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Values and Semivalues on Subspaces of Finite Games[1]

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Abstract: It is proved that every value or semivalue on a linear symmetric subspace of finite games is the restriction to this subspace of a semivalue on the space of all finite games.

The theorem is proved for the space of all finite games on a fixed finite set of players, and for the space of all games with a finite support on an infinite set of players (the universe of players).

Introduction

Let G_N be the space of all games in characteristic form on $N = \{1, 2, \ldots, n\}$, and let A_N be the subspace of G_N consisting of all additive games. A semivalue on G_N is (following [2]) an operator $\psi : G_N \to A_N$ which is a linear symmetric positive projection on A_N. A value on G_N is an efficient semivalue. A complete characterization of semivalues is given in [2]: An operator $\psi : G_N \to A_N$ is a semivalue iff there exist constants $(p_k)_{k=0}^{n-1}$ s.t. $p_k \geqslant 0$ for every $0 \leqslant k \leqslant n-1$, $\sum_{k=0}^{n-1} p_k \binom{n-1}{k} = 1$, and:

$$\psi v(i) = \sum_{k=0}^{n-1} p_k \sum_{\substack{S \subseteq N \setminus i \\ |S| = k}} (v(S \cup i) - v(S)), \quad v \in G_N, \ 1 \leqslant i \leqslant n. \qquad (*)$$

Moreover, the correspondence $\psi \leftrightarrow (p_k)_{k=0}^{n-1}$ is linear and 1-1.

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As for values, there exists a unique value – the Shapley value, which is obtained by letting $p_k = \frac{1}{n\binom{n-1}{k}}$ in (*). Given a single game $v \in G$, one may look for a concept of solution for v which depends only on v, or on games very similar to v, or on games generated by v, and not on all the games in G. Such an approach was already taken in [3]. It is proved there that the potential axiom determines the Shapley value on the set cosnsiting of v and all its subgames. In this paper we characterize values and semivalues on the smallest linear symmetric space which contains a given game or, more generally, on every linear symmetric subspace of G_N.

The notions of values and semivalues on subspaces of G_N are yet to be defined. No problem arises in the case of a subspace Q containing A_N. For such a subspace these notions may be defined exactly as for G_N. But when $Q \not\supseteq A_N$ the projection axiom is meaningless. Of course, it has an apparently natural generalization: $\psi\mu = \mu$, $\mu \in Q \cap A_N$. However, the concept of solution this generalized projection axiom leads to was found nonsatisfactory. Instead we suggest the Milnor axiom (see [4]): A player can't get more than its maximal marginal contribution or less than its minimal marginal contribution. We define a semivalue on an arbitrary subspace of G_N as a linear symmetric Milnor operator. We shall prove that if $Q \supseteq A_N$ then $\psi : Q \to A_N$ is a linear Milnor operator iff it is a linear positive projection. That is, our definition of a semivalue coincides in this case with the natural one.

We shall also prove that linear Milnor operators have the extention property: a linear Milnor operator on a subspace of G_N can be extended to a linear Milnor operator on G_N. This result serves also in the proof of the main theorem of Chapter 1: Every semivalue on a subspace of G_N can be extended to a semivalue on G_N. This is proved by a symmetrization process on the Milnor extentions of the given semivalue. The main theorem may be explained from another point of view: Let $\psi : G_N \to A_N$ Be a semivalue. Let $Q_\psi = \{v \in G_N : \psi v(N) = v(N)\}$. Obviously, for every $Q \subseteq Q_\psi$ $\psi : Q \to A_N$ is a value. By the main theorem, every value on a subspace of G_N is obtained in this manner.

In Chapter 2 we interpret a finite game as a game with a finite support on an infinite set of players (the universe of players) and prove our main theorem in this context.

The definition of a Milnor operator, as well as the proof of the extention property, are quite easy (one just has to use Zorn's Lemma instead of an induction process). The symmetrization process is, however, less trivial since the group of symmetries is an infinite one. We overcome this difficulty by using compactness arguments.

1 Values and Semivalues on Subspaces of G_N

Let $N = \{1, 2, \ldots, n\}$ be the players' set which will be fixed throughout this chapter. Denote by G_N the space of all games in characteristic form on N, and denote by A_N the subspace of G_N consisting of all additive games. We will identify A_N with the n-dimensional Euclidean space R^n. That is, for every u in R^n and every $S \subseteq N$, $u(S) = \sum_{i \in S} u_i$. A game $v \in G_N$ is a *monotomic game* if $v(S) \leqslant v(T)$ for every $S \subseteq T \subseteq N$. For every $M \subseteq G_N$, M^+ will denote the set of all monotonic games in M. For every u and v in G_N we will write $u \gtrsim v$ or $v \lesssim u$ if $u - v$ is a monotonic game. Clearly, for every λ and μ in A_N, $\lambda \gtrsim \mu$ iff $\lambda \geqslant \mu$ (that is, $\lambda_i \geqslant \mu_i$ for every $1 \leqslant i \leqslant n$). For every $v \in G_N$ we define v^* and v_* in A_N as follows:

$$v^*(i) = \max_{S \subseteq N \setminus i} (v(S \cup i) - v(S))$$

$$v_*(i) = \min_{S \subseteq N \setminus i} (v(S \cup i) - v(S)), \quad 1 \leqslant i \leqslant n.$$

That is $v^*(i)$ and $v_*(i)$ are the maximum marginal contributions of player i, respectively. It is clear that A_N with its usual order is a *complete* lattice. That is, every nonempty subset of A_N which is order bounded from above has a least upper bound (l.u.b.) and every nonempty subset of A_N which is order bounded from below has a greatest lower bound (g.l.b.). Indeed, if $\phi \neq M \subseteq A_N$ is bounded from above then $\lambda =$ l.u.b. (M) is the following additive game:

$$\lambda_i = \sup_{\mu \in M} \mu_i, \quad 1 \leqslant i \leqslant n.$$

By standard arguments using properties of max, min, l.u.b. and g.l.b., we have:

Lemma 1.1: Let u and v be in G_N and $u \in A_N$ then

1. $v^* \gtrsim v \gtrsim v_*$.

2. $v^* =$ g.l.b. $\{\lambda \in A_N : \lambda \gtrsim v\}$.

3. $v_* =$ l.u.b. $\{\lambda \in A_N : \lambda \lesssim v\}$.

4. $(-v)^* = -v_*$ and $(-v)_* = -v^*$

5. $(\alpha v)^* = \alpha v^*$ and $(\alpha v)_*$, $\alpha \geqslant 0$.

6. $(v+u)^* \leqslant v^* + u^*$ and $(v+u)_* \geqslant v_* + u_*$.

7. $\mu^* = \mu = \mu_*$

8. $(v+\mu)^* = v^* + \mu$ and $(v+\mu)_* = v_* + \mu$. □

The group of all symmetries of N will be denotes by H_N. For every θ in H_N we define a linear operator $\theta_* : G_N \to G_N$ as follows:

$$(\theta_* v)(S) = v(\theta S), \quad S \subseteq N \text{ and } v \in G_N.$$

Clearly the following holds.

$$\begin{aligned}(\theta_* v)^* &= \theta_* v^* \\ (\theta_* v)_* &= \theta_* v_*.\end{aligned} \tag{1.2}$$

A set $M \subseteq G_N$ is a *symmetric set* if for every $\theta \in H_N$ and $v \in M$, $\theta_* v \in M$. We will deal mainly with operators $\psi : Q \to A_N$, where Q is a linear symmetric subspace of G_N. Recall that ψ is a positive operator if $\psi v \geqslant 0$ whenever $v \gtrsim 0$ and that if $Q \supseteq A_N$ then ψ satisfies the projection axiom if $\psi\mu = \mu$ for every $\mu \in A_N$.

Definition 1.3: Let Q be a linear symmetric subspace of G_N s.t. $Q \supseteq A_N$ and let ψ : $Q \to A_N$. ψ is a *semivalue* if it is a linear, symmetric positive projection. ψ is a *value* if it is an efficient semivalue.

We now wish to extend the above definition to operators defined on subspaces of G_N not necessarily containing A_N. For that matter we present:

Milnor axiom:

$$v_* \leqslant \psi v \leqslant v^*.$$

We remark that for linear operators the Milnor axiom is equivalent to both the *upper Milnor axiom* ($\psi v \leqslant v^*$) and the *lower Milnor axiom* ($\psi v \geqslant v_*$).

Lemma 1.4: Let $\psi : Q \to A_N$ be a linear operator, where Q is a linear subspace of G_N s.t. $Q \supseteq A_N$. Then ψ satisfies the Milnor axiom iff ψ is a positive projection.

Proof: Assume first that ψ satisfies the Milnor axiom. By Corollary 1.1 for every $\mu \in A_N$,

$$\mu = \mu_* \leqslant \psi\mu \leqslant \mu^* = \mu.$$

Therefore ψ is a projection.

ψ is also positive, for if $v \in Q^+$, then since $v \gtrsim 0$ and $0 \in A_N$, Lemma 1.1 yields that $v_* \geqslant 0$. Thus by the Milnor axiom, $\psi v \geqslant v_* \geqslant 0$. As for the converse:

Assume ψ is positive projection and let $v \in Q$. By Lemma 1.1 $v \gtrsim v_*$. Therefore by the positivity and projection axioms,

$$\psi v \geqslant \psi v_* = v_*.$$

Hence ψ satisfies the lower Milnor axiom and since ψ is linear, the result follows. □

By Lemma 1.4 the following is a generalization of Definition 1.3:

Definition 1.5: Let Q be a linear symmetric subspace of G_N, and let $\psi : Q \to A_N$. Then:

ψ is a *semivalue* if it is linear, symmetric and satisfies the Milnor axiom.

ψ is a *value* if it is an efficient semivalue. □

In the following lemmas we deal with some extension properties of Milnor operators defined on subspaces of G_N.

Lemma 1.6: Let Q be a linear subspace of G_N, and let $\psi : Q \to A_N$ be a linear operator which satisfies the Milnor axiom. Then, ψ can be extended to a linear positive projection $\bar{\psi} : Q + A_N \to A_N$.

Proof: Define $\bar{\psi} : Q + A_N \to A_N$ as follows:

$$\bar{\psi}(v + \mu) = \psi v + \mu,$$

for $v \in Q$ and $\mu \in A_N$.

In order to prove that $\bar{\psi}$ is a well defined, linear positive projection it suffices to prove that $\psi v + \mu \geqslant 0$ whenever $v \in Q$, $\mu \in A_N$ and $v + \mu \gtrsim 0$. Indeed, if $v + \mu \gtrsim 0$ then $v \gtrsim -\mu$. Thus by Lemma 1.1 $v_* \geqslant -\mu$. Hence,

$$\psi v \geqslant v_* \geqslant -\mu,$$

which implies:

$$\psi v + \mu \leqslant 0. \qquad \square$$

Lemma 1.7: Let $Q \supseteq A_N$ be a linear subspace of G_N and let $\psi : Q \to A_N$ be a linear positive projection. Then ψ can be extended to a linear positive projection on G_N.

Proof: Obviously it suffices to prove that for every $u \notin Q$, ψ can be extended to a linear positive projection on $Q + \langle u \rangle$, where $\langle u \rangle$ is the linear space spanned by u.

Let $u \notin Q$. Define $\bar{\psi} : Q + \langle u \rangle \to A_N$ as follows:

$$\bar{\psi}(w + \alpha u) = \psi w + \alpha\lambda,$$

where $w \in Q$, α is a real number and

$$\lambda = \text{l.u.b.}\ \{\psi v : v \in Q \text{ and } v \lesssim u\}.$$

Since $u_* \in Q$ and $u_* \lesssim u$, the set $\{\psi v : v \in Q \text{ and } v \lesssim u\}$ is nonempty (it contains $\psi u_* = u_*$), and order bounded from avove by $\psi v^* = u^*$. Thus λ is a well defined element of A_N. (Recall that by Lemma 1.4 ψ satisfies the Milnor axiom.) Clearly $\bar{\psi}$ is a linear projection and we have to prove that it is positive. Assume $v + \alpha u \gtrsim 0$. If $\alpha > 0$ then $u \gtrsim -\frac{1}{\alpha} v$. Thus by the definition of λ, $\lambda \geqslant \psi\left(-\frac{1}{\alpha} v\right)$; which implies: $\psi v + \alpha\lambda \geqslant 0$. If $\alpha < 0$, then $u \lesssim -\frac{1}{\alpha} v$. Hence, for *every* w in Q s.t. $w \lesssim u$, $w \lesssim -\frac{1}{\alpha} v$. By the positive of ψ, $\psi w \leqslant \psi\left(-\frac{1}{\alpha} v\right)$, which yields that $\lambda \leqslant \psi\left(-\frac{1}{\alpha} v\right)$. Thus, $\psi v + \alpha\lambda \geqslant 0$. $\square$

Theorem 1.8 (Main Theorem of Chapter 1): Every semivalue on a linear symmetric supspace of G_N is the restriction to this subspace of some semivalue on G_N.

Proof: Let $\psi : Q \to A_N$ be a simivalue, where Q is a linear symmetric subspace of G_N. By Lemmas 1.6 and 1.7 ψ can be extended to a linear positive projection $\bar{\psi} : G_N \to A_N$. Define $\psi_1 : G_N \to A_N$ as follows:

$$\psi_1 = \frac{1}{n!} \sum_{\theta \in H_N} \theta_*^{-1} \bar{\psi} \theta_*.$$

Clearly ψ_1 is a semivalue on G_N and since $\bar{\psi}$ is symmetric on Q, $\psi_1 = \bar{\psi} = \psi$ on Q. □

Corollary 1.9: Let $\psi : Q \to A_N$ be a semivalue. Then there exist constants $(p_k)_{k=0}^{n-1}$ s.t. $p_k \geqslant 0$ for every $0 \leqslant k \leqslant n-1$, $\sum_{k=0}^{n-1} p_k \binom{n-k}{k} = 1$, and for every $v \in Q$ and every $i \in N$,

$$\psi v(i) = \sum_{k=0}^{n-1} p_k \sum_{\substack{S \subseteq N \backslash i \\ |S|=k}} (v(S \cup i) - v(S)).$$

Proof: Combine Theorem 1.8 with the result of [2] mentioned in the introduction. □

2 Games on the Universe of Players

Our terminology will be similar to the one given in [1] and in [2]. Let U be an infinite set of players (the universe of players). A set function $v : 2^U \to R$ is *monotonic* if $v(S) \leqslant v(T)$ for all $S \subseteq T \subseteq U$. A game on U is a set function $v : 2^U \to R$ with $v(\emptyset) = 0$, which is the difference of two monotonic set functions. The linear space of all games is denoted by $BV(U)$. The space of all additive games in $BV(U)$ is denoted by $FA(U)$. For every $M \subseteq BV(U)$, M^+ will denote the set of all monotonic games in M, and M^1 will denote the set of all $v \in M^+$ for which $v(U) = 1$. For every $v \in BV(U)$ and $T \subseteq U$ we define $v_T(S) = v(S \cap T)$, $S \subseteq U$. A set $T \subseteq U$ is a *support* of a game v if $v = v_T$. Let $M \subseteq BV(U)$. The set of all v in M for which $v = v_T$ is denoted by M_T. A game v is a *finite game* if it has a finite support. The space of all finite games is denoted by G, and the space of all finite additive games is denoted by A. That is, $A = FA(U) \cap G$. We will identify A and the set of all functions $\tau : U \to R$ with a finite support. More precisely:

for every such τ we associate $\mu \in A$ as follows:

$$\mu(S) = \sum_{i \in S \cap N} \tau(i), \quad S \subseteq U,$$

where N is a finite support of τ.

Let $\tilde{H}$ be the group of all symmetries of U. A set $T \subseteq U$ is a *support* of $\theta \in \tilde{H}$ if $\theta i = i$ for every $i \in T^c$. Let $T \subseteq U$. The subgroup of $\tilde{H}$ consisting of all θ which are supported in T is denoted by $\tilde{H}_T$. The subgroup of $\tilde{H}$ consisting of all symmetries with a finite support is denoted by H. Let $J \subseteq \tilde{H}$. A set $M \subseteq BV(U)$ is *J-symmetric* if $\theta_* v \in M$ for every $v \in M$ and every $\theta \in J$. It is *symmetric* if it is H-symmetric. The notions of J-symmetry and symmetry are defined in an obvious manner also for operators defined on J-symmetric subspaces of $BV(U)$. Obviously every H-symmetric subset of G is symmetric and every H-symmetric operator on a symmetric subspace of G is symmetric. For every μ_1 and μ_2 in A we define:

$$(\mu_1 \vee \mu_2)(i) = \max(\mu_1(i), \mu_2(i))$$

$$(\mu_1 \wedge \mu_2)(i) = \min(\mu_1(i), \mu_2(i)), \quad i \in U.$$

Clearly $\mu_1 \vee \mu_2$ and $\mu_1 \wedge \mu_2$ belong to A and they are the l.u.b. and g.l.b. of $\{\mu_1, \mu_2\}$ respectively. That is, A is a linear lattice. For every u and v in G we write $u \lesssim v$ or $v \gtrsim u$ if $v - u$ is a monotonic game. Obviously, for μ and λ in A, $\mu \gtrsim \lambda$ iff $\mu \vee \lambda$ (iff $\mu(i) \geqslant \lambda(i)$ for all i in U). For every $v \in G$ define:

$$v^*(i) = \sup_{S \subseteq U \backslash i} (v(S \cup i) - v(S))$$

$$v_*(i) = \inf_{S \subseteq U \backslash i} (v(S \cup i) - v(S)), \quad i \in U.$$

Clearly v^* and v_* are in A and for every support N of v, $v^* = (v^*)_N$, $v_* = (v_*)_N$ and

$$v^*(i) = \max_{S \subseteq N \backslash i} (v(S \cup i) - v(S))$$

$$v_*(i) = \min_{S \subseteq N \backslash i} (v(S \cup i) - v(S)), \quad i \in N.$$

We now define Milnor operators, values and semivalues as in Chapter 1. Obviously Lemmas 1.1, 1.4, 1.6 and 1.7 continue to hold in this context. (In Lemma 1.7 one has to use Zorn's Lemma instead of an induction process.)

A few more results are needed now. For every $\mu \in A$ define $\|\mu\| = \sum_{j \in U} |\mu(j)|$. Obviously $(A, \|\mu\|)$ is a normed space. For every $\mu_1 \leqslant \mu_2$ in A the set $B(\mu_1, \mu_2) = \{\mu \in A : \mu_1 \leqslant \mu \leqslant \mu_2\}$ is homeomorphic to the compact subset $\{x \in R^N : \mu_1(i) \leqslant x_i \leqslant \mu_2(i)\}$ of the euclidean space R^N, where N is a common finite support of μ_1 and μ_2. Therefore B is a compact set. By Tychonoff's theorem the topological product $L = \prod_{v \in G} B(v_*, v^*)$ is a compact set too.

Let K be the set of all linear positive projections $\psi : G \to A$ endowed with the topology of pointwise convergence. That is, for every directed set D, for every net $(\psi_\alpha)_{\alpha \in D}$ in K and for every $\psi \in K$,

$$\lim \psi_\alpha = \psi \text{ iff } \lim \|\psi_\alpha v - \psi v\| = 0 \quad \text{for every } v \in G.$$

Since by Lemma 1.4 every $\psi \in K$ satisfies the Milnor axiom, we can easily deduce that the transformation $\psi \to (\psi v)_{v \in G}$ is a homeomorphism of K onto a closed subset of L. Therefore: K is a compact topological space.

Theorem 2.1 (Main Theorem of Chapter 2): Every semivalue on a linear symmetric subspace of G is the restriction to this subspace of some semivalue on G.

Proof: Let $\psi_1 : Q \to A$ be a semivalue, where Q is a linear symmetric subspace of G. By Lemmas 1.6 and 1.7 ψ_1 can be extended to an element of K. Let $\tilde{K}$ be the compact subset of K consisting of all the extensions of ψ_1. For every finite subset N of U denote:

$$\tilde{K}_N = \{\psi \in \tilde{K} : \theta_*^{-1} \psi \theta_* = \psi \text{ for every } \theta \in H_N\}.$$

(Recall that $H_N = \{\theta \in H : \theta(i) = i \ \forall \ i \in N^C\}$.)

Obviously $\tilde{K}_N$ is a closed subset of $\tilde{K}$. We now show that it is not empty. Indeed, choose $\psi \in \tilde{K}$ and denote

$$\bar{\psi} = \frac{1}{n!} \sum_{\theta \in H_N} \theta_*^{-1} \psi \theta_*,$$

where $n = |N|$.

Clearly $\tilde{\psi} \in K_N$. Since for any finite number of finite subsets $N_1, N_2, \ldots, N_l$ of U, $\bigcap_{i=1}^{l} \tilde{K}_{N_i} = \tilde{K}_N$, where $N = \bigcap_{i=1}^{l} N_i$ we can use the finite intersection property of $\tilde{K}$ to deduce that $\bigcap_N \tilde{K}_N \neq \emptyset$. Obviously every ψ_2 in the intersection set is a semivalue on G which extends ψ_1. □

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