Journal of Mathematical Economics 17 (1988) 321–338. North-Holland

A PROBABILISTIC PROBLEM ARISING IN ECONOMICS*

Dov MONDERER

Northwestern University, Evanston, IL 60201, USA

Submitted February 1986, accepted January 1988

Let v be a market game derived from an l-glove market with population measure μ. That is, $v=\min(\lambda_1,\lambda_2,\ldots,\lambda_l)$, where $\lambda_1,\lambda_2,\ldots,\lambda_l$ are mutually singular probability measures which are absolutely continuous w.r.t. μ. It is proved that when $l \geqq 3$ the μ-asymptotic value of $v, \phi_\mu v$, exists iff at most one of the measures $\lambda_1,\ldots,\lambda_l$ is not in $L_2(\mu)$. A formula for $\phi_\mu v$ is given whenever it exists. In addition, a complete characterization of the range of the μ-values of v (where μ varies) is given.

1. Introduction

An l-glove market game has the form

$$v=\min(\lambda_1,\lambda_2,\ldots,\lambda_l),$$

where $\lambda_1,\lambda_2,\ldots,\lambda_l$ are mutually singular measures in NA^1.

Let $\mu\in\mathrm{NA}^1$ be a measure s.t. $\lambda_i\in L_1(\mu)$ for every $1\leqq i\leqq l$. In Hart (1980) it was proved that when $\lambda_i\in L_2(\mu)$ for all $1\leqq i\leqq l$, the μ-asymptotic value of $v, \phi_\mu v$, exists. In this paper we prove that this condition is almost necessary for the existence of $\phi_\mu v$. More precisely, we will prove that (when $l\geqq 3$) $\phi_\mu v$ exists iff *at most one of the measures* λ_i *is not in* $L_2(\mu)$. In addition we supply a formula for $\phi_\mu v$, whenever it exists.

Finally, a complete description of the set of all μ-values of v as μ varies is given. The result is a generalization of the L_2-case given in Monderer (1986c).

In order to prove our claims we are forced to handle a complicated problem in probability. A large part of this paper is devoted to the solution of this problem.

The paper is organized as follows: In section 2 we state and prove our probabilistic problem. In section 3 we present necessary and sufficient

---

*This is a revised version of working paper no. 111/86 of Everyman's University, January 86, Tel-Aviv. This paper is part of the author's Ph.D. done in Tel-Aviv University under the supervision of Professor Sergiu Hart. I wish to thank Prof. R.J. Aumann and Prof. S. Hart for very helpful discussions. Their contribution for the completion of this paper was invaluable.

conditions for the existence of the μ-asymptotic value of l-glove market game, and in section 4 we describe the range of the μ-asymptotic values of a given l-glove market. For a detailed discussion of l-glove markets and their solutions we refer the reader to Monderer (1986b).

2. A probabilistic problem

We use basically the terminology and notations of Aumann and Shapley (1974) and of chapters 1 and 2 of Monderer (1986c). (I, C, μ) will denote a fixed non-atomic standard probability space. Every subset of I under discussion is assumed to be measurable. Let $\lambda \in \mathrm{NA}$. We say that $\lambda \in L_p(\mu)$ (for $1 \leqq p \leqq \infty$) if $\lambda \ll \mu$ and $\mathrm{d}\lambda/\mathrm{d}\mu \in L_p(\mu)$. We interchangeably refer to $L_p(\mu)$ as a space of functions and as a space of measures. For every $1 \leqq p \leqq \infty$, $L_p(\mu)^+$ will denote the set of all positive measures in $L_p(\mu)$, and $L_p(\mu)^1$ will denote the space of all probabilities in $L_p(\mu)$.

For every partition π let (R_π, P_π) be the probability space of all $n!$ orders on $\pi (n = |\pi|)$, where the probability of each order is $1/n!$ For every $1 \leqq h \leqq n$ let T_h^R denote the hth member in the order R, and let Q_h^R denote the union of the first h members of R. That is,

$$Q_h^R = \bigcup_{i=1}^{h} T_i^R.$$

Let $\lambda_1, \lambda_2, \ldots, \lambda_l (l \geqq 2)$ be mutually singular measures in $L_l(\mu)^1$. For every partition π and every $1 \leqq i \leqq n$ let

$$A(\pi, \lambda_i, \lambda) = \frac{1}{n} \sum_{h=1}^{n} P_\pi \left(\lambda_i(Q_h^R) < \min_{j \neq i} \lambda_j(Q_h^R) \right), \tag{2.1}$$

where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ and $n = |\pi|$.

For every μ-admissible sequence $P = (\pi_m)_{m=1}^{\infty}$ let

$$\underline{A}(P, \lambda_i, \lambda) = \varliminf_{m \to \infty} A(\pi_m, \lambda_i, \lambda), \tag{2.2}$$

$$\bar{A}(P, \lambda_i, \lambda) = \varlimsup_{m \to \infty} A(\pi_m, \lambda_\lambda), \tag{2.3}$$

and if $\underline{A}(P, \lambda_i, \lambda) = \bar{A}(P, \lambda_i, \lambda)$ denote their common value by $A(P, \lambda_i, \lambda)$. If $A(P, \lambda_i, \lambda)$ exists for every μ-admissible sequence P, and does not depend on P, we denote it by $A(\lambda_i, \lambda)$. We say that $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ is an *admissible vector* if $A(\lambda_i, \lambda)$ exists for every $1 \leqq i \leqq l$.

In Theorem 1 we characterize admissible vectors.

Theorem 1. *Let* $\lambda_1, \lambda_2, \ldots, \lambda_l, l \geqq 2$, *be mutually singular measures in* $L_1(\mu)^1$. *Then* $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ *is an admissible vector iff* $l = 2$, *or* $l \geqq 3$ *and at most one of the measures* $\lambda_1, \lambda_2, \ldots, \lambda_l$ *is not in* $L_2(\mu)$.

Moreover:

– *If* $l = 2$, *then*

$$A(\lambda_1, \lambda) = A(\lambda_2, \lambda) = \tfrac{1}{2}.$$

– *If* $l \geqq 3$ *and for every* $1 \leqq i \leqq l$, $\lambda_i \in L_2(\mu)$, *then*

$$A(\lambda_i, \lambda) = N\left(\sqrt{a_i}x_i < \min_{j \neq i} \sqrt{a_j}x_j\right), \qquad 1 \leqq i \leqq l,$$

where N *is the standard normal distribution on* R_l *and for every* $l \leqq i \leqq l$,

$$a_i = \int \left(\frac{\mathrm{d}\lambda_i}{\mathrm{d}\mu}\right)^2 \mathrm{d}\mu.$$

– *If* $l \geqq 3$, $\lambda_1 \notin L_2(\mu)$ *and* $\lambda_i \in L_2(\mu)$ *for every* $2 \leqq i \leqq l$, *then*

$$A(\lambda_1, \lambda) = \tfrac{1}{2},$$

and for every $2 \leqq i \leqq l$,

$$A(\lambda_i, \lambda) = \tfrac{1}{2} A(\lambda_i, (\lambda_2, \lambda_3, \ldots, \lambda_l)).$$

The proof of Theorem 1 is quite complicated, and we divide it into several stages. We start with some notions and notations, and present a result of Rosen (1965) concerning sampling without replacement. This result will be needed several times in the proof.

For every $0 < \alpha < 1$ and every real number u let $X(\alpha, u)$ be a random variable satisfying

$$P\left(X(\alpha, u) = -u\sqrt{\frac{\alpha}{1-\alpha}}\right) = 1 - \alpha, \qquad P\left(X(\alpha, u) = u\sqrt{\frac{1-\alpha}{\alpha}}\right) = \alpha. \tag{2.4}$$

$F(\alpha, u)$ will denote the distribution function of such random variable.

For every $d = (d_i)_{i=1}^{\infty}$ in l_2 (i.e., $\sum_{i=1}^{\infty} d_i^2 < \infty$) let $F(\alpha, d)$ denote the distribution function of $X = \sum_{i=1}^{\infty} X(\alpha, d_i)$, where $(X(\alpha, d_i))_{i=1}^{\infty}$ is a sequence of independent random variables, each satisfying (2.4). It is known that X is well defined (that is, the series converges a.e.), and that $F(\alpha, d)$ is the limit in distribution of $(F(\alpha,d_1)*F(\alpha,d_2)*\cdots*F(\alpha,d_n))_{n=1}^{\infty}$, where $*$ denotes convolution.

For every c and d in l_2 denote also

$$F(\alpha,c,d)=F(\alpha,c)*F(\alpha,d).$$

Define now:

$$C=\{c\in l_2: c_1\leqq c_2<\leqq\cdots\leqq 0\},$$

$$D=\{d\in l_2: d_1\geqq d_2\geqq\cdots\geqq 0\},$$

$$M=C\times D.$$

M is a subset of the Hilbert space $l_2\times l_2$. We denote

$$M_1=\{(c,d)\in M: \|(c,d)\|\leqq 1\},$$

where $\|\cdot\|$ is the norm in $l_2\times l_2$:for $(x,y)\in l_2\times l_2$,

$$\|(x,y)\|^2=\|x\|^2+\|y\|^2.$$

M_1 is closed, convex and bounded, and thus weakly compact in $l_2\times l_2$. For every partition π of I, and every $\lambda\in L_1(\mu)$, denote

$$U_\pi(\lambda)=\max_{S\in\pi}\left|\lambda(S)-\frac{\lambda(I)}{n}\right|, \tag{2.5}$$

$$D_\pi(\lambda)^2=\sum_{S\in\pi}\left(\lambda(S)-\frac{\lambda(I)}{n}\right)^2, \tag{2.6}$$

where $n=|\pi|$.

Now let π be a partition of I, and λ a measure in $L_1(\mu)$ s.t. $D_\pi(\lambda)>0$. Let $S_1,S_2,\ldots,S_s$ be an ordering of all $S\in\pi$ for which $\lambda(S)<\lambda(I)/n$, and let $T_1,T_2,\ldots,T_t$ be an ordering of all $T\in\pi$ for which $\lambda(T)>\lambda(I)/n$, s.t.

$$\lambda(S_1)\leqq\cdots\leqq\lambda(S_s)<\frac{\lambda(I)}{n},$$

$$\lambda(T_1)\geqq\cdots\geqq\lambda(T_t)>\frac{\lambda(I)}{n}.$$

Denote $D(\lambda)=D_\pi(\lambda)$ and define

$$c_\pi(\lambda)=\left(\frac{\lambda(S_1)-\lambda(I)/n}{D(\lambda)},\frac{\lambda(S_2)-\lambda(I)/n}{D(\lambda)},\dots,\right.$$
$$\left.\frac{\lambda(S_s)-\lambda(I)/n}{D(\lambda)},0,0,\dots,0,\dots\right),\tag{2.7}$$

$$d_\pi(\lambda)=\left(\frac{\lambda(T_1)-\lambda(I)/n}{D(\lambda)},\frac{\lambda(T_2)-\lambda(I)/n}{D(\lambda)},\dots,\right.$$
$$\left.\frac{\lambda(T_t)-\lambda(I)/n}{D(\lambda)},0,0,\dots,0,\dots\right).\tag{2.8}$$

Obviously $(c_\pi(\lambda),d_\pi(\lambda))\in M_1$.

For every $1\leqq h\leqq n$ let $X(\lambda,\pi,h)$ and $Z(\lambda,\pi,h)$ be random variables defined on R_π by

$$X(\lambda,\pi,h)(R)=\lambda(Q_h^R),\tag{2.9}$$

$$Z(\lambda,\pi,h)(R)=\frac{\lambda(Q_h^R)-(h/n)\lambda(I)}{D(\lambda)\sqrt{(h/n)(1-h/n)}}\tag{2.10}$$

Remark. Often, where no confusion may result, we will omit (without further notice) indices and parameters to make reading easier.

For X and Z defined by (2.9) and (2.10) it is easy to see that

$$\mathrm{E}X=\frac{h}{n}\lambda(I),$$

and Theorem 1.1 in Rosen (1965) yields

$$\mathrm{var}(X)=\mathrm{E}\left(\left(X-\frac{h}{n}\lambda(I)\right)^2\right)=\frac{h}{n}\left(1-\frac{h}{n}\right)D(\lambda)^2,\tag{2.11}$$

i.e., $\mathrm{E}Z=0$ and $\mathrm{var}(Z)=1$.

Theorem 2 Let $\lambda\in L_1(\mu)$ and let $(\pi_m)_{m=1}^\infty$ be a μ-admissible sequence of partitions s.t. for every $m\geqq 1, D_m(\lambda)>0$, and s.t. the sequence $(c_m(\lambda),d_m(\lambda))_{m=1}^\infty$ converges weakly in $l_2\times l_2$ to some (c,d). In addition, for every $m\geqq 1$ let $h(m)$

be an integer, $1 \leqq h(m) \leqq n(m)$. *Then:*

– *If* $(c, d) = (0, 0)$ *and*

$$0 < \underline{\lim} \frac{h(m)}{n(m)} \leqq \overline{\lim} \frac{h(m)}{n(m)} < 1,$$

then

$$Z(\lambda, \pi_m, h(m)) \xrightarrow{\mathrm{D}} \mathrm{N}(0, 1).$$

– *If* $(c, d) \neq (0, 0)$ *and* $h(m)/n(m) \to \alpha$, $0 < \alpha < 1$, $\alpha \neq \frac{1}{2}$, *then*

$$Z(\lambda, \pi_m, h(m)) \xrightarrow{\mathrm{D}} \mathrm{N}(0, 1 - \|(c, d)\|^2) * F(\alpha, c, d),$$

where $\xrightarrow{\mathrm{D}}$ *denotes convergence in distribution.*

Proof. All this can be easily deduced from Theorem 12.1 in Rosen (1965). □

Proof of Stage I of Theorem 1 [*the case when all measures are in* $L_2(\mu)$]. In this stage we have to prove that if $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$, $l \geqq 2$, is a vector of mutually singular probability measures in $L_2(\mu)$, then λ is an admissible vector, and for every $1 \leqq i \leqq l$,

$$A(\lambda_i, \lambda) = \mathrm{N}\left(\sqrt{a_i} x_i < \min_{j \neq i} \sqrt{a_j} x_j\right),$$

where N is the standard normal distribution on R_l, and for every $1 \leqq i \leqq l$,

$$a_i = \int \left(\frac{\mathrm{d}\lambda_i}{\mathrm{d}\mu}\right)^2 \mathrm{d}\mu.$$

This result is an obvious consequence of Theorem 9.2 in Hart (1980). For a more detailed proof see Proof of Stage I in Monderer (1986c, p. 27). □

For the proof of Stage II of Theorem 1 (the case $l = 2$) we will need some further results, Theorem 3 and Corollary 2. These results, together with some others, will be needed also in the proofs of the remaining parts of Theorem 1.

Let, then, M_1^* be the set of all (c, d) in M_1 for which either $\|(c, d)\| < 1$, or for every $i \geqq 1, c_i < 0$, or for every $i \geqq 1, d_i > 0$, or $c = 0$, or $d = 0$. That is, if $(c, d) \in M_1$ and $(c, d) \notin M_1^*$ then $\|(c, d)\| = 1$, $c \neq 0$ and $d \neq 0$, and there exists a positive integer N s.t. for every $i \geqq N$, $c_i = d_i = 0$.

Theorem 3. *Let $(c,d)\in M_1^*$. Then there exists a finite set of A of numbers in the open interval $(0,1)$, s.t. for every measure $\mu_1\in L_1(\mu)$ which is not a scalar multiple of μ, for every μ-admissible sequence of partitions $(\pi_m)_{m=1}^{\infty}$ for which*

$$(c_m(\mu_1), d_m(\mu_1)) \xrightarrow{w} (c,d),$$

for every sequence of non-negative real numbers $(\delta_m)_{m=1}^{\infty}$ s.t. $\delta_m\to 0$, and for every sequence of positive integers $(h(m))_{m=1}^{\infty}$, $1\leqq h(m)\leqq n(m)$, satisfying

$$\frac{h(m)}{n(m)} \xrightarrow[m\to\infty]{} \alpha\notin A,$$

the following holds:

$$\lim_{m\to\infty} P_m(|Z(\mu_1,\pi_m,h(m))|\leqq\delta_m)=0. \tag{2.12}$$

Moreover:

- *If $(c,d)=(0,0)$ we may choose $A=\emptyset$.*
- *If either $\|(c,d)\|<1$, or for every $i\geqq 1$ $c_i<0$, or for every $i\geqq 1$ $d_i>0$, we may choose $A=\{\frac{1}{2}\}$.*

Proof. Let $\mu_1, P=(\pi_m)_{m=1}^{\infty}$ and $(\delta_m)_{m=1}^{\infty}$ be as stated in the theorem, and assume that $h(m)/n(m)\xrightarrow[m\to\infty]{}\alpha$, $0<\alpha<1$. If either $(c,d)=(0,0)$, or $(c,d)\neq(0,0)$ and $\alpha\neq\frac{1}{2}$, we get by Theorem 2 that $Z_m\xrightarrow{D} F$, where $Z_m=Z(\mu_1,\pi_m,h(m))$ and $F=\mathrm{N}(0,1-\|(c,d)\|^2)*F(\alpha,c)*F(\alpha,d)$. If F is a continuous function, then it is known that

$$P_m(|Z_m|\leqq\delta)\to F(\delta)-F(-\delta),$$

where the convergence is uniform in $-1<\delta<1$. This implies easily that

$$P(|Z_m|\leqq\delta_m)\to 0.$$

A sufficient condition for F to be continuous is that at least one of the distribution functions in the convolution is continuous. If $\|(c,d)\|<1$, then obviously $\mathrm{N}(0,1-\|(c,d)\|^2)$ is continuous.

Claim. If for every $i\geqq 1, c_i<0$, then $F(\alpha,c)$ is continuous (not necessarily absolutely continuous).

Proof of Claim. $F(\alpha,c)$ is the distribution function of $X=\sum_{i=1}^{\infty}X_i$, where

$(X_i)_{i=1}^{\infty}$ are independent random variables satisfying

$$P\left(X_i=-\sqrt{\frac{\alpha}{1-\alpha}}c_i\right)=1-\alpha, \qquad P\left(X_i=\sqrt{\frac{1-\alpha}{\alpha}}c_i\right)=\alpha. \tag{2.13}$$

By The Law of Pure Types [see Theorem 3.26 in Breiman (1968)], either $F(\alpha,c)$ is continuous or there exists a countable set D for which $P(X\in D)=1$.

If such D exists, then by Lévy (1931) there exists a sequence of real numbers $(a_i)_{i=1}^{\infty}$ s.t.

$$\sum_{i=1}^{\infty} P(X_i\neq a_i)<\infty.$$

In our case, since $c_i<0$ for every $i\geqq 1$, we have for every sequence $(a_i)_{i=1}^{\infty}$ of real numbers $(a_i)_{i=1}^{\infty}$:

$$\sum_{i=1}^{\infty} P(X_i\neq a_i)\geqq \sum_{i=1}^{\infty} \min(\alpha,1-\alpha)=\infty,$$

which implies that $F(\alpha,c)$ is a continuous function.

Similarly, of course, if for every $i\geqq 1, d_i>0$, then $F(\alpha,d)$ is continuous. So far we have proved that when $h(m)/n(m)\to\alpha, 0<\alpha<1$, then (2.12) holds if either $(c,d)=(0,0)$, or $\|(c,d)\|<1$ and $\alpha\neq\frac{1}{2}$, or for every $i\geqq 1, c_i<0$ and $\alpha\neq\frac{1}{2}$, or for every $i\geqq 1, d_i>0$ and $\alpha\neq\frac{1}{2}$.

Assume now that $\|(c,d)\|=1$ and $c=0$ or $d=0$, and also: neither $c_i<0$ for every $i\geqq 1$, nor $d_i>0$ for every $i\geqq 1$. W.l.g. assume that $c=0$ and that there exists $N\geqq 1$ s.t.

$$d_1\geqq d_2\geqq\cdots\geqq d_N>0=d_i, \qquad i>N.$$

For $\alpha\neq\frac{1}{2}$ denote

$$X_\alpha=\sum_{i=1}^{N} X(\alpha,d_i),$$

where $(X(\alpha,d_i))_{i=1}^{\infty}$ are independent random variables satisfying (2.13) with d_i in place of c_i. The distribution function of X_α is $F(\alpha,d)$ and if $h(m)/n(m)\to\alpha$ then $Z_m\xrightarrow{D}F(\alpha,d)$. Obviously X_α attains only a finite number of values with positive probability. Hence if $P(X_\alpha=0)=0$, (2.12) holds. Thus it suffices to show that the following set is finite:

$$A=\{0<\alpha<1, \alpha\neq\tfrac{1}{2}: P(X_\alpha=0)>0\}.$$

Indeed, if $P(X_\alpha=0)>0$ then there exists a non-trivial subset D of $\{1,2,\dots,N\}$ for which

$$\sqrt{\frac{1-\alpha}{\alpha}}\sum_{i\in D} d_i=\sqrt{\frac{\alpha}{1-\alpha}}\sum_{i\in D^c} d_i.$$

Since $d_i>0$ for every $1\leqq i\leqq N$, this type of equation has exactly one solution α, $0<\alpha<1$. Hence there are no more than 2^N-2 α's for which $P(X_\alpha=0)>0$.

Corollary 1. Let μ_1 *be a measure in* $L_1(\mu)$ *which is not a scalar multiple of* μ. *Let* $(\pi_m)_{m=1}^\infty$ *be a* μ*-admissible sequence of partitions with the property that every weakly convergent subsequence of* $(c_m(\mu_1),d_m(\mu_1))$ *converges to a point in* M_1^* *and let* $\delta_m\to 0$. *Then*

$$\lim_{m\to\infty}\frac{1}{n(m)}\sum_{h=1}^{n(m)} P_m(|Z(\mu_1,\pi_m,h)|\leqq\delta_m)=0. \tag{2.14}$$

Proof. The proof follows from Theorem 3. For a detailed proof see the proof of Corollary 3.25 in Monderer (1986b). □

In view of the last corollary, it is interesting to note:

Lemma 1. Let μ_1 *be a positive measure in* $L_1(\mu)$ *which is not a scalar multiple of* μ. *Then for every* μ*-admissible sequence of partitions* $(\pi_m)_{m=1}^\infty$,

$$c_m(\mu_1)\xrightarrow{w}0.$$

In particular, every weakly convergent subsequence of $(c_m(\mu_1),d_m(\mu_1))_{m=1}^\infty$ *converges to a point in* M_1^* *(since* $c=0$*).*

Proof. For every $S\in\pi_m$ s.t. $\mu_1(S)<\mu_1(I)/n$,

$$0\leqq\frac{\mu_1(I)/n-\mu_1(S)}{D_m(\mu_1)}\leqq\frac{\mu_1(I)}{nD_m(\mu_1)}=\frac{\mu_1(I)}{\sqrt{n}}\frac{1}{\sqrt{n}\,D_m(\mu_1)}\xrightarrow[m\to\infty]{}0$$

[by Lemma 3.15 in Monderer (1986b)].

Since in $l_2\times l_2$ weak convergence of a bounded sequence is equivalent to the convergence of every component,

$$c_m(\mu_1)\xrightarrow{w}0. \quad \square$$

Theorem 4. *Let $\mu_1 \in L_1(\mu)$ be a measure which is not a scalar multiple of μ. Let $(\pi_m)_{m=1}^{\infty}$ be a μ-admissible sequence of partitions. Let $(h(m))_{m=1}^{\infty}$, $1 \leq h(m) \leq n(m)$, a sequence of positive integers satisfying*

$$\frac{h(m)}{n(m)} \to \alpha,$$

where $0<\alpha<1$, $\alpha \neq \frac{1}{2}$. Let $(s_m)_{m=1}^{\infty}$ be any sequence of real numbers. Then

$$P_m(Z(\mu_1, \pi_m, h(m)) = s_m) \to 0.$$

Proof. See proof of Theorem 3.29 in Monderer (1986b). □

Corollary 2. *Let $\lambda=(\lambda_1, \lambda_2, \ldots, \lambda_l)$, $l \geqq 2$, be a vector of mutually singular measures in $L_1(\mu)$, and let $(\pi_m)_{m=1}^{\infty}$ be a μ-admissible sequence of partitions. Then:*

(*a*) $$\lim_{m \to \infty} \sum_{i=1}^{l} A(\pi_m, \lambda_i, \lambda) = 1.$$

(*b*) *If λ is an admissible vector, then*

$$\sum_{i=1}^{l} A(\lambda_i, \lambda) = 1.$$

Proof. (b) follows from (a). We thus prove (a). For every $i \neq j$ denote: $\gamma_{ij} = \lambda_i - \lambda_j$. It is easily verified that

$$0 \leqq 1 - \sum_{i=1}^{l} A(\pi_m, \lambda_i, \lambda) \leqq \frac{1}{n} \sum_{j=1}^{n} \sum_{i \neq j} P_m(\gamma_{ij}(Q_h)=0) \xrightarrow[m \to \infty]{} 0$$

[by Theorem 4]. □

Proof of Stage II in Theorem 1 (*the case* $l=2$). In this stage we have to show that if $\lambda=(\lambda_1, \lambda_2)$, then λ is an admissible vector, and

$$A(\lambda_i, \lambda) = \tfrac{1}{2}, \qquad i=1,2. \tag{2.15}$$

Indeed, combine Theorem 6 with Proposition 19.1 of Aumann and Shapley (1974). □

We now turn to prove the remaining parts of Theorem 1. We first show, in Corollary 3, that if λ is an admissible vector and $\lambda_i \notin L_2(\mu)$, then $A(\lambda_i, \lambda) = \frac{1}{2}$.

Since Corollary 2 implies that

$$\sum_{i=1}^{l} A(\lambda_i, \lambda) = 1,$$

this will yield that for $l \geqq 3$ it is not possible to have $\lambda_i \notin L_2(\mu)$ for more than two indices i. Then we will show by Lemma 3 that if λ is admissible then for every i, $A(\lambda_i, \lambda) > 0$. This will imply that in an admissible vector there can't be more than one measure not in $L_2(\mu)$. Finally, we will show, of course, that if exactly one measure is not in $L_2(\mu)$ then λ is indeed an admissible vector.

In order to prove all that we need Theorem 5, whose proof is given in Theorem 3.35 in Monderer (1986b). In Theorem 5 we prove the existence of μ-admissible sequences with certain properties. More explicitly: we have already shown that if $\lambda_1 \in L_2(\mu)$ (and λ_1 is not a scalar multiple of μ) then $D_m(\lambda_1) \approx 1/\sqrt{n}$ in the sense that there exists $0 < t < \infty$ s.t. $\sqrt{n} D_m(\lambda_1) \xrightarrow[m\to\infty]{} t$. We have also shown that $U_m(\lambda_1)/D_m(\lambda_1) \xrightarrow[m\to\infty]{} 0$, and that if $\lambda_1 \notin L_2(\mu)$, then $\sqrt{n} D_m(\lambda_1) \xrightarrow[m\to\infty]{} \infty$ (when λ_1 is positive). In Theorem 5 we prove that there exists always a μ-admissible sequence of partitions s.t. $D_{\pi_m}(\lambda_1)/c_m \xrightarrow[m\to\infty]{} 0$ and $U_m(\lambda_1)/D_m(\lambda_1) \xrightarrow[m\to\infty]{} 0$, whenever $\sqrt{n(m)} c_m \xrightarrow[m\to\infty]{} 0$. We shall need the following:

Theorem 5. *Let $\lambda_1, \lambda_2, \ldots, \lambda_l$, $l \geqq 2$, be mutually singular probability measures in $L_1(\mu)$, $\lambda_1 \notin L_2(\mu)$. Let $(I_1, I_2, \ldots, I_l)$ be a partition of I to pairwise disjoint supports of $(\lambda_1, \lambda_2, \ldots, \lambda_l)$. Then there exists a μ-admissible sequence $(\pi_m)_{m=1}^{\infty}$, refining the partition $(I_1, I_2, \ldots, I_l)$, s.t. for every $1 \leqq i \leqq l$ all the sets in $\pi_m(I_i)$ have the same μ-measure, and s.t. the following hold:*

$$\frac{U_m(\lambda_i)}{D_m(\lambda_i)} \xrightarrow[m\to\infty]{} 0, \qquad 1 \leqq i \leqq l, \tag{2.16}$$

$$\frac{D_m(\lambda_j)}{D_m(\lambda_1)} \xrightarrow[m\to\infty]{} 0, \qquad 2 \leqq j \leqq l. \qquad \square \tag{2.17}$$

Lemma 2. *Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$, $l \geqq 2$, be a vector of mutually singular probability measures in $L_1(\mu)$. Let $P = (\pi_m)_{m=1}^{\infty}$ be a μ-admissible sequence of partitions satisfying*

$$\frac{D_m(\lambda_i)}{D_m(\lambda_1)} \xrightarrow[m\to\infty]{} 0, \qquad 2 \leqq i \leqq l. \tag{2.18}$$

Then

$$\underline{A}(P, \lambda_1, \lambda) \geqq \tfrac{1}{2}.$$

Proof. Choose a sequence $\delta_m \to 0$, s.t. for every $2 \leqq i \leqq l$,

$$\frac{D_m(\lambda_i)}{\delta_m D_m(\lambda_1)} \xrightarrow[m\to\infty]{} 0$$

$$\left(\text{for example, } \delta_m = \sqrt{\max_{i\neq 1} \frac{D_m(\lambda_i)}{D_m(\lambda_1)}}\right).$$

For every $m \geqq 1$,

$$A(\pi_m, \lambda_1, \lambda) = \frac{1}{n} \sum_{h=1}^{n} P\left(\lambda_1(Q_h) < \min_{i\neq 1} \lambda_i(Q_h)\right)$$

$$= \frac{1}{n} \sum_{h=1}^{n} P\left(Y_m(1,h) < \min_{i\neq 1} Y_m(i,h)\right),$$

where for every $1 \leqq i \leqq l$,

$$Y_m(i,h) = \frac{\lambda_i(Q_h) - h/n}{D_m(\lambda_1)\sqrt{(h/n)(1-h/n)}}$$

(note that for $i=1$, $Y_m(1,h) = Z_m(\lambda_1, \pi_m, h)$). Now it follows easily that

$$A(\pi_m, \lambda_1, \lambda) \geqq \frac{1}{n} \sum_{h=1}^{n} P(Y_m(1,h) \leqq -\delta_m; \left|Y_m(i,h)\right| < \delta_m, i \geqq 2). \tag{2.19}$$

For every $1 \leqq h \leqq n(m)$ we have, by Chebychev's inequality,

$$P(\left|Y_m(i,h)\right| \geqq \delta_m) = P\left(\left|\lambda_i(Q_h) - \frac{h}{n}\right| \geqq \delta_m D_m(\lambda_1)\sqrt{\frac{h}{n}\left(1-\frac{h}{n}\right)}\right)$$

$$\leqq \frac{1}{\delta_m^2 D_m(\lambda_1)^2 (h/n)(1-h/n)} \mathrm{E}\left(\left(\lambda_i(Q_h) - \frac{h}{n}\right)^2\right)$$

$$= \frac{D_m(\lambda_i)^2}{\delta_m^2 D_m(\lambda_1)^2} \xrightarrow[m\to\infty]{} 0 \quad \text{[by (2.11)]}.$$

This, together with (2.19), implies

$$\underline{A}(P, \lambda_1, \lambda) \geqq \varliminf_{m\to\infty} \frac{1}{n} \sum_{h=1}^{n} P(Z_m(h) \leqq -\delta_m), \tag{2.20}$$

where

$$Z_m(h) = Y_m(1, h) = Z(\mu_1, \pi_m, h).$$

Now for every order R and every $1 \leqq h \leqq n$,

$$Z_m(n-h)(R^{-1}) \geqq \delta_m \quad \text{iff} \quad Z_m(h)(R) \leqq -\delta_m.$$

Thus (2.20) yields

$$\underline{A}(P, \lambda_1, \lambda) \geqq \tfrac{1}{2} \varliminf_{m \to \infty} \frac{1}{n} \sum_{h=1}^{n} (P(Z_m(h) \leqq -\delta_m) + P(Z_m(h) \geqq \delta_m))$$

$$= \tfrac{1}{2} - \tfrac{1}{2} \varlimsup_{m \to \infty} \frac{1}{n} \sum_{h=1}^{n} P(|Z_m(h)| < \delta_m) = \tfrac{1}{2} - 0 = \tfrac{1}{2}$$

(by Corollary 1 and Lemma 1)

(remember that λ_1 is a positive measure). ☐

Corollary 3. Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l)$ *be an admissible vector and* $\lambda_1 \notin L_2(\mu)$, *then* $A(\lambda_1, \lambda) = \frac{1}{2}$.

Proof. By Corollary 3.19 in Monderer (1986b) $A(\lambda_1, \lambda) \leqq \frac{1}{2}$. By Theorem 5 there exists a μ-admissible sequence of partitions P satisfying the conditions of Lemma 2 [i.e., satisfying (2.18)]. Thus by Lemma 2,

$$A(\lambda_1, \lambda) = \underline{A}(P, \lambda_1, \lambda) \geqq \tfrac{1}{2}. \quad \square$$

Lemma 3. Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l)$, $l \geqq 3$, *be a vector of mutually singular probability measures in* $L_1(\mu)$. *Assume that* $\lambda_1 \notin L_2(\mu)$ *and that* $\lambda_3, \lambda_4, \dots, \lambda_l$ *are all in* $L_2(\mu)$. *Let* $(I_1, I_2, \dots, I_l)$ *be a partition of* I *to pairwise disjoint supports of* $(\lambda_1, \lambda_2, \dots, \lambda_l)$, *and let* $P = (\pi_m)_{m=1}^{\infty}$ *be a* μ*-admissible sequence of partitions refining* $(I_1, I_2, \dots, I_l)$. *Then:*

(*a*) *If* $\lambda_2 \in L_2(\mu)$, *then*

$$\underline{A}(P, \lambda_1, \lambda) \geqq \tfrac{1}{2} A(\lambda_3, (\lambda_2, \dots, \lambda_l)) \geqq 1/2^l.$$

(*b*) *If* $\lambda_2 \notin L_2(\mu)$ *and the following hold:*

(1) $$\frac{D_m(\lambda_2)}{D_m(\lambda_1)} \to 0,$$

(2) $\dfrac{U_m(\lambda_j)}{D_m(\lambda_j)} \to 0, \qquad \text{for } j=1,2,$

then

$$\underline{A}(P, \lambda_3, \lambda) \geqq 1/2^l.$$

Proof. See proof of Lemma 3.44 in Monderer (1986b). □

Proof of Stage III of Theorem 1. In this stage we prove that if $\lambda=(\lambda_1, \lambda_2, \ldots, \lambda_l)$, $l \geqq 3$, is an admissible vector, then at most one of the measures $\lambda_1, \lambda_2, \ldots, \lambda_l$ is not in $L_2(\mu)$. Indeed, since λ is admissible then, by Corollary 2 $\sum_{i=1}^{l} A(\lambda_i, \lambda)=1$. By Corollary 3 $A(\lambda_i, \lambda)=\frac{1}{2}$ for every i s.t. $\lambda_i \notin L_2(\mu)$. Hence there can't be more than two measures not in $L_2(\mu)$.

Now assume there were exactly two measures not in $L_2(\mu)$. By Theorem 5 we can choose a μ-admissible sequence of partitions $P=(\pi_m)_{m=1}^{\infty}$ satisfying: $D_m(\lambda_2)/D_m(\lambda_1) \xrightarrow[m \to \infty]{} 0$ and $U_m(\lambda_j)/D_m(\lambda_j) \xrightarrow[m \to \infty]{} 0$ for $j=1,2$. Then it follows from Lemma 3 that

$$A(\lambda_3, \lambda)=\underline{A}(P, \lambda_3, \lambda) \geqq 1/2^l>0,$$

which contradicts the sum of $A(\lambda_i, \lambda)$, $1 \leqq i \leqq l$, being 1. □

For the proof of the last stage of Theorem 1 we need the following two lemmas whose proofs can be found in the proofs of Lemma 3.45 and Lemma 3.46 in Monderer (1986b).

Lemma 4. Let $I_2 \subset I, 0<\mu(I_2)<1$, *and let* λ_2 *be a probability measure in* $L_2(\mu)$ *concentrated on* I_2. *Let* $(\pi_m)_{m=1}^{\infty}$ *be a* μ*-admissible sequence of partitions, and let* $(J_2^m)_{m=1}^{\infty}$ *be a sequence of subsets of* I, *s.t. for every* $m \geqq 1, J_2^m$ *belong to the algebra generated by* π_m, B_m, *and s.t.*

$$\mu(J_2^m \Delta I_2) \xrightarrow[m \to \infty]{} 0.$$

Then

$$\frac{D_{\pi_m(J_1^m)}(\lambda_2)}{D_{\pi_m(J_2^m)}(\lambda_2)} \to 0,$$

where $J_1^m=(J_2^m)^c$.

Note that if λ_2 is not in $L_2(\mu)$, then the result of Lemma 4 is no longer true. For example, let $I=[0,1)$ and $I_2=[0,\frac{1}{2})$, let μ be the Lebesgue measure and let $\mathrm{d}\lambda_2/\mathrm{d}\mu = 1/x\ln^2 x$ on I_2, let P be the standard sequence of diadic partitions, and for every $m > 1$ let $J_2^m = [\frac{1}{2}, 1) \cup [0, \frac{1}{2^m})$.

But, as we are going to show in the next lemma, it is always possible to choose a suitable sequence of subsets.

Lemma 5. *Let* $I_1 \subset I, 0 < \mu(I_1) < 1$, *and let* λ_1 *be a probability measure in* $L_1(\mu)$ *which is not in* $L_2(\mu)$, *and which is concentrated on* I_1. *Let* $P = (\pi_m)_{m=1}^{\infty}$ *be a* μ*-admissible sequence of partitions. Then there exists a sequence* $(J_1^m)_{m=1}^{\infty}$ *of subsets of* I *s.t. for every* $m \geqq 1, J_1^m \in B_m$ (*where* B_m *is the algebra generated by* π_m), *and s.t. the following hold:*

$$(a) \qquad \mu(J_1^m \Delta I_1) \to 0,$$

$$(b) \qquad \frac{D_{\pi_m(J_2^m)}(\lambda_1)}{D_{\pi_m(J_1^m)}(\lambda_1)} \to 0,$$

where $J_2^m = (J_1^m)^c$. ☐

Proof of Stage IV of Theorem 1 [*the case when exactly one measure is not in* $L_2(\mu)$]. In this stage we have to show that if $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$, $l \geqq 3$, is a vector of mutually singular probability measures s.t. $\lambda_1 \notin L_2(\mu)$ and $\lambda_2, \ldots, \lambda_l$ are all in $L_2(\mu)$, then λ is an admissible vector and

$$A(\lambda_1, \lambda) = \tfrac{1}{2},$$

and for every $2 \leqq i \leqq l$,

$$A(\lambda_i, \lambda) = \tfrac{1}{2} A(\lambda_i, (\lambda_2, \ldots, \lambda_l)).$$

So let $P = (\pi_m)_{m=1}^{\infty}$ be any μ-admissible sequence of partitions. Since $\lambda_1 \notin L_2(\mu)$ and for every $i \geqq 2$, $\lambda_i \in L_2(\mu)$, $D_m(\lambda_i)/D_m(\lambda_1) \to 0$ for every $i \geqq 2$. From this and Lemma 2 it easily follows that

$$\underline{A}(P, \lambda_1, \lambda) \geqq \tfrac{1}{2},$$

which, together with Lemma 3.18 in Monderer (1986b) yields that

$$A(\lambda_1, \lambda) = \tfrac{1}{2}.$$

Now since $(\lambda_2, \ldots, \lambda_l)$ is an admissible vector, Corollary 2 implies

$$\sum_{i=2}^{l} A(\lambda_i, (\lambda_2, \ldots, \lambda_l)) = 1.$$

Thus in order to complete our proof it suffices to show that for every μ-admissible sequence P,

$$\underline{A}(P, \lambda_i, \lambda) \geqq \tfrac{1}{2} A(\lambda_i, (\lambda_2, \ldots, \lambda_l)), \qquad i \geqq 2.$$

This was already proved in Lemma 3 for P which refines $(I_1, I_2, \ldots, I_l)$. If P does not refine $(I_1, I_2, \ldots, I_l)$, then by Lemmas 4 and 5 there exists a sequence $(J_1^m)_{m=1}^{\infty}$ of subsets of I s.t. J_1^m belongs to the algebra generated by π_m, $\mu(J_1^m \Delta I_1) \to 0$, and the following hold:

$$\frac{D_{\pi_m(J_2^m)}(\lambda_1)}{D_{\pi_m(J_2^m)}(\lambda_1)} \to 0, \tag{2.21a}$$

$$\frac{D_{\pi_m(J_1^m)}(\lambda_i)}{D_{\pi_m(J_2^m}(\lambda_i)} \to 0, \qquad 2 \leqq i \leqq l, \tag{2.21b}$$

where $J_2^m = (J_1^m)^c$.

Now continue as in the proof of Lemma 3, with $K_m^1(R)$ being the number of sets in $\pi_m(J_1^m)$ [rather than in $\pi_m(I_1)$] appearing in the order R in one of the first $h(m)$ places. Everything goes well, except for (*) and (**).

As for (*) – it is not true now that given $K_m^1 = i$ the events $(Z_m^1 \geqq \delta_m)$ and $(\lambda_3(Q_{h(m)}) < \min_{j \neq 1,3} \lambda_j(Q_{h(m)}))$ are independent. But here (2.21a) and (2.21b) imply asymptotic independence, which is good enough for our purpose. The same holds for (**).

This concludes the proof of Theorem 1.

3. The existence of the μ-asymptotic value

Theorem 6. *Let* $v = \min(\lambda_1, \lambda_2, \ldots, \lambda_l)$, *where* $\lambda_1, \lambda_2, \ldots, \lambda_l$ *are mutually singular measures in* $L_1(\mu)^1$. *Then* $\phi_\mu v$ *exists iff* $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ *is an admissible vector. Moreover, if* $\phi_\mu v$ *exists, then*

$$\phi_\mu v = \sum_{i=1}^{l} A(\lambda_i, \lambda)\lambda_i.$$

Proof. The proof follows from Theorem 4.1 in Monderer (1986b). ☐

Theorem 7. *Let* $v = \min(\lambda_1, \lambda_2, \ldots, \lambda_l)$, *where* $\lambda_1 \lambda_2, \ldots, \lambda_l$ *are mutually singular measures in* $L_1(\mu)^1$. *Then* $\phi_\mu v$ *exists iff* $l = 2$, *or* $l \geqq 3$ *and at most one of the* *measures* $\lambda_1, \lambda_2, \ldots, \lambda_l$ *is not in* $L_2(\lambda)$. *Moreover:*

- *If* $l = 2$, *then*

$$\phi_\mu v = \tfrac{1}{2}\lambda_1 + \tfrac{1}{2}\lambda_2.$$

- *If* $l \geqq 3$ *and* $\lambda_i \in L_2(\mu)$ *for every* $1 \geqq i \geqq l$, *then*

$$\phi_\mu v = \sum_{i=1}^{l} \beta_i \lambda_i,$$

where $\beta_i = A(\lambda_i, \lambda) = \mathrm{N}(\sqrt{a_i x_i} < \min_{j \neq i} \sqrt{a_j x_j})$, *where* N *is the standard normal distribution on* R_l *and* $a_i = \int (\mathrm{d}\lambda_i / \mathrm{d}\mu)^2 \, \mathrm{d}\mu$.

- *If* $l \geqq 3$, $\lambda_1 \notin L_2(\mu)$ *and* $\lambda_2, \ldots, \lambda_l \in L_2(\mu)$, *then*

$$\phi_\mu v = \tfrac{1}{2}\lambda_1 + \tfrac{1}{2}\phi_\mu w,$$

where $w = \min(\lambda_2, \ldots, \lambda_l)$.

Proof. Combine Theorem 6 with Theorem 1. □

4. The range of the λ-values

Let $\lambda_1, \lambda_2, \ldots, \lambda_l$ be mutually singular NA^1-measures and let v be the market game whose core is the convex hull of the λ_i's. Denote:

$$R = \{\phi_\lambda v : \lambda \in \mathrm{NA}^1 \text{ is s.t. } \lambda_i \in L_1(\lambda)\ \forall 1 \leqq i \leqq n \text{ and } \phi_\lambda v \text{ exists}\}.$$

Also define for every $l \geqq 3$

$$K_l = \Big\{ \beta \in R_l^{++} : \sum_{i=1}^{l} \beta_i = 1,\ \sum_{j=1}^{k} \beta_{i_j} > 2^{-l+k}\ \forall 1 \leqq k \leqq l-1$$
$$\text{and for every } 1 \leqq i_1 < i_2 < \cdots < i_k \leqq l \Big\},$$

and for every $1 \leqq i \leqq l$

$$M_l^i = \{\beta \in R_l : \beta_i = \tfrac{1}{2} \text{ and } (\beta_1, \beta_2, \ldots, \beta_{i-1}, \beta_{i+1}, \ldots, \beta_l) \in \tfrac{1}{2} K_{l-1}\},$$

where $K_2 = \{(\frac{1}{2}, \frac{1}{2})\}$.

Finally let

$$M_l = \bigcup_{i=1}^{l} M_l^i.$$

Theorem 8. *Let* $\lambda_1, \lambda_2, \ldots, \lambda_l, l \geqq 3$, *be mutually singular* NA^1*-measures. Let* $v = \min(\lambda_1, \lambda_2, \ldots, \lambda_l)$. *Then*

$$R = \left\{ \sum_{i=1}^{l} \beta_i \lambda_i : \beta \in K_l \cup M_l \right\}.$$

Proof. Combine Theorem 7 with Main Theorem of Monderer (1986c). $\square$

References

Aumann, R.J. and L.S. Shapley, 1974, Values of nonatomic games (Princeton University Press, Princeton, NJ).

Billera, L.J. and J. Raanan, 1981, Cores of nonatomic linear production games, Mathematics of Operations Research 6, no. 3.

Billingsley, P., 1968, Convergence of probability measures (Wiley, New York).

Breiman, L., 1968, Probability (Addison Wesley, Reading, MA).

Halmos, P.R., 1950, Measure theory (Van Nostrand Reinhold, New York).

Hart, S., 1977, Asymptotic values of games with a continuum of players, Journal of Mathematical Economics 4, 57–80.

Hart, S., 1980, Measure-based values of market games, Mathematics of Operations Research 5, no. 2.

Kannai, Y., 1966, Values of games with a continuum of players, Israel Journal of Mathematics 4, 57–80.

Lévy, P, 1931, Sur les séries dont les termes sont des variables éventuelles indépendentes, Studia Mathematica 3, 119–155.

Mackey, G.W., 1957, Borel structures in groups and their duals, Transactions of the American Mathematical Society 85, 134–165.

Monderer, D., 1986a, Measure-based values of nonatomic games, Mathematics of Operations Research 11, no. 2, 321–335.

Monderer, D., 1986b, A probabilistic problem arising in economics, Working paper no. 111/86 (Everyman's University, Tel-Aviv).

Monderer, D., 1986c, Asymptotic measure-based values of nonatomic games, Mathematics of Operations Research, forthcoming.

Rosen, B., 1965, Limit theorems for sampling, Arkiv för Mathematik 5, 383–424.