfilled for simple random walk with the two populations being sufficiently far apart.

The proofs in [5] heavily rely on second moment methods. The main difficulty in the proofs here is the lack of second moments. (For this reason, presumably the statement of Theorem 1 fails under the weaker assumption that only  $\bar{G} < \infty$ .) The strategy of proof for Theorem 1 is therefore to introduce for  $K > 0$  an auxiliary process  $X^K$  whose jumps in each coordinate are suppressed when they lead out of the square  $[0, K]^2$ . This is done in such a way that the coordinate processes (minus the drifts) become square integrable orthogonal martingales. For these martingales, we use the conditions on  $G_{p, \log}$  to estimate the conditional quadratic variation process.

The proof of Theorem 2 also uses the auxiliary process  $X^K$  and its conditional quadratic variation process, but the arguments are more involved.

1.3. *Organization of the paper.* In Section 2, we prove Theorem 1. First, we derive basic properties of Brownian motion in  $[0, K]^2$  stopped upon hitting the boundary such as hitting distribution, moments and so on. Then we construct the auxiliary process  $X^K$  and conclude Theorem 1 via second moment estimates.

In Section 3, we prove Theorem 2.

**2. Coexistence of types, Proof of Theorem 1.** In Section 2.1, we perform some preliminary calculations for the variance of planar Brownian motion in  $[0, K]^2$  stopped upon hitting the boundary. In Section 2.2 we construct the auxiliary process  $X^K$ . In Section 2.3, we show that the coordinates of  $X^K$  are orthogonal martingales and we compute their conditional quadratic variation. In the final Section 2.4, we put the ends together to conclude the proof of Theorem 1.

2.1. *Brownian Motion in a Square.* Denote by  $Q$  the harmonic measure of planar Brownian motion in  $[0, \infty)^2$ . That is, if  $B = (B_1, B_2)$  is a Brownian motion in  $\mathbb{R}^2$  started at  $x \in [0, \infty)^2$  and

$$\tau_0 := \inf\{t > 0 : B_t \notin (0, \infty)^2\},$$

then we define

$$(2.1) \quad Q_x := \mathbf{P}_x[B_{\tau_0} \in \cdot].$$

If  $x = (u, v) \in (0, \infty)^2$ , then the harmonic measure  $Q_x$  has a one-dimensional Lebesgue density on  $E$  that can be computed explicitly

$$(2.2) \quad Q_{(u,v)}(d(\bar{u}, \bar{v})) = \begin{cases} \frac{4}{\pi} \frac{uv \bar{u}}{4u^2v^2 + (\bar{u}^2 + v^2 - u^2)^2} d\bar{u}, & \text{if } \bar{v} = 0, \\ \frac{4}{\pi} \frac{uv \bar{v}}{4u^2v^2 + (\bar{v}^2 + u^2 - v^2)^2} d\bar{v}, & \text{if } \bar{u} = 0. \end{cases}$$

Furthermore, trivially, we have that  $Q_x = \delta_x$  if  $x \in E$ .

In [10], it is shown that for  $x \in (0, \infty)^2$  and  $p \in (0, 2)$ , we have

$$(2.3) \quad \int y_i Q_x(dy) = x_i$$

and

$$(2.4) \quad \mathbf{E}_x[\tau_0^{p/2}] < \infty.$$

Applying the Burkholder-Davis-Gundy inequality, this implies that

$$(2.5) \quad \int y_i^p Q_x(dy) < \infty.$$

However, for  $p = 2$ , we have

$$(2.6) \quad \int y_i^2 Q_x(dy) = \infty.$$

Now consider planar Brownian motion  $B$  on  $[0, K]^2$  and its harmonic measure  $Q^K$  defined by

$$Q_x^K := \mathbf{P}_x[B_{\tau_K} \in \cdot],$$

where

$$\tau_K := \inf \{t > 0 : B_t \notin (0, K)^2\}.$$

Due to the obvious scaling, we can restrict ourselves mostly to  $K = 1$ . For simplicity, we define  $\tau := \tau_1$ .

**Lemma 2.1** *For all  $x \in [0, 1]^2$  and  $i = 1, 2$ , we have*

$$(2.7) \quad \int y_i Q_x^1(dy) = x_i.$$

Furthermore,

$$(2.8) \quad V(x) := \mathbf{E}_x[\tau] = \int (y_i - x_i)^2 Q_x^1(dy)$$

and

$$(2.9) \quad \int (y_1 - x_1)(y_2 - x_2) Q_x^1(dy) = 0.$$

**Proof.**  $(B_{i,t \wedge \tau})_{t \geq 0}$  is a bounded martingale and  $t \wedge \tau$  is its quadratic variation. Hence, (2.7) and (2.8) are simple consequences of the optional stopping theorem for martingales. Similarly, (2.9) follows from the fact that the product  $(B_{1,t \wedge \tau} B_{2,t \wedge \tau})_{t \geq 0}$  is a bounded martingale.  $\square$

**Lemma 2.2** *For  $x = (u, v) \in [0, 1]^2$ ,  $V(u, v)$  has the Fourier expansion*

$$(2.10) \quad \mathbf{E}_{(u,v)}[\tau] = V(u, v) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} c_{m,n} \sin((2m+1)\pi u) \sin((2n+1)\pi v),$$

where

$$(2.11) \quad c_{m,n} := \frac{32}{\pi^4} \frac{\frac{1}{2m+1} \frac{1}{2n+1}}{(2m+1)^2 + (2n+1)^2} \quad \text{for all } m, n \in \mathbb{N}_0.$$

**Proof.** It is well known that  $V$  is the unique solution of the Poisson equation with Dirichlet boundary condition

$$(2.12) \quad \begin{aligned} \frac{1}{2} \Delta g &= -1 \quad \text{in } (0, 1)^2, \\ g &= 0 \quad \text{at } \partial(0, 1)^2. \end{aligned}$$

Denote by  $g(u, v)$  the right hand side in (2.10). Clearly,  $g = 0$  at  $\partial(0, 1)^2$ . Furthermore, for  $u, v \in (0, 1)$ ,

$$(2.13) \quad \frac{1}{2} \Delta g(u, v) = - \left( \frac{4}{\pi} \sum_{m=0}^{\infty} \frac{\sin((2m+1)\pi u)}{2m+1} \right) \left( \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{\sin((2n+1)\pi v)}{2n+1} \right) = -1,$$

where the last equality follows from the fact that each factor is the Fourier series of the function on  $(0, 1)$  that is constant 1. Hence, we have  $g = V$ .  $\square$

**Corollary 2.3** *For planar Brownian motion  $B$  in  $[0, K]^2$ , we have*

$$\mathbf{E}_{(u,v)}[\tau_K] = K^2 V(u/K, v/K)$$

and

$$\mathbf{Cov}_{(u,v)}[B_{i,\tau_K}, B_{j,\tau_K}] = K^2 V(u/K, v/K) \mathbb{1}_{\{i=j\}}.$$

**Proof.** This follows from Brownian scaling. □

**Lemma 2.4** *For all  $u, v \in [0, 1]$ , we have*

$$(2.14) \quad V(u, v) \geq 2u(1-u)v(1-v).$$

**Proof.** Let  $f(u, v) = 2u(1-u)v(1-v)$ . Then

$$\frac{1}{2}\Delta f(u, v) = -2[u(1-u) + v(1-v)] \geq -1.$$

Hence, by the maximum principle,  $f$  is a sub-solution for the Poisson problem (2.12) which shows  $f \leq V$ . □

**Proposition 2.5** *Let  $K > 0$  and let  $B$  be Brownian motion in  $[0, K]^2$ . Then we have*

$$(2.15) \quad \begin{aligned} \mathbf{Cov}_{(u,v)}[B_{i,\tau_K}, B_{j,\tau_K}] &= V(u/K, v/K) K^2 \mathbb{1}_{\{i=j\}} \\ &\leq 8uv[1 + \log(K) + (\log(1/u) \wedge \log(1/v))] \mathbb{1}_{\{i=j\}} \quad \text{for all } u, v \in [0, K] \end{aligned}$$

and

$$(2.16) \quad \mathbf{Cov}_{(u,v)}[B_{i,\tau_K}, B_{j,\tau_K}] \geq \frac{1}{2}uv \mathbb{1}_{\{i=j\}} \quad \text{for all } u, v \in [0, K/2].$$

**Proof.** The first equality is due to Brownian scaling. Hence, it is enough to consider the case  $K = 1$ . Note that (2.16) is an immediate consequence of Lemma 2.4. Hence, we concentrate on showing (2.15). We have to show that (with  $V$  from (2.10))

$$(2.17) \quad V(u, v) \leq 8uv[1 + \log(1/u)].$$

By symmetry in  $u$  and  $v$ , this implies (2.15).

As  $V(u, v)$  is bounded by the expected time one-dimensional Brownian motion started at  $v$  needs to hit  $\{0, 1\}$ , we have  $V(u, v) \leq v(1-v) \leq v$  for all  $u, v \in [0, 1]$ . Hence, for  $u > 1/3$ , (2.17) holds with the factor 8 replaced by  $3/(1 + \log(3)) \leq 2$ .

We may and will now assume that  $u \leq 1/3$ . Let  $M \in \mathbb{N}$  be such that  $\frac{1}{2M+3} < u \leq \frac{1}{2M+1}$ . We will show that

$$(2.18) \quad V(u, v) \leq 8uv[1 + \log(M)] \quad \text{for } v \in [0, 1], M \in \mathbb{N}.$$

We estimate  $|\sin((2m+1)\pi u)| \leq \pi(2m+1)u$  for  $m \leq M-1$  and  $|\sin((2m+1)\pi u)| \leq 1$  for  $m \geq M$ , as well as  $|\sin((2n+1)\pi v)| \leq \pi(2n+1)v$ . Hence, we obtain

$$\begin{aligned}
V(u, v) &= \sum_{n=0}^{\infty} \sum_{m=0}^{M-1} c_{m,n} \sin((2m+1)\pi u) \sin((2n+1)\pi v) \\
&\quad + \sum_{n=0}^{\infty} \sum_{m=M}^{\infty} c_{m,n} \sin((2m+1)\pi u) \sin((2n+1)\pi v) \\
&\leq \frac{32uv}{\pi^2} \sum_{m=0}^{M-1} \sum_{n=0}^{\infty} \frac{1}{(2m+1)^2 + (2n+1)^2} + \frac{32v}{\pi^3} \sum_{m=M}^{\infty} \frac{1}{2m+1} \sum_{n=0}^{\infty} \frac{1}{(2m+1)^2 + (2n+1)^2} \\
&=: I_M(u, v) + J_M(u, v).
\end{aligned}$$

The two summands will be estimated separately. First note that

$$\begin{aligned}
I_M(u, v) &\leq \frac{32uv}{\pi^2} \sum_{m=0}^{M-1} \int_0^{\infty} \frac{1}{(2m+1)^2 + t^2} dt \\
&= \frac{16uv}{\pi} \sum_{m=0}^{M-1} \frac{1}{2m+1} \leq \frac{16uv}{\pi} [1 + \log(M)].
\end{aligned}$$

Similarly, we get (note that  $5M \geq 2M + 3 \geq 1/u$  by the assumption on  $M$ )

$$J_M(u, v) \leq \frac{16v}{\pi^2} \sum_{m=M}^{\infty} \frac{1}{(2m+1)^2} \leq \frac{4v}{\pi^2} \frac{1}{M} \leq \frac{20v}{\pi^2} \frac{1}{2M+3} \leq \frac{20}{\pi^2} uv.$$

Summing up and noting that  $16/\pi + 20/\pi^2 \leq 8$ , we obtain (2.18).  $\square$

*2.2. Construction of the Truncated Process.* The aim of this section is to construct a process  $X^K$  that approaches  $X$  as  $K \rightarrow \infty$  and which has finite second moments. The idea is to suppress the large jumps of  $X$  so that the remaining jumps have second moments. It turns out that if we proceed a bit more subtly, then we can obtain even that the coordinate processes of  $X^K$  are orthogonal square integrable martingales and that we can control the conditional quadratic variation process. The rough idea is as follows. The jumps of  $X$  can be interpreted as being driven by the positional changes of planar Brownian motion at its exit points from  $[0, \infty)^2$ . For the process  $X^K$ , we stop this planar Brownian motion when it exits  $[0, K]^2$ .

We could proceed in two ways to construct  $X^K$ :

1. We could imitate the SPDE construction of  $X$  (see [11]) by replacing the intensity measure on  $E$  of the Poisson point process by a suitable intensity measure on  $[0, K]^2 \setminus (0, K)^2$ .
2. We could imitate the Trotter type construction of  $X$  (see [12]) by replacing the harmonic measure  $Q$  on  $[0, \infty)^2$  by the harmonic measure  $Q^K$  on  $[0, K]^2$ .

Here we follow the latter approach. In [12], the following was done in order to construct  $X$ : For fixed  $\varepsilon > 0$ , consider the stochastic process  $X^\varepsilon$  with values in  $([0, \infty)^2)^S$  with the following dynamics:

- (i) Within each time interval  $[n\varepsilon, (n+1)\varepsilon)$ ,  $n \in \mathbb{N}_0$ ,  $X^\varepsilon$  is the solution of

$$dX_{i,t}^\varepsilon(k) = (\mathcal{A}X_{i,t}^\varepsilon)(k) dt \quad \text{for } t \in [n\varepsilon, (n+1)\varepsilon) \quad k \in S.$$

Clearly, the explicit solution is

$$X_{i,t}^\varepsilon(k) = (a_{t-n\varepsilon} X_{i,n\varepsilon}^\varepsilon)(k) \quad \text{for } t \in [n\varepsilon, (n+1)\varepsilon).$$

- (ii) At time  $n\varepsilon$ ,  $X^\varepsilon$  has a discontinuity. Independently, each coordinate  $X_{n\varepsilon-}^\varepsilon(k) = a_\varepsilon X_{(n-1)\varepsilon}^\varepsilon(k)$  is replaced by a random element of  $E$  drawn according to the distribution  $Q_{X_{n\varepsilon-}^\varepsilon(k)}$ .

In order for the solution in Step (i) to be well-defined, we have to impose some growth condition on the initial states (see [12, Theorem 1]). Since here we are interested in finite initial states, this growth condition is automatically fulfilled.

In [12], it was shown that  $X^\varepsilon$  converges as  $\varepsilon \rightarrow 0$  to  $X$  in the Skorohod space of paths  $[0, \infty) \rightarrow ([0, \infty)^2)^S$ . Define

$$\tau_K := \inf \{t > 0 : \langle X_{1,t} + X_{2,t}, 1 \rangle \geq K/2\}.$$

Clearly,  $\tau_K$  is a stopping time and by Doob's inequality, we get

$$\mathbf{P}[\tau_K < \infty] \leq 2 \frac{\langle X_{1,0} + X_{2,0}, 1 \rangle}{K}.$$

Now, assume that  $\langle x_1 + x_2, 1 \rangle < K$ . We construct  $X^{K,\varepsilon}$  with initial condition  $x$  just as  $X$  but with two differences:

- (1) In Step (ii) above, we replace  $E$  by  $[0, K]^2 \setminus (0, K)^2$  and  $Q$  by  $Q^K$ .
- (2) If  $\langle X_{1,n\varepsilon}^{K,\varepsilon} + X_{2,n\varepsilon}^{K,\varepsilon}, 1 \rangle > K/2$ , then Step (ii) above is omitted.

Note that Step (i) preserves the total mass, hence once the total mass exceeds  $K/2$ , the process  $X^{K,\varepsilon}$  is simply the discrete space heat flow with kernel  $\mathcal{A}$ . Denote by

$$\tau^{K,\varepsilon} := \inf \{t > 0 : \langle X_{1,t}^{K,\varepsilon} + X_{2,t}^{K,\varepsilon}, 1 \rangle \geq K/2\}$$

the time when this first happens. Note that due to the strong Markov property of planar Brownian motion, we have

$$Q_x = \int_{[0,K]^2 \setminus (0,K)^2} Q_x^K(dy) Q_y.$$

Hence,  $X^\varepsilon$  and  $X^{K,\varepsilon}$  can be coupled to coincide almost surely until  $\tau^{K,\varepsilon}$ . Since  $X^\varepsilon$  converges, this implies that also  $(X_{\tau_{K,\varepsilon} \wedge t}^{K,\varepsilon})_{t \geq 0}$  converges in the Skorohod space as  $\varepsilon \rightarrow 0$ . Since the  $\mathcal{A}$  heat flow clearly exists, in fact,  $X^{K,\varepsilon}$  converges as  $\varepsilon \rightarrow 0$  to some process  $X^K$ . Clearly,

$$M_{i,t}^{K,\varepsilon}(k) := X_{i,t}^{K,\varepsilon}(k) - \int_0^t (\mathcal{A}X_{i,s}^{K,\varepsilon})(k) ds, \quad i = 1, 2, k \in S,$$

are orthogonal square integrable martingales with conditional quadratic variation process (see (2.8) and (2.15))

$$\langle M_i^{K,\varepsilon}(k) \rangle_t = \sum_{n: n\varepsilon \leq t \wedge \tau^{K,\varepsilon}} K^2 V(X_{1,n\varepsilon-}^{K,\varepsilon}/K, X_{2,n\varepsilon-}^{K,\varepsilon}/K).$$

*Upper bound for the conditional quadratic variation.* Let  $I$  be the identity matrix. Note that for each  $n$  with  $(n-1)\varepsilon < \tau^{K,\varepsilon}$ , we either have  $X_{1,(n-1)\varepsilon}^{K,\varepsilon}(k) = 0$  (which implies  $X_{1,n\varepsilon-}^{K,\varepsilon}(k) \leq (a_\varepsilon - I)X_{1,(n-1)\varepsilon}^{K,\varepsilon}(k)$ ) or  $X_{2,(n-1)\varepsilon}^{K,\varepsilon}(k) = 0$  (which implies  $X_{2,n\varepsilon-}^{K,\varepsilon}(k) \leq (a_\varepsilon - I)X_{2,(n-1)\varepsilon}^{K,\varepsilon}(k)$ ). Also note that by Proposition 2.5, we have  $K^2V(u/K, v/K) \leq u h_K(v)$  and  $K^2V(u/K, v/K) \leq v h_K(u)$ , where

$$(2.19) \quad h_K(u) := 8u(1 + \log(K/u)).$$

Hence, we get

$$\begin{aligned} & \langle M_i^{K,\varepsilon}(k) \rangle_t \\ & \leq \varepsilon \sum_{n: n\varepsilon \leq t \wedge \tau^{K,\varepsilon}} \varepsilon^{-1}(a_\varepsilon - I)X_{1,(n-1)\varepsilon}^{K,\varepsilon}(k)h_K(X_{2,n\varepsilon-}^{K,\varepsilon}(k)) + \varepsilon^{-1}(a_\varepsilon - I)X_{2,(n-1)\varepsilon}^{K,\varepsilon}(k)h_K(X_{1,n\varepsilon-}^{K,\varepsilon}(k)), \end{aligned}$$

Since bounded  $L^2$ -martingales converge to bounded  $L^2$ -martingales, and since  $\varepsilon^{-1}(a_\varepsilon - I)(k, l) \xrightarrow{\varepsilon \rightarrow 0} \mathbf{A}(k, l)$  for  $k \neq l$ , we get that

$$(2.20) \quad M_{i,t}^K(k) := X_{i,t}^K(k) - \int_0^t (\mathbf{A}X_{i,s}^K)(k) ds, \quad i = 1, 2, k \in S,$$

are orthogonal square integrable martingales with conditional quadratic variation processes

$$(2.21) \quad \langle M_i^K(k) \rangle_t \leq \int_0^t \left( \mathbf{A}X_{1,s}^K(k)h_K(X_{2,s}^K(k)) + \mathbf{A}X_{2,s}^K(k)h_K(X_{1,s}^K(k)) \right) ds, \quad i = 1, 2, k \in S.$$

*Lower bound for the conditional quadratic variation.* Define

$$(2.22) \quad \tau^K := \inf \{t > 0 : \langle X_{1,t}^K + X_{2,t}^K, 1 \rangle \geq K/2\}.$$

By Doob's inequality,

$$\mathbf{P}[\tau^K < \infty] \leq 2 \frac{\langle X_{1,0} + X_{2,0}, 1 \rangle}{K}.$$

Furthermore,  $X^K$  coincides with  $X$  (in distribution) until time  $\tau^K$ . By Proposition 2.5, we have

$$\begin{aligned} \langle M_i^{K,\varepsilon}(k) \rangle_t & \geq \frac{1}{2} \sum_{n: n\varepsilon \leq t \wedge \tau^{K,\varepsilon}} X_{1,n\varepsilon-}^{K,\varepsilon}(k)X_{2,n\varepsilon-}^{K,\varepsilon}(k) \\ & = \frac{1}{2} \sum_{n: n\varepsilon \leq t \wedge \tau^{K,\varepsilon}} a_\varepsilon X_{1,(n-1)\varepsilon}^{K,\varepsilon}(k)a_\varepsilon X_{2,(n-1)\varepsilon}^{K,\varepsilon}(k). \end{aligned}$$

Recall that  $a_\varepsilon(k, l) = e^{-\varepsilon} \mathbb{1}_{\{k=l\}} + e^{-\varepsilon} \varepsilon \mathbf{A}(k, l) + \dots$ . Since  $X_{1,(n-1)\varepsilon}^{K,\varepsilon}(k)X_{2,(n-1)\varepsilon}^{K,\varepsilon}(k) = 0$  for all  $n \leq \tau^{K,\varepsilon}$ , we have

$$\langle M_i^{K,\varepsilon}(k) \rangle_t \geq \frac{1}{2} \varepsilon e^{-\varepsilon} \sum_{n: n\varepsilon \leq t \wedge \tau^{K,\varepsilon}} [\mathbf{A}X_{1,(n-1)\varepsilon}^{K,\varepsilon}(k)X_{2,(n-1)\varepsilon}^{K,\varepsilon}(k) + X_{1,(n-1)\varepsilon}^{K,\varepsilon}(k) \mathbf{A}X_{2,(n-1)\varepsilon}^{K,\varepsilon}(k)],$$

where we also used  $\mathbf{A}X_{i,(n-1)\varepsilon}^{K,\varepsilon}(k)X_{3-i,(n-1)\varepsilon}^{K,\varepsilon}(k) = \mathbf{A}X_{i,(n-1)\varepsilon}^{K,\varepsilon}(k)X_{3-i,(n-1)\varepsilon}^{K,\varepsilon}(k)$  for  $(n-1)\varepsilon < \tau^{K,\varepsilon}$ .

Since  $(X_i^{K,\varepsilon})_{\varepsilon > 0}$  is a convergent sequence of bounded square integrable martingales, also the conditional quadratic variation processes converge and we infer for  $t \geq s \geq 0$ ,

$$(2.23) \quad \langle M_i^K(k) \rangle_t - \langle M_i^K(k) \rangle_s \geq \frac{1}{2} \int_{s \wedge \tau^K}^{t \wedge \tau^K} \left( \mathbf{A}X_{1,r}^K(k)X_{2,r}^K(k) + \mathbf{A}X_{2,r}^K(k)X_{1,r}^K(k) \right) dr.$$

2.3. *Truncated Process and Martingales.* In order not to interrupt the flow of the argument later, we start here with a lemma.

**Lemma 2.6** *Let  $Y$  and  $Z$  be non-positively correlated non-negative random variables and assume that  $h : [0, \infty) \rightarrow [0, \infty)$  is concave and monotone increasing. Then  $\mathbf{E}[Yh(Z)] \leq \mathbf{E}[Y]h(\mathbf{E}[Z])$ .*

**Proof.** If  $\mathbf{E}[Z] = 0$ , then we even have equality. Now, assume that  $\mathbf{E}[Z] > 0$ . By concavity of  $h$ , there exists a real number  $b \in \mathbb{R}$  such that for all  $z \geq 0$ ,

$$h(z) \leq h(\mathbf{E}[Z]) + (z - \mathbf{E}[Z])b.$$

Since  $h$  is nondecreasing, we have  $b \geq 0$  and thus

$$\mathbf{E}[Yh(Z)] \leq \mathbf{E}[Y(h(\mathbf{E}[Z]) + (Z - \mathbf{E}[Z])b)] \leq \mathbf{E}[Y]h(\mathbf{E}[Z]).$$

□

Let  $l_1, l_2 \in S$  and let  $X^K$  be the truncated process with initial state  $X_0^K = (\mathbb{1}_{\{l_1\}}, \mathbb{1}_{\{l_2\}})$ .

Writing (2.20) in the form

$$X_{i,t}^K(k) = X_{i,0}^K(k) + \sum_{l \in S} \int_0^t a_{t-s}(k, l) dM_{i,s}^K(l),$$

and recalling that the  $M_i^K(l)$  are orthogonal martingales, we get that the random variables  $X_{1,s}^K(k)$  and  $X_{2,s}^K(l)$  are uncorrelated for  $k, l \in S$ . Hence,  $\mathbf{A}X_{i,s}^K(l)$  and  $X_{3-i,s}^K(l)$  are uncorrelated. Note that  $x \mapsto h_K(x)$  (defined in (2.19)) is concave and monotone increasing for  $x \leq K$ . Hence, by Lemma 2.6, we get that

$$(2.24) \quad \mathbf{E}[\langle M_i^K(l) \rangle_t] \leq \sum_{j=1}^2 \int_0^t h_K(\mathbf{E}[X_{j,s}^K(l)]) \mathbf{E}[\mathbf{A}X_{3-j,s}^K(l)] ds.$$

Denote by

$$(2.25) \quad M_i^K = \sum_{k \in S} M_i^K(k) = \langle X_i^K, 1 \rangle, \quad i = 1, 2,$$

the total mass process of  $X_i^K$ . Then (2.24) implies

$$(2.26) \quad \begin{aligned} \mathbf{Var}[M_{i,t}^K] &= \mathbf{E}[\langle M_i^K \rangle_t] \\ &= \sum_{l \in S} \mathbf{E}[\langle M_i^K(l) \rangle_t] \\ &\leq \sum_{j=1}^2 \int_0^\infty \sum_{l \in S} h_K((a_s X_{j,0})(l)) (\mathbf{A}a_s X_{3-j,0})(l) ds. \end{aligned}$$

Using again the concavity of  $h_K$  and Jensen's inequality (for the probability measure  $l \mapsto (\mathbf{A}a_s)(l, k)$ ), we get

$$(2.27) \quad \mathbf{Var}[M_{i,t}^K] \leq \sum_{j=1}^2 \int_0^\infty \sum_{k \in S} X_{3-j,0}(k) h_K\left(\sum_{l \in S} (\mathbf{A}a_s)(l, k) (a_s X_{j,0})(l)\right) ds.$$

Now, recall that the initial states are  $X_{i,0} = \mathbb{1}_{\{l_i\}}$  and that  $p_s = (a_s(\mathbf{A} + \mathbf{A}^T)a_s^T)$ . Hence, we obtain

$$(2.28) \quad \mathbf{Var}[M_{i,t}^K] \leq \sum_{j=1}^2 \int_0^\infty h_K(p_s(l_1, l_2)) ds \leq 8 \log(K) G_{p,\log}(l_1, l_2).$$

Hence,  $M_1^K$  and  $M_2^K$  are (orthogonal)  $L^2$ -bounded martingales and thus converge almost surely and in  $L^2$  to some random variables  $M_{1,\infty}^K$  and  $M_{2,\infty}^K$  with  $\mathbf{E}[M_i^K] = 1$  and  $\mathbf{Var}[M_{i,\infty}^K] \leq 8 \log(K) G_{p,\log}(l_1, l_2)$ .

2.4. *Proof of Theorem 1.* Clearly, the product  $M_1^K \cdot M_2^K$  is a uniformly integrable martingale, hence

$$(2.29) \quad \mathbf{E}[M_{1,\infty}^K M_{2,\infty}^K] = \mathbf{E}[M_{1,0}^K M_{2,0}^K] = 1.$$

Write

$$\hat{M}_i^K := M_{i,\infty}^K \mathbb{1}_{\{\tau_K < \infty\}}.$$

Since we have  $X^K = X$  on  $\tau^K = \infty$ , in order to show Theorem 1, it is enough to show that  $\mathbf{E}[M_{1,\infty}^K M_{2,\infty}^K \mathbb{1}_{\{\tau_K = \infty\}}] > 0$ . To this end, we compute

$$\mathbf{E}[M_{1,\infty}^K M_{2,\infty}^K \mathbb{1}_{\{\tau_K = \infty\}}] = \mathbf{E}[M_{1,\infty}^K M_{2,\infty}^K] - \mathbf{E}[M_{1,\infty}^K \hat{M}_2^K] - \mathbf{E}[\hat{M}_1^K M_{2,\infty}^K] + \mathbf{E}[\hat{M}_1^K \hat{M}_2^K].$$

By (2.29), it is thus enough to show that

$$(2.30) \quad \mathbf{E}[M_{1,\infty}^K \hat{M}_2^K] < \frac{1}{2} \quad \text{and} \quad \mathbf{E}[\hat{M}_1^K M_{2,\infty}^K] < \frac{1}{2}.$$

Although  $M_{1,\infty}^K$  and  $M_{2,\infty}^K$  are uncorrelated, this is not true for  $\hat{M}_1^K$  and  $M_{2,\infty}^K$  (at least we cannot show this). Hence, we have to use a slightly more subtle argument. We employ the Cauchy-Schwarz inequality to estimate

$$(2.31) \quad \begin{aligned} \mathbf{E}[M_{1,\infty}^K \hat{M}_2^K]^2 &\leq \mathbf{E}[(M_{1,\infty}^K)^2] \mathbf{E}[(\hat{M}_2^K)^2] \\ &= (1 + \mathbf{Var}[M_{1,\infty}^K]) \mathbf{E}[(\hat{M}_2^K)^2] \\ &\leq (1 + 8 \log(K) G_{p,\log}(l_1, l_2)) \frac{9}{4} K^2. \end{aligned}$$

Recall that  $K > 2$  is fixed. Now, choose  $l_1$  and  $l_2$  such that  $G_{p,\log}(l_1, l_2)$  gets so small that the right hand side in (2.31) is bounded by  $\frac{1}{4}$ . Similarly, we get  $\mathbf{E}[\hat{M}_1^K M_{2,\infty}^K]^2 < \frac{1}{4}$ .

This shows (2.30) and thus completes the proof of Theorem 1.  $\square$

**3. Non-coexistence of types, Proof of Theorem 2.** The strategy of proof is described in the following two steps.

**Step 1.** Replace the process  $X$  by the approximate process  $X^K$  constructed in Section 2.2. If  $X$  would have coexistence of types, then so would  $X^K$  (for some large  $K$ ).

**Step 2.** Since  $M_{i,t}^K = \langle X_{i,t}^K, 1 \rangle$ ,  $t \geq 0$ , is a convergent martingale with bounded jump size, also the conditional quadratic variation process  $\langle M_{1,\cdot}^K \rangle$  converges. We derive a lower bound for  $\langle M_{1,\cdot}^K \rangle$  and show that due to the recurrence of  $\mathcal{A}$ , this lower bound would diverge with positive probability if  $X^K$  had coexistence of types. Hence, there can be no coexistence of types for  $X^K$  and thus neither for  $X$ .

3.1. *Step 1: The approximate process.* Assume that for  $X$ , coexistence of types is possible. We will lead this assumption to a contradiction.

Recall that  $M_{i,t} := \langle X_{i,t}, 1 \rangle$ ,  $t \geq 0$ ,  $i = 1, 2$ , are the total mass processes. Coexistence of types means that there exists a deterministic initial state  $X_0$  such that  $M_{i,t} < \infty$ ,  $i = 1, 2$ , and such that

$$\lim_{t \rightarrow \infty} M_{1,t} M_{2,t} > 0 \quad \text{with positive probability.}$$

(Recall from the discussion prior to Theorem 1 that the total mass processes are nonnegative martingales and are hence convergent.) We use Lemma 3.1 below (with  $Z_t = M_{1,t} M_{2,t}$ ) to infer that there exists a  $\delta > 0$ , such that

$$(3.1) \quad \mathbf{P}[M_{1,t} M_{2,t} \geq \delta \text{ for all } t \geq 0] \geq 6\delta.$$

**Lemma 3.1** *Let  $(Z_t)_{t \geq 0}$  be a nonnegative right continuous supermartingale. Then*

$$\inf\{Z_t : t \geq 0\} > 0 \quad \text{a.s. on the event } \left\{ \lim_{t \rightarrow \infty} Z_t > 0 \right\}.$$

**Proof.** Let  $(\mathcal{F}_t)_{t \geq 0}$  denote the natural filtration of  $Z$ . By the martingale convergence theorem,  $(Z_t)$  converges almost surely to some limit  $Z_\infty$ . Fix numbers  $T, S > 0$  with  $T > S + 1$ . For  $\varepsilon > 0$ , define the bounded stopping time  $\tau_\varepsilon := \inf\{t \geq 0 : Z_t \leq \varepsilon\} \wedge (S + 1)$ . By the optional sampling theorem for right continuous supermartingales (see, e.g., [6, Theorem II.2.13]), we get

$$Z_{\tau_\varepsilon} \geq \mathbf{E}[Z_T | \mathcal{F}_{\tau_\varepsilon}] \quad \text{a.s.}$$

By Markov's inequality, we infer for  $\delta > 0$ , that  $\mathbf{P}[Z_T \geq \delta | \mathcal{F}_{\tau_\varepsilon}] \leq Z_{\tau_\varepsilon}/\delta$  and, in particular,

$$\mathbf{P}[Z_T \geq \delta \text{ and } \tau_\varepsilon \leq S] \leq \frac{\varepsilon}{\delta}.$$

Letting  $\varepsilon \downarrow 0$  and then  $\delta \downarrow 0$  yields

$$\mathbf{P}[Z_T > 0 \text{ and } \inf\{Z_t : t \in [0, S]\} = 0] = 0.$$

Letting  $T \rightarrow \infty$  gives

$$\mathbf{P}[Z_\infty > 0 \text{ and } \inf\{Z_t : t \in [0, S]\} = 0] = 0.$$

Since  $(Z_t)$  converges, this implies

$$\mathbf{P}[Z_\infty > 0 \text{ and } \inf\{Z_t : t \geq 0\} = 0] = 0.$$

□

Recall  $\delta$  defined in (3.1) and define

$$K := \frac{2}{\delta}(M_{1,0} + M_{2,0}).$$

Let  $X^K$  denote the truncated process defined in Section 2.2 with  $X_0^K = X_0$ . Recall that  $(M_{i,t}^K)_{t \geq 0}$  is the total mass process of  $X_i^K$  and that it is a martingale. Hence, we have

$$(3.2) \quad \mathbf{P}[F] \geq 1 - \delta,$$

where

$$(3.3) \quad F := \{M_{1,t}^K + M_{2,t}^K \leq K/2 \text{ for all } t \geq 0\}.$$

Also

$$\mathbf{P}[M_{1,t} + M_{2,t} \leq K/2 \text{ for all } t \geq 0] \geq 1 - \delta.$$

We can couple  $X^K$  and  $X$  such that both processes coincide on  $F$ . Hence, for

$$B := \{M_{1,t}^K M_{2,t}^K \geq \delta \text{ for all } t \geq 0\},$$

we have

$$(3.4) \quad \mathbf{P}[B] \geq \mathbf{P}[F \cap B] \geq \mathbf{P}[M_{1,t} M_{2,t} \geq \delta \text{ for all } t \geq 0] - \mathbf{P}[F^c] \geq 5\delta.$$

*3.2. Step 2: The lower bound for the conditional quadratic variation.* Denote by  $(\langle M_{1,\cdot}^K \rangle_t)_{t \geq 0}$  the conditional quadratic variation process of  $(M_{1,t}^K)_{t \geq 0}$ . Since  $M_1^K$  is a martingale whose jumps are bounded (by  $K$ ), convergence of  $M_1^K$  implies almost sure convergence of  $\langle M_{1,\cdot}^K \rangle_t \rightarrow \langle M_{1,\cdot}^K \rangle_\infty < \infty$  as  $t \rightarrow \infty$ . Hence, for any  $\varepsilon > 0$  and  $\delta > 0$ , there exists a  $T_0$  such that

$$(3.5) \quad \mathbf{P}[\langle M_{1,\cdot}^K \rangle_t - \langle M_{1,\cdot}^K \rangle_{T_0} > \varepsilon] \leq \delta \quad \text{for all } t \geq T_0.$$

The aim of this section is to show that (3.4) leads to a contradiction to (3.5) which shows that the assumption that for  $X$  coexistence of types would be possible was wrong.

Fix  $\delta > 0$ . We choose an appropriate  $\varepsilon > 0$  for a contradiction via the following procedure. Recall (1.19) and define

$$(3.6) \quad A^*(k) := 1 - \mathbf{A}(k, k) \quad \text{and} \quad A^* := \inf_{k \in S} A^*(k) > 0.$$

Recall  $c$  from (1.20) and define

$$(3.7) \quad c' := \frac{A^* c}{16}.$$

Let

$$(3.8) \quad R := 2 + \frac{2K^2}{\delta^3 c'}.$$

Furthermore, define

$$(3.9) \quad \varepsilon := \frac{\delta^2 c'}{R + 1}.$$

Note that

$$(3.10) \quad \varepsilon - c' \delta^2 = -R\varepsilon \quad \text{and} \quad \frac{K^2}{R^2 \varepsilon} \leq \delta.$$

Recall from (2.20) that  $M_1^K(k)$ ,  $k \in S$ , are orthogonal martingales and that  $M_1^K = \sum_{k \in S} M_1^K(k)$ . Hence, by (2.23), for  $t \geq T_0$ , we have

$$\begin{aligned} \langle M_{1,\cdot}^K \rangle_t - \langle M_{1,\cdot}^K \rangle_{T_0} &= \sum_{k \in S} (\langle M_1^K(k) \rangle_t - \langle M_1^K(k) \rangle_{T_0}) \\ &\geq \frac{1}{2} \sum_{k \in S} \int_{T_0 \wedge \tau^K}^{t \wedge \tau^K} (\mathbf{A}X_{1,s}^K(k)X_{2,s}^K(k) + \mathbf{A}X_{2,s}^K(k)X_{1,s}^K(k)) ds, \end{aligned}$$

where  $\tau^K$  is defined in (2.22). Define

$$(3.11) \quad Z_t = \frac{1}{2} \sum_{k \in S} \int_{T_0}^t X_{1,s}^K(k) \mathcal{A}X_{2,s}^K(k) + X_{2,s}^K(k) \mathcal{A}X_{1,s}^K(k) ds \quad \text{for } t \geq T_0.$$

Note that  $\tau^K = \infty$  on  $F$ . Hence, for all  $t \geq T_0$ ,

$$(3.12) \quad \langle M_{1,\cdot}^K \rangle_t - \langle M_{1,\cdot}^K \rangle_{T_0} \geq Z_t \quad \text{on } F.$$

**Lemma 3.2** *For any  $k_1, k_2 \in S$ ,  $k_1 \neq k_2$ , let  $N_t(k_1, k_2)$  be given by*

$$N_t(k_1, k_2) = \sum_{\substack{l, l' \in S \\ l \neq l'}} \int_{T_0}^t a_{t-s}(k_1, l') a_{t-s}(k_2, l) \left( X_{1,s}^K(l') dM_{2,s}^K(l) + X_{2,s}^K(l) dM_{1,s}^K(l') \right) \quad \text{for } t \geq T_0.$$

Then for  $t \geq T_0$ , on  $F$  we have

$$\begin{aligned} X_{1,t}^K(k_1) X_{2,t}^K(k_2) &= a_{t-T_0} X_{1,T_0}^K(k_1) a_{t-T_0} X_{2,T_0}^K(k_2) \\ &\quad - \sum_{l \in S} \int_{T_0}^t a_{t-s}(k_1, l) a_{t-s}(k_2, l) \left( X_{1,s}^K(l) \mathcal{A}X_{2,s}^K(l) + X_{2,s}^K(l) \mathcal{A}X_{1,s}^K(l) \right) ds + N_t(k_1, k_2). \end{aligned}$$

**Proof.** This is an immediate consequence of (2.20), the fact that

$$X_{1,t}^K(k) X_{2,t}^K(k) = 0 \quad \text{for all } k \in S, \text{ if } t \leq \tau^K,$$

and of  $\tau^K = \infty$  on  $F$ . □

**Lemma 3.3** *On the event  $F$ , the following decomposition holds:*

$$(3.13) \quad Z_t = Z_t^{(1)} - Z_t^{(2)} + Z_{1,t}^{(3)} + Z_{2,t}^{(3)} \quad \text{for } t \geq T_0,$$

where

$$\begin{aligned} Z_t^{(1)} &= \sum_{l \in S} \int_{T_0}^t a_{s-T_0} X_{1,T_0}^K(l) \left( \bar{\mathcal{A}}(a_{s-T_0} X_{2,T_0}^K)(l) + A^*(l) a_{s-T_0} X_{2,T_0}^K(l) \right) ds, \\ Z_t^{(2)} &= \sum_{l \in S} \sum_{k \in S} \int_{T_0}^t \int_{T_0}^s a_{s-r}(l, k) [(\bar{\mathcal{A}} a_{s-r}(\cdot, k))(l) + A^*(l) a_{s-r}(l, k)] \\ &\quad (X_{1,r}^K(k) \mathbf{A}X_{2,r}^K(k) + \mathbf{A}X_{1,r}^K(k) X_{2,r}^K(k)) dr ds, \\ Z_{i,t}^{(3)} &= \sum_{l \neq l'} \bar{\mathbf{A}}(l, l') \sum_{k \in S} \int_{T_0}^t \int_{T_0}^s a_{s-r}(l, k) \left( a_{s-r} X_{3-i,r}^K(l') - a_{s-r}(l', k) X_{3-i,r}^K(k) \right) dM_{i,r}^K(k) ds. \end{aligned} \tag{3.14}$$

**Proof.** By definition,

$$\begin{aligned}
(3.15) \quad Z_t &= \frac{1}{2} \sum_{i=1}^2 \sum_{k \in S} \int_{T_0}^t X_{i,s}^K(k) \mathcal{A} X_{3-i,s}^K(k) ds \\
&= \frac{1}{2} \sum_{i=1}^2 \sum_{k \in S} \int_{T_0}^t X_{i,s}^K(k) \sum_{l \in S} (\mathbf{A}(k, l) - \mathbb{1}_{\{k=l\}}) X_{3-i,s}^K(l) ds \\
&= \sum_{\substack{k_1, k_2 \in S \\ k_1 \neq k_2}} \int_{T_0}^t \bar{\mathbf{A}}(k_1, k_2) X_{1,s}^K(k_1) X_{2,s}^K(k_2) ds,
\end{aligned}$$

where the last inequality follows since  $X_{1,t}^K(k) X_{2,t}^K(k) = 0$  for all  $k \in S$ ,  $t \geq 0$  on  $F$ . Now, by Lemma 3.2, we have

$$\begin{aligned}
(3.16) \quad Z_t &= \sum_{\substack{k_1, k_2 \in S \\ k_1 \neq k_2}} \int_{T_0}^t \bar{\mathbf{A}}(k_1, k_2) a_{s-T_0} X_{1,T_0}^K(k_1) a_{s-T_0} X_{2,T_0}^K(k_2) ds \\
&\quad - \sum_{\substack{k_1, k_2 \in S \\ k_1 \neq k_2}} \int_{T_0}^t \bar{\mathbf{A}}(k_1, k_2) \sum_{l \in S} \int_{T_0}^s a_{s-r}(k_1, l) a_{s-r}(k_2, l) \\
&\quad \quad \quad \times \left( X_{1,r}^K(l) \mathcal{A} X_{2,r}^K(l) + X_{2,r}^K(l) \mathcal{A} X_{1,r}^K(l) \right) dr ds \\
&\quad + \sum_{\substack{k_1, k_2 \in S \\ k_1 \neq k_2}} \int_{T_0}^t \bar{\mathbf{A}}(k_1, k_2) N_s(k_1, k_2) ds.
\end{aligned}$$

Let us consider the first term on the right hand side of the above equation. We easily get that

$$\begin{aligned}
(3.17) \quad &\sum_{\substack{k_1, k_2 \in S \\ k_1 \neq k_2}} \int_{T_0}^t \bar{\mathbf{A}}(k_1, k_2) a_{s-T_0} X_{1,T_0}^K(k_1) a_{s-T_0} X_{2,T_0}^K(k_2) ds \\
&= \sum_{k_1 \in S} \int_{T_0}^t a_{s-T_0} X_{1,T_0}^K(k_1) \left( \bar{\mathbf{A}} a_{s-T_0} X_{2,T_0}^K(k_1) + A^*(k_1) a_{s-T_0} X_{2,T_0}^K(k_1) \right) ds \\
&= Z_t^{(1)},
\end{aligned}$$

and we are done with the first term in the decomposition. Similarly, for the second term, we have

$$\begin{aligned}
&\sum_{\substack{k_1, k_2 \in S \\ k_1 \neq k_2}} \int_{T_0}^t \bar{\mathbf{A}}(k_1, k_2) \sum_{l \in S} \int_{T_0}^s a_{s-r}(k_1, l) a_{s-r}(k_2, l) \\
&\quad \quad \quad \times \left( X_{1,r}^K(l) \mathcal{A} X_{2,r}^K(l) + X_{2,r}^K(l) \mathcal{A} X_{1,r}^K(l) \right) dr ds \\
&= \sum_{l \in S} \sum_{k_1 \in S} \int_{T_0}^t \int_{T_0}^s a_{s-r}(k_1, l) \left[ \bar{\mathbf{A}} a_{s-r}(\cdot, l)(k_1) + A^*(k_1) a_{s-r}(k_1, l) \right] \\
&\quad \quad \quad \times \left( X_{1,r}^K(l) \mathcal{A} X_{2,r}^K(l) + X_{2,r}^K(l) \mathcal{A} X_{1,r}^K(l) \right) dr ds \\
&= Z_t^{(2)}.
\end{aligned}$$

For the third term, we have:

$$\begin{aligned}
 & \sum_{\substack{k_1, k_2 \in S \\ k_1 \neq k_2}} \int_{T_0}^t \bar{\mathbf{A}}(k_1, k_2) N_s(k_1, k_2) ds \\
 &= \sum_{\substack{k_1, k_2 \in S \\ k_1 \neq k_2}} \int_{T_0}^t \bar{\mathbf{A}}(k_1, k_2) \sum_{\substack{l, l' \in S \\ l \neq l'}} \int_{T_0}^s a_{s-r}(k_1, l') a_{s-r}(k_2, l) \left( X_{1,r}^K(l') dM_{2,r}^K(l) + X_{2,r}^K(l) dM_{1,r}^K(l') \right) ds \\
 &= \sum_{\substack{k_1, k_2 \in S \\ k_1 \neq k_2}} \bar{\mathbf{A}}(k_1, k_2) \int_{T_0}^t \int_{T_0}^s \left\{ \sum_{l \in S} a_{s-r}(k_2, l) (a_{s-r} X_{1,r}^K(k_1) - a_{s-r}(k_1, l) X_{1,r}^K(l)) dM_{2,r}^K(l) \right. \\
 (3.18) \quad & \left. + \sum_{l' \in S} a_{s-r}(k_1, l') (a_{s-r} X_{2,r}^K(k_2) - a_{s-r}(k_2, l') X_{2,r}^K(l')) dM_{1,r}^K(l') \right\} ds \\
 &= \sum_{\substack{k_1, k_2 \in S \\ k_1 \neq k_2}} \bar{\mathbf{A}}(k_1, k_2) \int_{T_0}^t \int_{T_0}^s \left\{ \sum_{l \in S} a_{s-r}(k_2, l) (a_{s-r} X_{1,r}^K(k_1) - a_{s-r}(k_1, l) X_{1,r}^K(l)) dM_{2,r}^K(l) \right. \\
 & \left. + \sum_{l \in S} a_{s-r}(k_2, l) (a_{s-r} X_{2,r}^K(k_1) - a_{s-r}(k_1, l) X_{2,r}^K(l)) dM_{1,r}^K(l) \right\} ds \\
 &= Z_{1,t}^{(3)} + Z_{2,t}^{(3)},
 \end{aligned}$$

where the third equality follows by the symmetry of  $\bar{a}$ .  $\square$

**Lemma 3.4** Recall  $\bar{G}^*$  from (1.15). For all  $t \geq T_0$ , on  $F$  we have

$$Z_t^{(2)} \leq \frac{\bar{G}_{2(t-T_0)}^*}{2} Z_t.$$

**Proof.** Recall (1.13) and (1.14) and note that (with  $I$  the unit matrix)

$$(3.19) \quad \int_0^t a_r^T \bar{\mathcal{A}} a_r dr = \frac{1}{2} (\bar{a}_{2t} - I).$$

Hence (using that  $A^*(l) \leq 1$ ),

$$\begin{aligned}
 (3.20) \quad Z_t^{(2)} &\leq \sum_{k \in S} \int_{T_0}^t \int_r^t [a_{s-r}^T \bar{\mathcal{A}} a_{s-r} + \bar{a}_{2(s-r)}] (k, k) ds (X_{1,r}^K(k) \mathbf{A} X_{2,r}^K(k) + \mathbf{A} X_{1,r}^K(k) X_{2,r}^K(k)) dr \\
 &= \sum_{k \in S} \int_{T_0}^t \left( \frac{1}{2} (\bar{a}_{2(t-r)}(k, k) - 1) + \int_r^t \bar{a}_{2(s-r)}(k, k) ds \right) \\
 &\quad [X_{1,r}^K(k) \mathbf{A} X_{2,r}^K(k) + \mathbf{A} X_{1,r}^K(k) X_{2,r}^K(k)] dr.
 \end{aligned}$$

Then use the fact that  $\bar{a}_t(k, k) - 1 \leq 0$  and  $\mathbf{A} X_{i,r}^K(k) X_{3-i,r}^K(k) = \mathcal{A} X_{i,r}^K(k) X_{3-i,r}^K(k)$  on  $F$  to get that

on  $F$ ,  $Z_t^{(2)}$  is bounded above by

$$\begin{aligned}
& \frac{1}{2} \sum_{k \in S} \int_{T_0}^t \bar{G}_{2(t-r)}(k, k) \left( \mathcal{A}X_{1,r}^K(k)X_{2,r}^K(k) + X_{1,r}^K(k)\mathcal{A}X_{2,r}^K(k) \right) dr \\
& \leq \frac{1}{2} \bar{G}_{2(t-T_0)}^* \int_{T_0}^t \sum_{k \in S} \left( \mathcal{A}X_{1,r}^K(k)X_{2,r}^K(k) + X_{1,r}^K(k)\mathcal{A}X_{2,r}^K(k) \right) dr \\
(3.21) \quad & = \frac{1}{2} \bar{G}_{2(t-T_0)}^* Z_t.
\end{aligned}$$

□

Next we will handle  $Z_i^{(3)}$ .

**Lemma 3.5** *For all  $t \geq 0$  and for  $i = 1, 2$ , we have*

$$(3.22) \quad Z_{i,t}^{(3)} = \sum_{k \in S} \int_{T_0}^t h_i(k, t, r) dM_{i,r}^K(k),$$

where for  $t$  sufficiently large

$$|h_i(k, t, r)| \leq \bar{G}_{2(t-T_0)}^* M_{3-i,r}^K \quad \text{for all } r \in [T_0, t].$$

**Proof.** First, by the stochastic Fubini Theorem (see, e.g., Theorem 64 in [16]), we can change the order of integration in order to get

$$\begin{aligned}
h_i(k, t, r) &= \sum_{\substack{l, l' \in S \\ l \neq l'}} \bar{\mathbf{A}}(l, l') \int_r^t a_{s-r}(l, k) (a_{s-r} X_{3-i,r}^K(l') - a_{s-r}(l', k) X_{3-i,r}^K(k)) ds \\
&= \int_r^t a_{s-r}^T \bar{\mathcal{A}} a_{s-r} X_{3-i,r}^K(k) - a_{s-r}^T \bar{\mathcal{A}} a_{s-r}(k, k) X_{3-i,r}^K(k) ds \\
&\quad + \int_r^t \sum_{l \in S} A^*(l) a_{s-r}(l, k) [a_{s-r} X_{3-i,r}^K(l) - a_{s-r}(l, k) X_{3-i,r}^K(k)] ds \\
&=: I_1 + I_2.
\end{aligned}$$

For the first integral, using (3.19), we get

$$0 \leq \frac{1}{2} \bar{a}_{2(t-r)} X_{3-i,r}^K(k) - \frac{1}{2} \bar{a}_{2(t-r)}(k, k) X_{3-i,r}^K(k) = I_1 \leq \frac{1}{2} M_{3-i,r}^K.$$

For the second integral, we obtain similarly

$$\begin{aligned}
0 \leq I_2 &\leq \int_r^t \sum_{l \in S} a_{s-r}(l, k) a_{s-r} X_{3-i,r}^K(l) ds \\
&= \int_r^t \bar{a}_{s-r} X_{3-i,r}^K(k) ds \\
&= \frac{1}{2} \bar{G}_{2(t-r)}^* M_{3-i,r}^K \leq \frac{1}{2} \bar{G}_{2(T_0-r)}^* M_{3-i,r}^K.
\end{aligned}$$

Combining the estimates for  $I_1$  and  $I_2$ , we get

$$|h_i(k, t, r)| \leq \frac{1}{2} (\bar{G}_{2(t-T_0)}^* + 1) M_{3-i,r}^K \leq \bar{G}_{2(t-T_0)}^* M_{3-i,r}^K,$$

and we get the bound for  $h_i(k, t, r)$  for  $t$  sufficiently large.  $\square$

We have to introduce some notation and define a number of additional constants. Define the event

$$(3.23) \quad C = \left\{ \sum_{|k| \leq L} X_{i,T_0}^K(k) \geq \frac{1}{2} M_{i,T_0}^K \quad \text{for } i = 1, 2 \right\}.$$

Since  $M_{i,T_0}^K < \infty$ ,  $i = 1, 2$ , there exists an  $L > 0$  such that

$$(3.24) \quad \mathbf{P}[C] \geq 1 - \delta.$$

**Lemma 3.6** *There exists a  $T_1 \geq T_0$  such that for all  $t > T_1$ ,*

$$Z_t^{(1)} \geq c' \bar{G}_{2(t-T_0)}^* M_{1,T_0}^K M_{2,T_0}^K \quad \text{on } C \cap F.$$

**Proof.** In order to simplify the notation, let  $Z_t^{(1)} = Z_t^{(1,1)} + Z_t^{(1,2)}$ , where

$$\begin{aligned} Z_t^{(1,1)} &:= \sum_{l \in S} \int_{T_0}^t a_{s-T_0} X_{1,T_0}^K(l) \bar{A}(a_{s-T_0} X_{2,T_0}^K)(l) ds, \\ Z_t^{(1,2)} &:= \sum_{l \in S} \int_{T_0}^t a_{s-T_0} X_{1,T_0}^K(l) A^*(l) a_{s-T_0} X_{2,T_0}^K(l) ds. \end{aligned}$$

We start with showing that  $Z_t^{(1,1)} \geq 0$ . To this end, using (3.19), we compute

$$\begin{aligned} Z_t^{(1,1)} &= \sum_{l \in S} \int_{T_0}^t X_{1,T_0}^K(l) [a_{s-T_0}^T \bar{A} a_{s-T_0}] X_{2,T_0}^K(l) ds \\ &= \frac{1}{2} \sum_{l \in S} \bar{X}_{1,T_0}^K(l) (\bar{a}_{2(t-T_0)} - I) X_{2,T_0}^K(l) \\ &= \frac{1}{2} \sum_{l \in S} \bar{X}_{1,T_0}^K(l) \bar{a}_{2(t-T_0)} X_{2,T_0}^K(l) \geq 0 \quad \text{on } F, \end{aligned}$$

since  $X_{1,T_0}^K(k) X_{2,T_0}^K(k) = 0$  on  $F$  for all  $k \in S$ .

The bound for  $Z^{(1,2)}$  follows similarly to Dawson and Perkins [5, p. 1109]. Recall  $c$  from (1.20). For  $k, l \in S$  such that  $|k|, |l| \leq L$ , let  $T(k, l)$  be large enough such that

$$\frac{\bar{G}_t(k, l)}{G_t^*} \geq \frac{c}{2} \quad \text{for all } t \geq T(k, l).$$

Define

$$(3.25) \quad T_1 = T_0 + \max_{|k|, |l| \leq L} T(k, l).$$

Then for any  $t \geq T_1$ , we have that on  $C$

$$\begin{aligned} Z_t^{(1,2)} &\geq \frac{A^*}{2} \sum_{k, l \in S} \bar{G}_{2(t-T_0)}(k, l) X_{1,T_0}^K(k) X_{2,T_0}^K(l) \\ &\geq \frac{A^*}{2} \left( \sum_{|k|, |l| \leq L} X_{1,T_0}^K(k) X_{2,T_0}^K(l) \right) \min_{|k|, |l| \leq L} \bar{G}_{2(t-T_0)}(k, l) \\ &\geq \frac{A^* c}{16} M_{1,T_0}^K M_{2,T_0}^K \bar{G}_{2(t-T_0)}^*, \end{aligned}$$

where the last inequality follows by (3.23) and (3.25). Recalling (3.7) we get that on  $C$

$$Z_t^{(1,2)} \geq c' \bar{G}_{2(t-T_0)}^* M_{1,T_0}^K M_{2,T_0}^K.$$

Since  $Z_T^{(1,1)} \geq 0$  on  $F$ , this finishes the proof of Lemma 3.6.  $\square$

From Lemma 3.4 and Lemma 3.6, we get that on  $F \cap C$ ,  $Z_t$  is bounded below by

$$(3.26) \quad Z_t \geq c' \bar{G}_{2(t-T_0)}^* M_{1,T_0}^K M_{2,T_0}^K - \frac{\bar{G}_{2(t-T_0)}^*}{2} Z_t + \bar{Z}_t^{(3)},$$

where  $\bar{Z}_t^{(3)} = Z_{1,t}^{(3)} + Z_{2,t}^{(3)}$ . Let  $\alpha := 1/(1 + \bar{G}_{2(t-T_0)}^*/2)$ . From (3.26), we get that on  $F \cap C$

$$(3.27) \quad Z_t \geq \alpha c' \bar{G}_{2(t-T_0)}^* M_{1,T_0}^K M_{2,T_0}^K + \alpha \bar{Z}_t^{(3)}.$$

Then (recall from (3.12) that  $Z_t \leq \langle M_{1,\cdot}^K \rangle_t - \langle M_{1,\cdot}^K \rangle_{T_0}$  on  $F$ )

$$(3.28) \quad \begin{aligned} & \mathbf{P}[\langle M_{1,\cdot}^K \rangle_t - \langle M_{1,\cdot}^K \rangle_{T_0} \leq \varepsilon, F \cap C] \\ &= \mathbf{P}[Z_t \leq \varepsilon, \langle M_{1,\cdot}^K \rangle_t - \langle M_{1,\cdot}^K \rangle_{T_0} \leq \varepsilon, F \cap C] \\ &\leq \mathbf{P}[\alpha \bar{Z}_t^{(3)} \leq \varepsilon - \alpha c' \bar{G}_{2(t-T_0)}^* M_{1,T_0}^K M_{2,T_0}^K, \langle M_{1,\cdot}^K \rangle_t - \langle M_{1,\cdot}^K \rangle_{T_0} \leq \varepsilon, F \cap C]. \end{aligned}$$

We assume that  $t$  is sufficiently large so that  $\bar{G}_{2(t-T_0)}^* \geq 2$ , hence

$$(3.29) \quad 1 \leq \alpha \bar{G}_{2(t-T_0)}^* = \frac{2 \bar{G}_{2(t-T_0)}^*}{2 + \bar{G}_{2(t-T_0)}^*} \leq 2.$$

By (3.29), (3.4) and (3.10), we get

$$\begin{aligned} & \mathbf{P}[\alpha \bar{Z}_t^{(3)} \leq \varepsilon - \alpha c' \bar{G}_{2(t-T_0)}^* M_{1,T_0}^K M_{2,T_0}^K, \langle M_{1,\cdot}^K \rangle_t - \langle M_{1,\cdot}^K \rangle_{T_0} \leq \varepsilon, F \cap C] \\ &\leq \mathbf{P}[\alpha \bar{Z}_t^{(3)} \leq \varepsilon - \alpha c' \bar{G}_{2(t-T_0)}^* \delta^2, \langle M_{1,\cdot}^K \rangle_t - \langle M_{1,\cdot}^K \rangle_{T_0} \leq \varepsilon, F \cap C] + 1 - 5\delta \\ &\leq \mathbf{P}[\alpha \bar{Z}_t^{(3)} \leq \varepsilon - c' \delta^2, \langle M_{1,\cdot}^K \rangle_t - \langle M_{1,\cdot}^K \rangle_{T_0} \leq \varepsilon, F \cap C] + 1 - 5\delta \\ &= \mathbf{P}[\alpha \bar{Z}_t^{(3)} \leq -R\varepsilon, \langle M_{1,\cdot}^K \rangle_t - \langle M_{1,\cdot}^K \rangle_{T_0} \leq \varepsilon, F \cap C] + 1 - 5\delta \\ &\leq R^{-2} \varepsilon^{-2} \alpha^2 \mathbf{E} \left[ (\bar{Z}_t^{(3)})^2 \mathbb{1}_{\{\langle M_{1,\cdot}^K \rangle_t - \langle M_{1,\cdot}^K \rangle_{T_0} \leq \varepsilon\}} \mathbb{1}_F \right] + 1 - 5\delta. \end{aligned}$$

Using Lemma 3.5, this inequality can be continued by

$$(3.30) \quad \begin{aligned} &\leq R^{-2} \varepsilon^{-1} (K/2)^2 (\alpha \bar{G}_{2(t-T_0)}^*)^2 + 1 - 5\delta \\ &\leq \frac{K^2}{R^2 \varepsilon} + 1 - 5\delta \\ &\leq 1 - 4\delta. \end{aligned}$$

Combining (3.28), (3.30), (3.2) and (3.24), we get

$$\begin{aligned} \mathbf{P}[\langle M_{1,\cdot}^K \rangle_t - \langle M_{1,\cdot}^K \rangle_{T_0} \leq \varepsilon] &\leq \mathbf{P}[\langle M_{1,\cdot}^K \rangle_t - \langle M_{1,\cdot}^K \rangle_{T_0} \leq \varepsilon, F \cap C] + \mathbf{P}[F^c] + \mathbf{P}[C^c] \\ &\leq 1 - 4\delta + \delta + \delta = 1 - 2\delta. \end{aligned}$$

This is a contradiction to (3.5) and hence finishes the proof of Theorem 2.  $\square$

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