

VARIATIONS ON THE SHAPLEY VALUE

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1. Introduction.

The variations that are the subject of this discussion are those axiomatically characterized solutions that are obtained by varying the set of axioms which define the Shapley value, or the domain over which this value is defined, or both.

We first capture axiomatically, in Section 3, those solutions that preserve one of the essential features of the Shapley value, namely, that they are given, for each player, by some averaging of the player's marginal contributions to coalitions, where the probabilistic weights depend on the coalitions only and not on the game. We call the family of all such solutions *probabilistic values*. The Shapley value is the unique probabilistic value that is efficient and symmetric. In the following sections we characterize and discuss two families of solutions: *quasivalues*, which are efficient probabilistic values, and *semivalues*, which are symmetric probabilistic values.

In Section 4 we discuss quasivalues. These solutions are related to the descriptions of the Shapley value as a *random-order value*: it is the expected marginal contributions of the players when they are randomly ordered. Indeed a solution is a quasivalue if and only if it is a random-order value. The Shapley value is the special random-order value in which the random orders are uniformly drawn.

Quasivalues can be equivalently described by choosing at random the "arrival" times of the players, rather than their order, and taking the expected marginal contribution of a player to the players who arrived before him. In particular, the Shapley value is obtained when the arrival times are independent and each is uniformly distributed in the unit interval. When arrival times are independent, the quasivalue has another representation as a *path value*: the value can be computed by integrating the derivative of the multilinear extension of the game along a certain

path.

In Section 5 we characterize and discuss a family of quasivalues—the *weighted Shapley values*—which was introduced by Shapley alongside the Shapley value.

Semivalues are discussed in Section 6. The symmetry axiom, which plays a central role in the characterization of semivalues, is reflected in the probabilities that define the value: they depend only on the size of coalitions. A semivalue does not, of course, have to be efficient. We show that the specific deviation of a semivalue from efficiency defines it uniquely. That is, if the magnitude of the deviation from efficiency is the same for two semivalues in each game, then they coincide.

A probabilistic value defined on the set of all games is necessarily additive. When the set of games is restricted to simple games (that is, games in which the worth of a coalition is either 0 or 1), then additivity cannot be applied. Yet, all the generalizations discussed so far can be axiomatically defined for this class of games when the *transfer* axiom replaces that of additivity. This is done in Section 7.

Section 8 deals with solutions that generalize the Shapley value by changing the domain over which the solution is defined. In such generalizations the solution is defined on pairs, consisting of a game and some structure on the set of players. The Shapley value is a special case of such a generalization in the sense that it coincides with the solution on the restricted domain in which the second argument is fixed to be the “trivial” one. Under this category we survey mostly solutions in which the structure is a partition of the set of the players, and a solution in which the structure is a graph the vertices of which are the players.

2. Preliminaries. Let N be a finite set with n elements, $n \geq 1$. Elements of N are called *players*. Any subset of N is called a *coalition*. For a coalition S , we denote $N \setminus S$ by S^c . We denote by $C = C(N)$ the set of all coalitions, and by $C_0 = C_0(N)$ the set of all nonempty coalitions.

A *game* on N is a set function $v : C \rightarrow R$, where R denotes the set of real numbers, with $v(\emptyset) = 0$. We will write $v(S \cup i)$ for $v(S \cup \{i\})$, and $v(S \setminus i)$ for $v(S \setminus \{i\})$. A game v is *additive* if for every pair of disjoint coalitions $S, T \in C$, $v(S \cup T) = v(S) + v(T)$, and it is *superadditive* if for each pair of such coalitions, $v(S \cup T) \geq v(S) + v(T)$. It is *monotone* if $v(S) \leq v(T)$ for every $S \subseteq T$. A game v is *simple* if $v(S) \in \{0, 1\}$ for every $S \in C$.

Player i is a *null* player in v if $v(S \cup i) - v(S) = 0$ for every $S \subseteq N \setminus i$. A coalition M is a *carrier* of v if M^c consists of null players. Hence, M is a carrier for v if and only if $v(S \cap M) = v(S)$ for all $S \subseteq N$. Player i is a *dummy* player in v if there exists an $a \in R$ such that $v(S \cup i) - v(S) = a$ for every $S \subseteq N \setminus i$.

The set of all games is denoted $G = G(N)$ and the set of all additive games is denoted $A = A(N)$. We naturally identify an additive game $v \in A$ with the vector $x \in R^N$ defined by $x_i = v(i)$ for every $i \in N$. For each such $x \in R^N$ and for every $S \in C$ we denote $x(S) = \sum_{i \in S} x_i$.

A *permutation* of N is a one-to-one function that maps N onto N . The set of all permutations on N is denoted by $\Theta = \Theta(N)$. The set of all one-to-one functions $\tau : N \rightarrow \{1, \dots, n\}$ is denoted by $OR = OR(N)$. Each $\tau \in OR$ is identified with a complete order on N . In this order, i comes before j if $\tau(i) < \tau(j)$. For a game v and a permutation θ we define $\theta^*v \in G$ by $\theta^*v(S) = v(\theta(S))$ for all $S \in C$.

Let $H \subseteq G$. A *solution* on H is a function $\psi : H \rightarrow A$. We will use the terms

operator and solution interchangeably. The domain of a solution is G whenever it is not explicitly specified otherwise. It is useful to single out a list of possible properties of solutions (axioms) that have been discussed in the literature.

A.1 ψ satisfies the *additivity axiom* or is *additive* if $\psi(u + v) = \psi(u) + \psi(v)$ for every $u, v \in G$.

A.2 ψ is *homogeneous* if $\psi(\alpha v) = \alpha\psi(v)$ for every $v \in G$ and $\alpha \in R$.

A.3 ψ satisfies the *linearity axiom*, or is *linear*, if it is additive and homogeneous.

A.4 ψ satisfies the *projection axiom*, or is a *projection*, if $\psi(\mu) = \mu$ for every $\mu \in A$.

A.5 ψ satisfies the *null player axiom* if $\psi v(i) = 0$ for every $v \in G$ and every null player i in v .

A.6 ψ satisfies the *dummy player axiom* if $\psi v(i) = a$ for $v \in G$ and for every dummy player i such that $v(S \cup i) - v(S) = a$ for every $S \subseteq N \setminus i$.

A.7 ψ satisfies the *positivity axiom*, or ψ is *positive*, if $\psi v(i) \geq 0$ for every i whenever v is a monotone game.

A.8 ψ satisfies the *Milnor axiom*, or is a *Milnor operator*, if for every v and $i \in N$,

$$\min_{S \subseteq N \setminus i} (v(S \cup i) - v(S)) \leq \psi v(i) \leq \max_{S \subseteq N \setminus i} (v(S \cup i) - v(S)).$$

A.9 ψ satisfies the *symmetry axiom* or is *symmetric* if $\psi(\theta^* v) = \theta^* \psi v$ for every $v \in G$ and every permutation θ .

A.10 ψ satisfies the *efficiency axiom* or is *efficient* if $\psi v(N) = v(N)$ for every $v \in G$.

A.11 ψ satisfies the *carrier axiom* if $\psi v(M) = v(M)$ for every $v \in G$ and every

carrier M of v .

We state the above axioms for solutions defined on G , but will sometimes use them for solutions defined on strict subsets of G . In such a case, an axiom is to be understood as holding whenever possible. Thus, for example, if ψ is a solution on $H \subset G$, the projection axiom should be read as $\psi\mu = \mu$ whenever $\mu \in H \cap A$.

3. Probabilistic Values. For a finite set M we denote by $\Delta(M)$ the set of all probability measures on M . That is, $p \in \Delta(M)$ if $p : M \rightarrow [0, 1]$ and $\sum_{m \in M} p(m) =$

1. We denote by E_p the expectation operator with respect to p .

The set of all subsets of $N \setminus i$ is denoted by $C^i = C^i(N)$. For each game v , we denote by $D^i v$ the set function defined on C^i by $D^i v(S) = v(S \cup i) - v(S)$. We call $D^i v(S)$ the *marginal contribution* of i to S .

A solution ψ is called a *probabilistic value*, if for each player i there exists a probability $p^i \in \Delta(C^i)$, such that $\psi v(i)$ is the expected marginal contribution of i with respect to p^i . Namely,

$$(3.1) \quad \psi v(i) = E_{p^i}(D^i v) = \sum_{S \in C^i} p^i(S)(v(S \cup i) - v(S)).$$

The Shapley value, which inspired this definition, is the probabilistic value defined by the probabilities p^i defined by $p^i(S) = 1/(n \binom{n-1}{s})$, where s denotes the number of elements in S . We denote the Shapley value by $\varphi = \varphi^N$. It can be easily verified that the probability measures p^i in (3.1) are uniquely determined by ψ . Probabilistic values can be axiomatically characterized as follows.

Theorem 1. *Let ψ be a solution. Then the following three assertions are equivalent:*

- (1) ψ is a probabilistic value.

(2) ψ is a linear positive operator that satisfies the dummy axiom.

(3) ψ is a linear Milnor operator.

Weber (1988) proved the equivalence of (1) and (2). Monderer (1988, 1990) proved the equivalence of (2) and (3).

Shapley (1953a) proved that the Shapley value is the unique solution that satisfies the additivity, symmetry and carrier axioms.¹ The following is one of many other characterizations of the Shapley value. We single out this particular characterization (which is proved in Weber (1988)) because it can serve as the basis for a unified approach to analyzing the variations on the Shapley value.

Theorem 2. *The Shapley value is the unique solution which is a symmetric and efficient probabilistic value.*

The main variations on the Shapley value retain the condition of being a probabilistic value (i.e., that a player's value depends linearly and positively on his marginal contributions), while relaxing one of the other two conditions in Theorem 2. We call an efficient probabilistic value a *quasivalue*, and a symmetric probabilistic value a *semivalue*. Quasivalues are discussed in Sections 4 and 5, semivalues in Sections 6 and 7.

Remarks.

- (1) As probabilistic values are characterized by axioms that do not mix values of different players, they can be defined and characterized for each player separately. Such an approach was taken by Weber (1988).

¹He proved this result for solutions defined on the set of superadditive games, with finite support, on a universal set of players; but his proof, as is well known, can also be used to characterize the Shapley value on the space $G(N)$ of all games on N .

(2) Weber (1988) proved that Theorem 1 holds for solutions defined on the class of monotone games. One can deduce from Monderer (1988) and from a remark in Weber that the equivalence of (1) and (3) in Theorem 1 continues to hold for solutions defined on the class of superadditive games.

4. Efficient Probabilistic Values (Quasivalues). We begin with an important characterization of quasivalues. For each $\tau \in OR$ we define the operator φ^τ by

$$\varphi^\tau v(i) = v(\{j \in N : \tau(j) \leq \tau(i)\}) - v(\{j \in N : \tau(j) < \tau(i)\}).$$

If we think of τ as the order in which players enter the “room” where the game takes place, then $\varphi^\tau(i)$ is the marginal contribution of i to the coalition of players that preceded him. For every $b \in \Delta(OR)$ we define a solution φ^b by

$$\varphi^b v(i) = \sum_{\tau \in OR} \varphi^\tau v(i) b(\tau).$$

Thus, $\varphi^b v(i)$ is the expected marginal contribution of i , where the order of entering the room is selected according to b .

A solution ψ is called a *random-order value* if there exists $b \in \Delta(OR)$ such that $\psi = \varphi^b$.

It is easy to verify that every random-order value is a quasivalue. Indeed, given $b \in \Delta(OR)$ we can define for each $i \in N$ and $S \in C^i$,

$$(4.1) \quad p^i(S) = b(\{\tau \in OR : \tau(j) < \tau(i) \text{ iff } j \in S\}).$$

If we insert the p^i 's from (4.1) into (3.1), the resulting probabilistic value ψ coincides with φ^b . The following theorem of Weber (1988) shows that the converse is also true.

Theorem 3. *A solution is a quasivalue if and only if it is a random-order value.*

Note that the Shapley value is a random-order value with $b(\tau) = 1/n!$ for every $\tau \in OR$.²

It can be easily verified that for every $b \in \Delta(OR)$ there exist n random variables $(z_i)_{i \in N}$ that are jointly distributed in the cube $[0, 1]^N$ with a density function and therefore with absolutely continuous marginal distribution functions $(\alpha_i)_{i \in N}$, such that for every $\tau \in OR$

$$(4.2) \quad b(\tau) = \text{prob}(z_{\tau^{-1}(i)} < z_{\tau^{-1}(j)} \quad \text{for every } 1 \leq i < j \leq n).$$

This enables us to interpret φ^b in terms of random arrival times, as follows. We think of the value of the random variable z_i as the time of arrival of player i in the “room”. The marginal contribution of i , when he arrives at time t , is his marginal contribution to the set of other players who arrived up to time t . Define

$$(4.3) \quad N(t) = \{j \in N : z_j \leq t\}.$$

Then $N(t)$ is a random coalition in C and it can be easily verified that

$$(4.4) \quad \varphi^b v(i) = \int_0^1 E(v(N(t) \cup i) - v(N(t) \setminus i) | z_i = t) d\alpha_i(t),$$

where the integral in (4.4) is the Riemann-Stieltjes integral. Thus, we have shown that a solution ψ is a random order value, if and only if there exist n random variables $(z_i)_{i \in N}$, which are jointly distributed in the cube $[0, 1]^N$, and have a

²Observe that for sufficiently large n the affine map $b \rightarrow \varphi^b$ is not one-to-one. Indeed, the affine dimension of the convex set $\Delta(OR)$ is $n! - 1$, while the affine dimension of the set of quasivalues is less than $n(2^{n-1} - 1)$, because each quasivalue ψ is uniquely determined by a vector (p^1, p^2, \dots, p^n) in $\times_{i \in N} \Delta(C^i)$ which, in addition, satisfies the necessary linear conditions for the efficiency of ψ . It is still to be checked whether the uniform distribution over OR is the only one that gives rise to the Shapley value.

density function, such that $\psi v(i)$ is given by the right-hand side of (4.4), where $(\alpha_i)_{i \in N}$ are the marginal distribution functions.

Owen (1972) defined the multilinear extension of a game v . Each coalition S can be identified with an extreme point, e_S , of the cube $[0, 1]^N$, where e_S is the characteristic function of S . Therefore, each game v can be considered as a function on the extreme points of the cube and hence it can be uniquely extended to a multilinear function on the cube, which is denoted by \bar{v} . Owen showed that if the random variables that define b via (4.2) are independent, then for every i :

$$(4.5) \quad \varphi^b v(i) = \int_0^1 \frac{\partial \bar{v}}{\partial x_i}(\alpha(t)) d\alpha_i(t).$$

In particular, the Shapley value is determined by n i.i.d. random variables that are uniformly distributed in $[0, 1]$. Therefore,

$$\varphi v(i) = \int_0^1 \frac{\partial \bar{v}}{\partial x_i}(te) dt,$$

where $e = e_N$.

The presentation of the solution φ^b , in (4.5), gives rise to another natural solution. A solution ψ is a *path value* if there exists an absolutely continuous nondecreasing path $\alpha : [0, 1] \rightarrow [0, 1]^N$ with $\alpha(0) = 0$ and $\alpha(1) = e$ such that for every game v and every player i , $\psi v(i)$ is defined by the right-hand side of (4.5). Alternatively, ψ is a path value if there exists $b \in \Delta(OR)$ such that $\psi = \varphi^b$, and there exist n independent random variables that are distributed in $[0, 1]$ with absolutely continuous distribution functions such that (4.2) holds. It can be verified that not every quasivalue is a path value.

Remarks.

- (1) Gilboa and Monderer (1991) discussed quasivalues defined on subspaces of games. They gave necessary and sufficient conditions that ensure that such quasivalues can be extended to quasivalues on G .
- (2) Monderer (1992) used the concept of quasivalue in order to provide a game-theoretic approach to the stochastic choice problem. He used Theorem 3 to confirm a conjecture of Block and Marschak (1960), which had already been proved before by Falmagne (1978). The relationship between the stochastic choice problem and quasivalues was further analyzed by Gilboa and Monderer (1992).

5. Weighted Values. Weighted values comprise a special class of quasivalues, and more specifically of path values. We describe two forms of such values. Both were introduced by Shapley (1953a,b), alongside the standard Shapley value, and were axiomatized by Kalai and Samet (1987).

Let $\omega \in R_{++}^N$ (that is, $\omega_i > 0$ for every $i \in N$). Recall that a unanimity game u_S , for $S \in C_0$, is defined by $u_S(T) = 1$ for $T \supseteq S$ and $u_S(T) = 0$ otherwise, and that the set of unanimity games, for all $S \in C_0$ is a basis for the linear space G of games. The *positively weighted value* φ^ω (which is called the ω -value) is the unique linear operator satisfying

$$\varphi^\omega u_S(i) = \begin{cases} \frac{\omega_i}{\omega(S)}, & \text{for } i \in S \\ 0, & \text{for } i \notin S \end{cases}$$

for each unanimity game u_S . If $\omega_i = \frac{1}{n}$ for every $i \in N$, then φ^ω is the Shapley value. Owen (1972) has shown that positively weighted values are path values, the path α being defined by

$$(5.1) \quad \alpha_i(t) = t^{\omega_i} \quad \text{for every } t \in [0, 1].$$

Note that the mapping $\omega \rightarrow \varphi^\omega$ is homogeneous of degree zero, that is, $\varphi^{\lambda\omega} = \varphi^\omega$ for every $\lambda > 0$. Therefore, we can restrict our attention to vectors ω in the relative interior of $\Delta(N)$, which we call *weight vectors*. We turn now to an axiomatic characterization of positively weighted values. A solution ψ is *strictly positive* if for every monotone game v without null players, $\psi v(i) > 0$ for every $i \in N$. It is easily verified that a probabilistic value ψ is strictly positive, if and only if $p^i(S) > 0$ for every i and every $S \in C^i$, where $\{p^i : i \in N\}$ is the collection of probability measures that define ψ via (3.1). A coalition S is said to be a *coalition of partners* or a *p-type coalition*, in the game v , if, for every $T \subset S$ and each $R \subseteq S^c$, $v(R \cup T) = v(R)$. A solution ψ satisfies the *partnership axiom* if, whenever S is a p-type coalition,

$$\psi v(i) = \psi(\psi v(S)u_S)(i) \quad \text{for every } i \in S.$$

The proof of the following theorem is given in Kalai and Samet (1987):

Theorem 4. *A solution is a positively weighted value, if and only if it is a strictly positive probabilistic value that satisfies the partnership axiom.*

The set of positively weighted values is not closed in the space of solutions. Obviously every solution which is the limit of positively weighted values is a probabilistic value that satisfies the partnership axiom. We turn now to characterize such solutions by introducing the concept of weight systems. A *weight system* is a vector $w = (w^S)_{S \in C_0}$ in $\times_{S \in C_0} \Delta(S)$ that satisfies

$$(5.2) \quad w_i^S = \frac{w_i^T}{w^T(S)}$$

whenever $i \in S \subseteq T$ and $w^T(S) > 0$. Note that any weight vector ω can be identified with the weight system w , where $w_i^S = \frac{\omega_i}{\omega(S)}$ for every $i \in S \in C_0$. The

set of all weight systems is denoted by \mathcal{W} . For $w \in \mathcal{W}$ the *weighted value* φ^w (which is called the w -value) is defined as the unique linear operator which is defined for every unanimity game u_S by

$$(5.3) \quad \varphi^w u_S(i) = \begin{cases} w_i^S, & \text{for } i \in S \\ 0, & \text{for } i \notin S. \end{cases}$$

The w -value of a unanimity game in (5.3) can be interpreted as follows. Every weight system w defines an ordered partition (N_1, \dots, N_k) of N (where k depends on w). The players in N_1 are those players i for whom $w_i^N > 0$. The coalition N_h , for $1 < h \leq k$, consists of all the players i in $(\cup_{l=1}^{h-1} N_l)^c$, for which $w_i^{(\cup_{l=1}^{h-1} N_l)^c} > 0$.

For $S \in C_0$, let $h = h(S)$ be the first index for which $S \cap N_h \neq \emptyset$. Then φ^w allocates among the players in $S \cap N_h$ their share, $u_S(N) = 1$, proportional to their weights in w^{N_h} while $\varphi^w u_S(i) = 0$ for all other players.

To show that every weighted value is a path value, Kalai and Samet (1987) defined independent random variables z_i , $i \in N$ with pairwise disjoint supports, such that if $i_h \in N_h$, for $h = 1, \dots, k$, then $\text{prob}(z_{i_1} > z_{i_2} > \dots > z_{i_k}) = 1$. The distribution of each z_i is defined on its support in $[0, 1]$ by a linear transformation of the distribution given in (5.1) with $\omega_i = w_i^{(\cap_{l=1}^{h-1} N_l)^c}$ when $i \in N_h$. Kalai and Samet (1987) proved:

Theorem 5. *A solution is a weighted value if and only if it is a quasivalue that satisfies the partnership axiom.*

Hart and Mas-Colell (1989) presented two new approaches to value theory: the potential approach and the consistency approach. They showed that these approaches are naturally extended to the case of positively weighted values.

Here we present their theory using a slightly different notation. For a game v and a coalition S we denote the restriction of v to S by v^S . That is, $v^S \in G(S)$ and $v^S(T) = v(T)$ for every $T \in C(S)$. Let $AG = \cup_{S \subseteq N} G(S)$. A function $P : AG \rightarrow R$ is a ω -potential function, for a weight vector $\omega \in \Delta(N)$ (that is, $\omega_i > 0$ for every $i \in N$), if for every $v \in G$

$$(5.4) \quad P(v^\emptyset) = 0, \quad \text{and} \quad \sum_{i \in S} (P(v^S) - P(v^{S \setminus i})) = v(S), \quad \text{for every } S \neq \emptyset.$$

Note that if (5.4) holds, then it also holds for every $v \in G(T)$ for $S \subseteq T \subset N$.

Hart and Mas-Colell proved that there exists a unique ω -potential function, denoted by P_ω , and called the ω -potential function of G . It satisfies

$$P_\omega(v^S) = \int_0^1 \frac{\bar{v}(\alpha(t)e_S)}{t} dt,$$

where $\alpha(t)(i) = t^{\omega_i}$ for every $i \in N$, and where for $x, y \in R^N$, xy is the vector defined by $(xy)_i = x_i y_i$ for every $i \in N$. The e -potential function, P_e , is called the potential function (or simply the potential) of G . Note that $P_\omega v = E(\frac{1}{t} \int_0^1 v(N(t)) dt)$, where $N(t)$ is given by (4.3) with the random variables $(z_i)_{i \in N}$ defined by (5.1).

Hart and Mas-Colell also proved:

Theorem 6. *Let $\omega \in \Delta(N)$ be a weight vector. Then for every game v and every $i \in N$,*

$$\varphi^\omega v(i) = \omega_i [P_\omega(v^N) - P_\omega(v^{N \setminus i})].$$

We now present the consistency property of weighted values. A function $\psi : AG \rightarrow \cup_{S \in C} A(S)$ is called an *extended solution* if $\psi v^S \in A(S)$ for every $S \in C$. Thus, an extended solution is a collection of solutions ψ^S on $G(S)$ such that $\psi^S v^S =$

ψv^S for every coalition S . Let ψ be an extended solution and let $T \subseteq S$. We define the reduced game $v_T^{(S,\psi)} \in G(T)$ as follows:

$$v_T^{(S,\psi)}(F) = v(F \cup (S \setminus T)) - \sum_{i \in S \setminus T} \psi v^{F \cup (S \setminus T)}(i), \quad \text{for all } F \subseteq T.$$

An extended solution ψ is *consistent* if for every game $v \in G$, coalition S , and coalition $T \subseteq S$,

$$\psi v_T^{(S,\psi)}(j) = \psi v^S(j) \quad \text{for every } j \in T.$$

Hart and Mas-Colell prove that for a fixed weight vector ω , the ω -weighted Shapley value, regarded in a natural way as an extended solution (that is, $(\varphi^\omega)^S = \varphi^{\omega^S}$, where ω^S is the restriction of ω to S), is consistent. Moreover, they proved that if an extended solution ψ is consistent and coincides with the ω -weighted extended value on two-person games, then it is the ω -weighted extended value on AG . They further used the consistency axiom to give an axiomatic characterization for positively weighted values.

Theorem 7. *An extended solution is a ω -value, for some weight vector ω , if and only if it is consistent, and for two-person games it is efficient, TU-invariant³, and monotonic.*

We end this discussion with a result concerning the weighted values of a game as a set-valued solution. Whatever the meaning of the weight vector w , it reflects certain properties or information about the players, not derived from the game, but rather given exogenously. If we lack this information, we can consider the less informative set-valued solution $\mathcal{W}(v) = \{\varphi^w v : w \in \mathcal{W}\}$. Let $C(v)$ denote the

³That is, $\psi(av^{\{i,j\}} + \mu^{\{i,j\}}) = a\psi v^{\{i,j\}} + \mu^{\{i,j\}}$, for every $i \neq j$, $a > 0$ and $\mu \in A$.

core of v and let $AC(v)$ be the anti-core of v , which consists of all $x \in A$ such that $x(S) \leq v(S)$ for every S and $x(N) = v(N)$. Monderer, Samet, and Shapley (1992) proved:

Theorem 8. *For every game v ,*

$$C(v) \cup AC(v) \subseteq \mathcal{W}(v).$$

That there is such a general relationship between the core and values is somewhat surprising in light of the difference in concept behind these solutions. Theorem 8 generalizes a theorem of Weber (1988), who proved that $C(v) \subseteq Q(v)$, where $Q(v)$ is the Weber set, i.e., the set of all $\psi(v)$ where ψ ranges over all quasivalues.

Remarks.

- (1) Monderer, Samet and Shapley (1992) discussed geometrical and topological properties of the set of weight systems \mathcal{W} (which is homeomorphic to the set of weighted values, because the map $w \rightarrow \varphi^w$ is one-to-one). \mathcal{W} can easily be seen to be homeomorphic to the set of conditional systems that was discussed in the literature on non-cooperative game theory (e.g., see Myerson (1986) and McLennan (1989a,b)). The main theorem of McLennan (1989b) states that this set is homeomorphic to a ball. Monderer, Samet and Shapley (1992) proved that for a strictly convex game v , \mathcal{W} is mapped homeomorphically onto the core of v by the mapping $w \rightarrow \varphi^w v$. This gives a new proof for McLennan's theorem. Moreover, the well-studied geometry of the core of strictly convex games (see Shapley (1971)) is reflected in the structure of \mathcal{W} .

(2) Shapley (1981) proposed a family of weighted cost allocation schemes which, when translated into the language of cooperative game theory, yield the dual-weighted values. The dual game, v^* , of a game v is defined by: $v^*(S) = v(N) - v(S^c)$. For $w \in \mathcal{W}$, the dual w -weighted value is defined by: $\varphi_*^w v = \varphi^w v^*$. Shapley gave an axiomatization for solutions $\psi : G \times W \rightarrow A$ of the form $(v, \omega) \rightarrow \varphi_*^\omega v$. Kalai and Samet (1987) axiomatized the family of dual-weighted values, explored the relationship between weighted and dual-weighted values, and used this relationship in order to give an axiomatization of the Shapley value that does not use the symmetry axiom. Dual-weighted values are also discussed in Monderer, Samet and Shapley (1992), where it is shown that a theorem analogous to Theorem 8 can be stated for these solutions.

6. Symmetric Probabilistic Values (Semivalues). As semivalues are not efficient (except for the Shapley value), they cannot be considered as mechanisms for benefit or cost sharing. However, they can be interpreted as indices of power. The structure of power is usually described by a simple game, and therefore it is of interest to study semivalues as solutions defined on such games. But before doing so, we first analyze semivalues on all of G .

Recall that the probabilities $(p^i)_{i \in N}$, where $p^i \in \Delta(C^i)$, by which a probabilistic value ψ can be expressed, as in (3.1), are uniquely determined by ψ . If in addition ψ is symmetric, i.e., it is a semivalue, then it can easily be seen that $p^i(S)$ depends only on the number of elements of S . That is, there exists a function $\beta : \{0, 1, \dots, n -$

$1\} \rightarrow [0, 1]$ such that for every i ,

$$(6.1) \quad p^i(S) = \beta(s),$$

where s denotes the number of elements in S . Obviously,

$$(6.2) \quad \sum_{s=0}^{n-1} \beta(s) \binom{n-1}{s} = 1.$$

Thus, we get the following characterization of semivalues given in Dubey, Neyman and Weber (1981):

Theorem 9. *A solution ψ is a semivalue if and only if there exists a function*

$$\beta : \{0, 1, \dots, n-1\} \rightarrow [0, 1],$$

that satisfies (6.2) such that $\psi = \varphi^\beta$, where for every game v and player i ,

$$(6.3) \quad \varphi^\beta v(i) = \sum_{S \in C^i} \beta(s) (v(S \cup i) - v(S)), \quad i \in N.$$

Moreover, the correspondence $\beta \rightarrow \varphi^\beta$ is one-to-one.

Note that the Shapley value is φ^β for $\beta(s) = \frac{1}{n \binom{n-1}{s}}$ for every $0 \leq s \leq n-1$.

The Banzhaf value⁴ is the semivalue defined by $\beta(s) = \frac{1}{2^{n-1}}$.

By removing the efficiency axiom from the list of axioms defining the Shapley value, we obtain the family of semivalues. Thus, for a semivalue ψ , $\psi v(N)$ is not necessarily $v(N)$. It is interesting, then, to note that it is precisely what ψ gives to the grand coalition N , in the various games, which determines ψ . This is what we state and prove next.

⁴See Section 7.

Theorem 10. *Let ψ_1 and ψ_2 be two semivalues⁵ that satisfy for each game v ,*

$$(6.4) \quad \psi_1 v(N) = \psi_2 v(N).$$

Then $\psi_1 = \psi_2$.

Proof. Let \mathcal{S} be the linear space of operators spanned by the set of semivalues. By Theorem 9, \mathcal{S} is n -dimensional. For every game v we define a linear functional $f_v : \mathcal{S} \rightarrow R$, by $f_v(\psi) = \psi v(N)$. Fix an order τ on N , and for $k = 1, \dots, n$, let v_k be the unanimity game on $\{\tau^{-1}(1), \dots, \tau^{-1}(k)\}$. It is easily verified that f_{v_1}, \dots, f_{v_n} are linearly independent. Therefore, if $\psi \in \mathcal{S}$ and $f_{v_k}(\psi) = 0$ for $k = 1, \dots, n$, then $\psi = 0$. Thus, if ψ_1 and ψ_2 satisfy (6.4), then $\psi_1 - \psi_2 = 0$. \square

Dubey, Neyman and Weber (1981) characterized semivalues defined on games with finite support on an infinite universe of players. Let U be an infinite set of players. Denote by $G_f = G_f(U)$ the space of all games on U with a finite carrier, and by $A_f = A_f(U)$ the space of all additive games in G_f . A solution, in this context, is a function $\psi : G_f \rightarrow A_f$. The symmetry of a solution is defined as before, with slight and obvious changes. Probabilistic solutions are defined as follows. For every finite subset of players M and for every $v \in G(M)$, denote by v_M the natural extension of v to $C(U)$, that is, $v_M(T) = v(T \cap M)$ for $T \subseteq U$. Every solution ψ on G_f defines a solution ψ^M on $G(M)$ by $\psi^M v(i) = \psi v_M(i)$ for every $v \in G(M)$ and every $i \in M$. We say that ψ is a probabilistic value on $G(U)$ if for every finite subset of players, M , ψ^M is a probabilistic value. Defining a semivalue

⁵This theorem can be proved for more general solutions, namely, for non-positive semivalues which are linear, symmetric solutions that satisfy the dummy axiom. Roth (1977) proved a special case of this stronger version of Theorem 10. He showed that the Banzhaf value is the only non-positive semivalue ψ that satisfies $\psi v(N) = \varphi^{bz} v(N)$ for every game v , where φ^{bz} is the Banzhaf value.

as a probabilistic and symmetric solution, note that ψ is a semivalue if and only if ψ^M is a semivalue for every finite coalition M .

Theorem 11. *Let U be an infinite set of players and let ψ be a solution on G_f . Then ψ is a semivalue if and only if there exists a Borel probability measure ξ on $[0, 1]$ such that for every finite coalition N with n elements, and for every $v \in G_f$ with carrier N , $\psi v = \psi_\xi v$, where,*

$$\psi_\xi v(i) = \sum_{S \subseteq N \setminus i} \beta^n(s) (v(S \cup i) - v(S)), \quad i \in N,$$

with

$$(6.5) \quad \beta^n(s) = \int_0^1 t^s (1-t)^{n-s-1} d\xi(t) \quad \text{for } 0 \leq s \leq n-1.$$

Moreover, the correspondence $\xi \rightarrow \psi_\xi$ is one-to-one.

Note that (6.5) can be rewritten in terms of random independent arrival times like (4.4),

$$\psi v(i) = \int_0^1 E(v(N(t) \cup i) - v(N(t) \setminus i) | z_i = t) d\xi(t),$$

where $N(t)$ is the random set defined in (4.5) with respect to n *i.i.d.* random variables z_1, \dots, z_n , uniformly distributed on $[0, 1]$. Observe, however, that unlike (4.4) the integration here is with respect to ξ and not z_i .

Remarks.

- (1) Dubey, Neyman and Weber (1981) discussed continuity properties of semi-values on G_f . For every $v \in G_f$ the variation norm of v is : $\|v\| = \inf\{u(N) + w(N)\}$, where the supremum is over all monotone games $u, w \in G_f$ such that $u - w = v$. With this norm both G_f and A_f are normed

spaces. The continuity of semivalues is defined with respect to this norm.

It is proved that a semivalue ψ_ξ is continuous if and only if ξ is absolutely continuous with respect to the Lebesgue measure m on $[0, 1]$, and its Radon-Nikodym derivative with respect to m is bounded, when considered as a member of $L_\infty(0, 1)$, that is, $\|\frac{d\xi}{dm}\|_\infty < \infty$. Moreover, the mapping $g \rightarrow \psi_{\xi(g)}$, where $\xi(g)(A) = \int_A g dm$, is an isometry in the natural sense.

- (2) Monderer (1988) discussed semivalues defined on symmetric subspaces of G . He proved that every such semivalue can be extended (not necessarily in a unique way) to a semivalue on G . Note that if ψ is a semivalue and G_ψ is the subspace of games v for which $\psi v(N) = v(N)$, then the restriction of ψ to any symmetric subspace of G_ψ is a Shapley value on this subspace. Monderer's theorem proves that every Shapley value on a symmetric subspace of games is obtained in this manner.

7. Indices of Power. We denote the class of simple games by SG . A coalition S is a *winning coalition* in a simple game v if $v(S) = 1$, and it is a *losing coalition* if $v(S) = 0$. Of special interest are monotone simple games. In such games, a superset of a winning coalition is a winning coalition, and a subset of a losing coalition is a losing one.⁶ This property reflects the natural idea, especially for voting games, that by adding members to a coalition it becomes stronger. The set of monotone simple games is denoted by MSG and $MSG \setminus \{0\}$ is denoted by MSG^+ . The set of superadditive simple games is denoted by SSG , and SSG^+ is $SSG \setminus \{0\}$.

A solution defined on a subset of simple games is called an *index of power*.

⁶Many authors refer to a game $v \in SG$ as a 0–1 game, and reserve the notion “simple game” for what we called a monotone simple game, and some of those authors even exclude the game $v = 0$ from the set of simple games.

We refer the reader to Dubey and Shapley (1979), and Owen (1982) for a partial reference to articles about the use of indices of power in various areas, such as political science, law and electrical engineering.

Other than the linearity axiom, all axioms in Section 2 are applicable (and make sense with respect) to indices of power. We now present the *transfer* axiom, introduced by Dubey (1975),⁷ which in some sense substitutes for the linearity axiom for indices of power. For any two games v and w we define the games $(v \vee w)$ and $(v \wedge w)$ by: $(v \vee w)(S) = \max(v(S), w(S))$ and $(v \wedge w)(S) = \min(v(S), w(S))$. Note that both games are simple whenever v and w are simple games.

A.12 A solution ψ defined on a set M of simple games satisfies the *transfer* axiom if for every $v, w \in M$ for which $v \vee w$ and $v \wedge w$ belong to M ,

$$\psi(v) + \psi(w) = \psi(v \vee w) + \psi(v \wedge w).$$

The next theorem, which is proved in Weber (1988), characterizes probabilistic values, and hence quasivalues and semivalues on simple games.

Theorem 12. *An index of power defined on the set of simple games, or on the set of monotone simple games, is a probabilistic value if and only if it satisfies the transfer, dummy, and positivity axioms. Consequently, an index of power is a quasivalue on these sets of games if and only if it satisfies the transfer, dummy, positivity and efficiency axioms, and it is a semivalue if and only if it satisfies the transfer, dummy, positivity and symmetry axioms.*

Note that every probabilistic value ψ that is defined on a set of simple games, which contains the unanimity games, can be uniquely extended to a probabilistic

⁷The name was suggested by Weber (1988).

value on G . Moreover, if ψ is a quasivalue on such a set of simple games, then its extension is a quasivalue on G and if ψ is a semivalue, then its extension is a semivalue. This follows from the fact that the symmetry and efficiency axioms are “linear” axioms, in the sense that if they hold for a symmetric linear base of games then they hold for every game. The extension property of power indices implies, in particular, that every quasivalue on simple games is a random-order value (see Theorem 3).

Representations of indices of power as random-order values, or probabilistic values, have a special flavor. Let $v \neq 0$ be a simple monotone game, and let $\tau \in OR$. The *pivot* for τ in v is the unique player $j = j_\tau$ for whom the set R_j , of players preceding j in τ , is a losing coalition, while $R_j \cup j$ is winning. Thus, for a probability measure b on the set of orders $\Delta(OR)$, $\varphi^b v(i)$ is the b -probability that i is the pivot. In particular, the Shapley value (which is also called the Shapley-Shubik index of power - see Shapley and Shubik (1954)) of a player i in v is his probability of being the pivot when all orders have equal probability. We say that i is a *swing* for a coalition S in v , if $v(S) = 0$ and $v(S \cup i) = 1$. Let ψ be a probabilistic value defined by the probabilities p^1, \dots, p^n on C^1, \dots, C^n , respectively. Then $\psi v(i)$ is the p^i -probability that i is a swing for a random coalition of the other voters.⁸

The Banzhaf index of power (Banzhaf (1965)) has been extensively discussed in the literature. Dubey and Shapley (1979) characterized the Banzhaf index on MSG^+ in the same way that Roth (1977) characterized the Banzhaf value on G (see Footnote 5). Lehrer (1988) gave another characterization, based on the behavior

⁸See Straffin (1988) for other probabilistic interpretations of the Shapley-Shubik and the Banzhaf indices of power.

of an extended index of power when applied to games in which coalitions of the original game are amalgamated into one player, as follows. For every $\emptyset \subset T \subset N$ choose a player i_T in T . Let v be a simple game and let T be a nonempty coalition. Define two games $v_{[T]}$ and $v_{[T]}^m$ on the set of players $T^c \cup \{i_T\}$ as follows : $v_{[T]}(S) = v_{[T]}^m(S) = v(S)$ for every $S \subseteq T^c$, and for every such S ,

$$v_{[T]}(S \cup i_T) = v(S \cup T)$$

and

$$v_{[T]}^m(S \cup i_T) = \max_{\emptyset \subset B \subseteq T} v(S \cup B).$$

Theorem 13. *Let ψ be an extended semivalue on the sets of simple games on subsets of N ,⁹ and let $(i_T)_{\emptyset \subset T \subset N}$ be a collection of players such that $i_T \in T$. Then ψ is the Banzhaf index of power if and only if for every two-player coalition $T = \{i, j\}$,*

$$\psi v(i) + \psi v(j) \leq \psi v_{[T]}(i_T).$$

Moreover, ψ is the Banzhaf index if and only if for each such T ,

$$\psi v(i) + \psi v(j) \leq \psi v_{[T]}^m(i_T).$$

Remarks.

- (1) Einy (1987) characterized semivalues on $MSG(U)$, when U is an infinite universe set of players. Actually he proved the “semivalues” part of Theorem 12 in such a setup. In his papers (1987,1988), he also provided a formula

⁹That is, $\psi : \cup_{S \subseteq N} SG(S) \rightarrow \cup_{S \subseteq N} A(S)$, $\psi v \in A(S)$ for $v \in SG(S)$, and the restriction of ψ to $SG(S)$ is a semivalue for every $S \subseteq N$.

for the Banzhaf index of power of a monotone simple game v in terms of the minimal winning coalitions of this game.

- (2) For every positive weight vector $w = (w_1, \dots, w_n)$, and $0 < c < 1$ we define the weighted majority game, $v_{w,c}$ with a quota c as follows:

$$v_{w,c}(S) = \begin{cases} 1, & \text{if } w(S) \geq c \\ 0, & \text{if } w(S) < c. \end{cases}$$

Dubey and Shapley (1979) have proved the following intriguing equality.

For *any* probabilistic index of power ψ , $\int_0^1 \psi v_{w,c}(i) dc = w_i$.

8. Values with A Social Structure.

In all the variations on the Shapley value discussed in this section, there is some structure on the set of the players which is involved in determining the value of a game. All are extensions of the Shapley value, because when the structure on N is the “trivial” one, where triviality depends on the type of structure used in the solution, then the value is the Shapley value.

Aumann and Drèze (1974) were the first to introduce such a structure into the study of value, by changing the efficiency axiom. Instead of assuming that the grand coalition, N , forms and allocates its worth, $v(N)$, among its players, they assumed that there is a fixed partition $\pi = \{B_1, \dots, B_m\}$ of N , which they called a *coalitional structure*, such that each coalition, B_k , forms and allocates its value, $v(B_k)$, among its members. Using the following axioms they characterized a unique solution called the π -value.

A solution ψ is π -efficient if for every game v , $\psi v(B_k) = v(B_k)$ for all $1 \leq k \leq m$. A permutation θ is π -invariant if $\theta(B_k) = B_k$ for every k , and ψ is π -symmetric if $\psi(\theta^*v) = \theta^*\psi v$ for every π -invariant permutation θ . Aumann and Drèze (1974)

proved:

Theorem 14. *There exists a unique probabilistic value φ^π which is π -symmetric and π -efficient, called the π -value, and it is given by:*

$$(8.1) \quad \varphi^\pi v(j) = \varphi v^{B_k}(j) \quad \text{for } j \in B_k.$$

Note that by (8.1), for $i \in B_k$, $\varphi^\pi v(i)$ depends only on the game v^{B_k} . Hence, the power of a player $i \in B_k$ to negotiate with players outside B_k is not taken into account when computing his share of $v(B_k)$.

Myerson (1977) studied a value, called the *fair allocation rule*, where the social structure is given by a graph with vertex set N . An arc (i, j) in graph g can be thought of as signifying some social relation, say a neighborhood or a channel of communication, between players i and j . For each coalition S , we denote by $g|_S$ the restriction of the graph g to the vertices in S . By $\pi_{g|_S}$ we denote the partition of S into the connected components of $g|_S$.

The axiomatization of the fair allocation rule is accomplished by fixing the game v , and varying the social structure. A function $\phi : GR \rightarrow A$, where GR is the set of all graphs on N , is a *fair allocation rule* if it satisfies two properties: it is *relatively efficient*, by which we mean that $\phi g(S) = v(S)$ for every connected component of g ; and it is *fair* in the sense that for any two graphs g and h such that $h = g \cup \{(i, j)\}$,

$$\phi h(i) - \phi g(i) = \phi h(j) - \phi g(j).$$

Thus, ϕ is fair if, by adding an arc to a graph, the players forming this arc equally gain or lose.

Myerson (1977) proved:

Theorem 15. *For any game $v \in G$, there exists a unique function $\phi^m : GR \rightarrow A$ which is a fair allocation, and it is given by,*

$$\phi_v^m(g) = \varphi(v|g),$$

where φ is the Shapley value, and $v|g$ is the game which satisfies for each coalition S ,

$$(v|g)(S) = \sum_{T \in \pi_{g|S}} v(T).$$

Note that each of the coalitions in the partition of N to its connected components, i.e., the partition $\pi_{g|N}$, forms and allocates its worth among its members. Moreover, like the case of the π -value, the value of a player, in the fair allocation rule, is determined solely by the restriction of the game to the coalition in $\pi_{g|N}$, of which he is a member.

The following solution, introduced by Owen (1977), is based on a coalition structure, i.e., a partition $\pi = \{B_1, \dots, B_m\}$ of N . But unlike the previous two solutions, it is a quasivalue, and as such it is efficient in the usual sense, that is, the players allocate the worth of the grand coalition N . By Theorem 3, a quasivalue is a random-order value, and the value proposed by Owen (1977) is the random-order value φ^{b_π} , defined by the probability distribution $b_\pi \in \Delta(OR)$, which we describe next.

We say that an order τ is consistent with π if for each $k = 1, \dots, m$, all players in B_k arrive “together” in the following sense. If i arrives before j , and both are in B_k , then all the players arriving between i and j are also in B_k . Let $b_\pi \in \Delta(OR)$ be the probability measure that is uniformly distributed over all orders that are consistent with π . That is, it assigns the probability $\frac{1}{m!b_1! \dots b_m!}$ to each such order,

where b_k denotes the number of players in B_k .

This value, called the π -coalitional structure (cs) value was characterized in various setups by Owen (1977) and by Hart and Kurz (1983). We denote it by φ_π^{cs} and need some additional notations before we can present a modified version of their axiomatization. Let π be a partition of N . For each $v \in G$ we define a *partitional game* v_π , the players of which are the coalitions in π , as follows. For each $F \subseteq \pi$,

$$v_\pi(F) = v(\cup_{B \in F} B).$$

We say that B_1 and B_2 are symmetric players in v_π if for every $F \subseteq \pi$ that does not contain B_1 and B_2 , $v_\pi(F \cup \{B_1\}) = v_\pi(F \cup \{B_2\})$. A solution ψ satisfies the π -coalitional symmetry axiom if $\psi v(B_1) = \psi v(B_2)$ whenever B_1 and B_2 are symmetric players in v_π . The proof of the following theorem can be derived from Owen (1977) and Hart and Kurz (1983):

Theorem 16. *Let ψ be a solution and let π be a partition. Then ψ is the π coalitional structure value if and only if ψ is a random-order value that satisfies the π -symmetry and the π -coalitional symmetry axioms.*

Owen (1977) and Hart and Kurz (1983) explored some interesting properties of the Owen value. One such property is that the total value of a coalition B in π equals the Shapley value of the player B in the game v_π . That is,

$$\varphi_\pi^{cs} v(B) = \varphi v_\pi(B) = \sum_{i \in B} \varphi v_\pi(i).$$

The next question is how do the players in B distribute the total amount available to them, $\varphi v_\pi(B)$. To answer this, we define for $B \in \pi$ a game $v_{(\pi, B)}$ in $G(B)$. Let

$S \subseteq B$ and denote by $\pi_B|S$ the partition refining π by splitting B into S and $B \setminus S$.

The game $v_{(\pi,B)}$ is defined by:

$$v_{(\pi,B)}(S) = \varphi^{b_{\pi_B|S}} v(S), \quad S \subseteq B.$$

Theorem 17. *For each $i \in B$,*

$$\varphi^{b_\pi} v(i) = \varphi^{v_{(\pi,B)}}(i).$$

Owen showed that Theorem 17 holds for $i \in S$ if $\pi_B|S$ is defined as the partition of $N \setminus (B \setminus S)$ obtained from π by replacing B with S (i.e., the players in $B \setminus S$ simply disappear). Hart and Kurz showed that Theorem 17 also holds when we define the game $v_{(\pi,B)}$ by defining $\pi_B|S$ as the partition obtained from π by replacing B with S and the singletons of $B \setminus S$.

Remarks.

- (1) Myerson (1980) extended the notion of the cooperation graph to a cooperation hypergraph, where a link can be formed among the members of a coalition with more than two players.
- (2) Amer and Carreras (1995) extended the notion of the cooperation graph to the weighted cooperation graph, called a *cooperation index*. More formally, a cooperation index is a function $w : C \rightarrow [0, 1]$, where $w(S)$ may be interpreted as the probability of a communication link among the members of S . Naturally, it is required that $w(\{i\}) = 1$ for every player i . Amer and Carreras (1997) further discussed cooperation indices in the context of weighted Myerson values.

- (3) Owen (1977) and Hart and Kurz (1983) gave an axiomatic characterization of the coalitional structure value as an operator which is defined on pairs (v, π) , using axioms that consider changes in the values of such an operator when both the game and the partition are changed.
- (4) McLean (1991) defined and analyzed coalitional structure random-order values, as follows. Let π be a partition of N and let $b \in \Delta(OR)$. b is *consistent with* π if $b(\tau) = 0$ for every order τ which is not consistent with π . A random order value is a π *random order coalitional structure value* if it is defined by some $b \in \Delta(OR)$ which is consistent with π . Levy and McLean (1989) defined and characterized weighted coalitional structure values.
- (5) McLean and Ye (1996) developed the theory of coalitional structure semi-values for finite games.
- (6) Winter (1992) characterized the Owen value by a consistency property which generalizes the one given in Hart and Mas-Colell (1989). He also generalized the potential approach and showed that the Owen value, Myerson value and A-D value can be developed through this approach.
- (7) Winter (1989) developed the theory of values defined on pairs (v, p) , where $v \in G$ and $p = (\pi_1, \dots, \pi_m)$ is a decreasing sequence of partitions.¹⁰
- (8) Lucas (1963) and Myerson (1976) dealt with values of generalized games, where the worth of a coalition also depends on the partition into coalitions generated by the players.

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