POTENTIALS AND WEIGHTED VALUES OF NON-ATOMIC GAMES*

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Abstract.

The "potential approach" to value theory for finite games was introduced by Hart and Mas-Colell (1989). Here this approach is extended to non-atomic games. On appropriate spaces of differentiable games there is a unique potential operator, that generates the Aumann and Shapley (1974) value. As a corollary we obtain the uniqueness of the Aumann-Shapley value on certain subspaces of games. Next, the potential approach is applied to the weighted case, leading to "weighted non-atomic values". It is further shown that the asymptotic weighted value is well-defined, and that it coincides with the weighted value generated by the potential.

1. Introduction.

A *multi-person game in coalitional form* consists of a set of *players* together with the specification of the total amount that each subset of players – called a *coalition*

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– can obtain (this is called the *transferable utility* case). One of the most appealing and successful solution concepts is that of *value*, originally due to Shapley (1953). It associates to each player in such a game a real number, which may be viewed as a kind of "expected outcome".

Recently, a new approach to value theory, called the *potential approach*, has been introduced by Hart and Mas-Colell (1989). It is based on the idea that one attaches a real number to each game, such that the resulting marginal contributions of all players always yield an efficient outcome. That is, these marginal contributions add up to the worth of the grand coalition. It was proved by Hart and Mas-Colell that this requirement characterizes precisely the Shapley value.

In this paper we consider "large games" where the set of players is modelled by a non-atomic continuum. Aumann and Shapley (1974) presented three approaches to value theory in such games, namely: the axiomatic approach, the random order approach, and the asymptotic approach. Here we add to these the potential approach. Specifically, the potential Pv of a game v (in an appropriate space of differentiable games) satisfies the following: The directional derivative $\partial(Pv)(I, S)$ of Pv at the grand coalition I in the direction S is the Aumann-Shapley value (recall that the Aumann-Shapley value ϕv of v satisfies $\phi v(S) = \int_0^1 \partial v(tI, S)dt$, and note that the potential Pv of v turns out to be $Pv(S) = \int_0^1 \frac{v(tS)}{t} dt$ for every coalition S). Moreover, we use the potential approach to prove the uniqueness of the Aumann-Shapley value on restrictable subspaces of pNA_∞ that contain NA. This is a generalization of the analogous theorems proved for finite games by Hart and Mas-Colell (1989) and by Neyman (1989). All this is done in Section 3, after the required preliminaries in Section 2.

The power and usefulness of the potential approach is then exhibited in the study of *weighted* values. Assume that a positive weight $w(s)$ is attached to each player s. The appropriate *weighted w-potential* $P_w v$ of v turns out to be

$$P_w v(S) = \int_0^1 \frac{v(t^w S)}{t} dt \quad \text{for every coalition } S,$$

where $(t^w S)(s) = t^{w(s)}$ for $s \in S$ and $= 0$ otherwise.

This enables us to define the *w-value* $\phi_w v$ of a non-atomic game v as the (directional) derivative of the w-potential, i.e. $\phi_w v(S) := \partial(P_w v)(I, wS)$. See Section 4.

Next we discuss the asymptotic approach to weighted values of non-atomic games. Here the continuum of players is partitioned into finitely many disjoint "blocks", each one of which becomes a player in the finite game generated by this partition. One then considers the values of these finite games, and their limit as the partitions become finer and finer. When there are no weights this procedure is well defined. However, in the weighted case, one needs also to assign a weight to each block Z of the partition. It is not clear at all how to do it (unless of course we are in the very special case where all players in a block have identical weights).

In Section 5 we solve this problem by showing that if the weights of all blocks Z are defined as the average weights $\frac{1}{\lambda(Z)} \int_Z w(s) d\lambda(s)$ with respect to an *arbitrary* "population measure" λ, then the resulting asymptotic w-value is *independent* of the particular λ chosen. Moreover, it coincides with the w-value obtained by the potential approach.

Other work dealing with the potential approach in large games is Pazgal (1991), which considers potential and consistency in cost allocation problems, and Hart and Mas-Colell (1995a, 1995b, 1996) which study the non-transferable utility case (where the potential leads to the egalitarian solutions and the Harsanyi NTU-value).

2. Differentiable Games.

We use the standard notations established in the book of Aumann and Shapley (1974). Let (I, C) be a standard measurable space; I is the *set of players*, and C the *set of coalitions*. A *game* v is a real-function on C with $v(\emptyset) = 0$. For a coalition S we define the *restriction* of v to S by v_S. That is, $v_S(T) = v(S \cap T)$ for every $T \in C$. A set of games M is *restrictable* if $v_S \in M$ for every $v \in M$ and for every $S \in C$. Let $B = B(I, C)$ be the Banach space of all real-valued, bounded, and Borel measurable functions on (I, C). The supremum norm of $g \in B$ is denoted by $\|g\|$. The pointwise multiplication of f and g in B is denoted by fg. That is, $(fg)(s) := f(s)g(s)$ for every $s \in I$. The integral of $f \in B$ with respect to a measure γ on (I, C) is denoted by $\gamma(f)$. We identify a set S with its indicator function 1_S, where $1_S(s) = 1$ for $s \in S$ and $= 0$ otherwise. The set of non-negative functions in B is denoted by B^+, and B^1 denotes the set of all $f \in B$ satisfying $0 \leq f \leq 1$. Let NA be the space of all non-atomic measures on (I, C). Recall that pNA is the $\| \quad \|_{BV}$ closure of polynomials of non-atomic probability measures, and that pNA' is the $\| \quad \|_{sup}$ closure of pNA. By Aumann and Shapley (see Proposition 22.16 and the discussion at Page 149), every $v \in pNA'$ has a unique uniformly NA-continuous extension to B^1. We denote this extension also by v. The group of all measurable isomorphisms of (I, C) is denoted by G. For $f \in B$ and $\theta \in G$, we denote $f \circ \theta^{-1}$ by $\theta_* f$.

We come now to our first definition. We will use the notation $\partial v(f, g)$ for the directional derivative of v at f in the direction g. A game $v \in pNA'$ is *continuously differentiable* if there exists a real-valued function $\partial v(.,.)$ on $B^1 \times B^+$ that satisfies the following three properties:

(2.1) For every $f \in B^1$ there exists $\gamma \in NA$ such that $\partial v(f, g) = \gamma(g)$ for every $g \in B^+$.

(2.2) For every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $f \in B^1$ and $g \in B^+$ with $(f + g) \in B^1$ and $\|g\| < \delta$,

$$|v(f + g) - v(f) - \partial v(f, g)| \leq \varepsilon \|g\|.$$

(2.3) $\partial v(.,.)$ is uniformly NA-continuous and bounded on $B^1 \times B^1$.

Note that, if v is continuously differentiable, then ∂v is well-defined; that is, there exists exactly one function $\partial v(.,.)$ satisfying (2.1)-(2.3).

We elaborate now on conditions (2.1)-(2.3): γ of (2.1) is the gradient of v at f. Condition (2.2) says that v is uniformly Frechet differentiable on B^1 in non-negative directions and, in particular,

$$\partial v(f, g) = \lim_{\varepsilon \to 0^+} \frac{v(f + \varepsilon g) - v(f)}{\varepsilon}$$

whenever $f \in B^1$, $g \in B^+$, and $f + \varepsilon g \in B^1$ for sufficiently small $\varepsilon > 0$. Also,

$$\partial v(f, g) = \lim_{\varepsilon \to 0^+} \frac{v(f) - v(f - \varepsilon g)}{\varepsilon}$$

whenever $f \in B^1$, $g \in B^+$, and $f - \varepsilon g \in B^1$ for sufficiently small $\varepsilon > 0$. A detailed formulation of the continuity condition (2.3) is: For every $\varepsilon > 0$ there exist $\gamma_1, \gamma_2, \ldots, \gamma_l \in NA$ and $\delta > 0$ such that for every $f_1, f_2, g_1, g_2 \in B^1$ with $|\gamma_j(f_1 - f_2)| < \delta$ and $|\gamma_j(g_1 - g_2)| < \delta$ for all $1 \leq j \leq l$,

$$|\partial v(f_1, g_1) - \partial v(f_2, g_2)| < \varepsilon.$$

Furthermore, let $M < \infty$ be a bound in (2.3), then the linearity of $\partial v(f, g)$ in g implies

(2.4) $$|\partial v(f, g)| \leq M \|g\| \quad \text{for all } f \in B^1 \text{ and } g \in B^+.$$

Altogether, "uniformly continously differentiable" (or even "uniformly continously uniformly differentiable") would have been a more appropriate term; we use "continuous differentiability" for simplicity.

Let D be the set of all continuously differentiable games in pNA'. We omit the obvious proof of the following lemma.

Lemma 2.1. *D is a linear symmetric restrictable subspace of pNA'. Moreover, for every $v \in D$, $f \in B^1$, and $g \in B^+$,*

$$\partial\theta_* v(f,g) = \partial v(\theta_* f, \theta_* g) \quad \text{for every } \theta \in G,$$

and

$$\partial v_S(f,g) = \partial v(f 1_S, g 1_S) \quad \text{for every } S \in C.$$

Denote by AC_∞ the set of all games $v \in BV$ for which there exists $\mu \in NA^+$ such that

$$|v(S) - v(T)| \leq \mu(S) - \mu(T) \quad \text{for all } T \subseteq S \text{ in } C. \tag{2.5}$$

For every $v \in AC_\infty$ define

$$\|v\|_\infty = \inf\{\mu(I) : \mu \quad \text{satisfies (2.5)}\}.$$

It can be easily verified (see Monderer (1990)) that $(AC_\infty, \| \quad \|_\infty)$ is a Banach space, that $AC_\infty \subset AC$, that $\|v\|_{BV} \leq \|v\|_\infty$ for every $v \in AC_\infty$, and that for every $v \in AC_\infty \cap pNA'$ and for every $\mu \in NA^+$ satisfying (2.5),

$$|v(f) - v(g)| \leq \mu(f) - \mu(g) \quad \text{for all } g \leq f \text{ in } B^1. \tag{2.6}$$

Let $D_\infty = D \cap AC_\infty$ be the space of continuously differentiable games in AC_∞.

Lemma 2.2. *D_∞ is $\| \quad \|_\infty$-closed in AC_∞. Moreover, for every $v \in D_\infty$*

$$|\partial v(f,g)| \leq \|v\|_\infty \|g\| \quad \text{for all } f \in B^1 \text{ and } g \in B^+. \tag{2.7}$$

Proof. Let $v \in D_\infty$. For every $f \in B^1$ with $1-\alpha \leq f \leq \alpha$ for some $0.5 < \alpha < 1$ and for every $g \in B^+$,

$$\left|\frac{v(f+\varepsilon g) - v(f)}{\varepsilon}\right| \leq \mu(g)$$

for all sufficiently small $|\varepsilon| > 0$, where μ is given by (2.5).

Therefore (2.7) is satisfied for such f and g. By the uniform continuity of $\partial v(.,.)$, (2.7) is satisfied for every $f \in B^1$ and $g \in B^+$. To prove that D_∞ is $\| \quad \|_\infty$-closed in AC_∞, let $(v_n)_{n=1}^\infty$ be a sequence in D_∞ that converges in the $\| \quad \|_\infty$ norm to $u \in AC_\infty$. As convergence in the $\| \quad \|_\infty$ norm implies convergence in the max norm, $u \in pNA'$. For all $n, m \geq 1$, and for all $f \in B^1$ and $g \in B^+$, applying (2.7) to $v_n - v_m$ yields:

$$|\partial v_n(f,g) - \partial v_m(f,g)| \leq \|v_n - v_m\|_\infty \|g\|.$$

Therefore the sequence of real numbers $(\partial v_n(f,g))_{n=1}^\infty$ converges for every such f and g, say to $\partial u(f,g)$. It can now be directly verified that $\partial u(.,.)$ satisfies (2.1)-(2.3). Hence, $u \in D_\infty$. $\square$

Obviously every polynomial of nonatomic vector measures belongs to AC_∞. Denote by pNA_∞ the $\| \quad \|_\infty$-closure of the linear space P of all polynomials of nonatomic probability measures. Obviously, $pNA_\infty \subset pNA \subset pNA'$.

Lemma 2.3. *Every game in pNA_∞ is continuously differentiable. That is, $pNA_\infty \subseteq D_\infty$.*

Proof. Let $\lambda \in NA^1$ and let n be a positive integer. For $v = \lambda^n$ we define $\partial v(f,g) = n\lambda(f)^{n-1}\lambda(g)$ for every $f \in B^1$, $g \in B^+$. Obviously, $\partial v(.,.)$ satisfies (2.1)–(2.3) implying $v \in D$. As D_∞ is linear and $\| \quad \|_\infty$-closed, $pNA_\infty \subseteq D_\infty$. $\square$

3. Potentials.

In this section we define the "potential" of a non-atomic game, and show that it yields the Aumann-Shapley value (in appropriate spaces). Recall (Hart and Mas-Colell (1989)) that for a finite game v with a player set N, the potential $u = Pv$ of v is uniquely defined by

$$(*) \qquad \sum_{s \in S} (u(S) - u(S \setminus \{s\})) = v(S) \quad \text{for all } S \subseteq N.$$

In the non-atomic case, the difference $u(S) - u(S \setminus \{s\})$ is replaced by the directional derivative $\partial u(S, \{s\})$, and the left-hand side of (*) becomes $\partial u(S,S)$ when $\partial u(f,g)$ is additive in g (as is the case for u in D; recall (2.1)). Thus we have the following formal definition.

Let $v \in pNA'$. We say that u is a *potential* for v if u is continuously differentiable (i.e., $u \in D$) and

$$\partial u(S,S) = v(S) \quad \text{for every } S \in C.$$

The set of all games in pNA' that have a potential is denoted by POT. There is no loss of generality in restricting ourselves to games in pNA'. If $v \in BV$, and $u \in D$ is a potential for v, then by the NA-continuity of $\partial u(.,.)$, v is NA-continuous. Therefore by Mertens (1980) (see also Tauman (1982)), $v \in pNA'$. For a real-valued function h on $[0,1]$, the integral $\int_0^1 h(t)dt$ is to be understood as a Lebesgue integral.

Lemma 3.1. *Every game $v \in POT$ has a unique potential Pv given by*

$$Pv(S) = \int_0^1 \frac{v(tS)}{t} dt \quad \textit{for every } S \in C.$$

Proof. Let $u \in D$ be a potential for v. By the NA-continuity of both v and $\partial u(.,.)$,

$$\partial u(f, f) = v(f) \quad \text{for every } f \in B^1. \tag{3.1}$$

Let $S \in C$, then by (3.1),

$$\partial u(tS, tS) = v(tS) \quad \text{for every } 0 \leq t \leq 1.$$

Therefore,

$$\partial u(tS, S) = \frac{v(tS)}{t} \quad \text{for every } 0 < t \leq 1. \tag{3.2}$$

Since $u \in D$, (3.2) implies that $\frac{v(tS)}{t}$ is continuous on the closed interval $[0, 1]$, when we define its value at $t = 0$ by $\partial u(0, S)$. Hence, by integrating both sides of (3.2) over $[0, 1]$, while keeping in mind that $\frac{d}{dt}(u(tS)) = \partial u(tS, S)$ and $u(0) = 0$, we obtain:

$$u(S) = \int_0^1 \frac{v(tS)}{t} dt. \quad \square$$

Corollary 3.2. *POT is a linear symmetric restrictable subspace of pNA'. Moreover, $P : POT \longrightarrow D$ is a linear positive operator, and for every $v \in POT$,*

$$P(\theta_* v) = \theta_*(Pv) \quad \textit{for all } \theta \in G,$$

and

$$P(v_S) = (Pv)_S \quad \textit{for all } S \in C.$$

Lemma 3.3. *Every continuously differentiable game has a potential. That is, $D \subseteq POT$, and for every $v \in D$,*

$$\partial(Pv)(f,g) = \int_0^1 \partial v(tf,g)dt \quad \textit{for all } f \in B^1,\ g \in B^+.$$

Moreover, if $v \in D_\infty$, then $Pv \in D_\infty$ and $\|Pv\|_\infty \leq \|v\|_\infty$.

Proof. Let $v \in D$. Then $\frac{v(tS)}{t}$ is continuous on the closed interval $[0,1]$, when its value at $t = 0$ is defined as $\partial v(0,S)\ \left(= \lim_{t\to 0^+} \frac{v(tS)}{t}\right)$. Therefore, the following operator Q is well-defined:

$$Qv(S) = \int_0^1 \frac{v(tS)}{t}dt \quad \text{for all } S \in C. \tag{3.3}$$

It is now easily verified that $Qv \in D$; indeed,

$$\partial(Qv)(f,g) := \int_0^1 \partial v(tf,g)dt$$

for every $f \in B^1$ and $g \in B^+$, satisfies (2.1)-(2.3).

Let $v \in D_\infty$. Let $\gamma \in NA^+$ satisfy (2.6) for v. Then it satisfies (2.6) for Pv by (3.3). Therefore $Pv \in AC_\infty$, and $\|Pv\|_\infty \leq \|v\|_\infty$. $\square$

Corollary 3.4. *$pNA_\infty \subseteq POT$. Moreover $Pv \in pNA_\infty$ for every $v \in pNA_\infty$.*

Remark. The following example shows that the potential operator cannot be naturally extended from pNA_∞ to pNA, even if we relax some of its differentiability properties (i.e., we do not require $Pv \in D$): Let $\lambda \in NA^1$. Define $v(S) = f(\lambda(S))$, where $f(t) = -\frac{1}{\ln(\frac{1}{2}t)}$ for every $0 < t \leq 1$ and $f(0) = 0$. We show that f is absolutely continuous on $[0,1]$. Indeed, $f'(t) = \frac{1}{t(\ln(\frac{1}{2}t))^2}$ is integrable on $[0,1]$, and $f(t) = \int_0^t f'(s)ds$ for every $t \in [0,1]$, implying the absolute continuity of f. Therefore, by Theorem C in Aumann and Shapley, $v \in pNA$. However, if $\lambda(S) > 0$,

$$\int_0^1 \frac{v(tS)}{t}dt = \int_0^1 \frac{1}{-t\ln(\frac{1}{2}t\lambda(S))}dt = \infty.$$

Hence, the potential formula of Lemma 3.1 cannot be extended to v.

For every $v \in POT$ and $S \in C$ define

$$\psi v(S) := \partial(Pv)(I, S).$$

Proposition 3.5. *ψ is a value on POT. Moreover, the restriction of ψ to pNA_∞ is the Aumann-Shapley value ϕ.*

Proof. The proof that ψ is a value follows from Corollary 3.2. By Monderer (1990), ϕ is the unique value on pNA_∞. Therefore $\psi = \phi$ on pNA_∞. $\square$

In the following theorem we prove the uniqueness of the value on restrictable subspaces of pNA_∞ that contain NA (the same proof works also for restrictable subspaces of D_∞ that contain NA). This is the non-atomic version of the analogous result for finite games proved by Hart and Mas-Colell (1989) and by Neyman (1989).

Theorem 3.6. *Let Q be a linear symmetric restrictable subspace of pNA_∞ that contains NA. Then there exists a unique value on Q. This value is the Aumann-Shapley value.*

Proof. Let $\bar{\phi}$ be a value on Q. By Monderer (1990) there exists a Borel probability measure ξ on $[0, 1]$ such that for every $w \in Q$,

$$\bar{\phi}w(S) = \int_0^1 \partial w(tI, S) d\xi(t) \quad \text{for all } S \in C. \tag{3.4}$$

For every $w \in Q$ define

$$\bar{P}w(S) = \int_0^1 \frac{w(tS)}{t} d\xi(t) \quad \text{for all } S \in C.$$

Now, $\bar{P}w \in D$:

$$\partial(\bar{P}w)(f, g) := \int_0^1 \partial w(tf, g) d\xi(t), \tag{3.5}$$

for every $f \in B^1, g \in B^+$, satisfies (2.1)-(2.3). Hence, by (3.5) and (3.4), for every $S \in C$,

$$\partial(\bar{P}w)(S,S) = \int_0^1 \partial w(tS,S)d\xi(t) = \int_0^1 \partial w_S(tI,I)d\xi(t) = \bar{\phi}w_S(I) = w(S).$$

Therefore $\bar{P}w$ is a potential for w, and by Lemma 3.1, $\bar{P}w = Pw$.

By (3.4), $\partial(\bar{P}w)(I,S) = \bar{\phi}w(S)$ for every $w \in Q$ and $S \in C$. By Lemma 3.5, $\partial(Pw)(I,S) = \phi w(S)$ for every such w and S. Hence $\bar{\phi} = \phi$ on Q. □

4. Weighted Potentials and Weighted Values.

We now generalize the approach of the previous section to the "weighted" case. That is, in addition to the game, to each player s there is given a fixed weight $w(s) > 0$. For finite games, the corresponding weighted potential $u = P_w v$ is uniquely defined by

$$(**) \qquad \sum_{s \in S} w(s)(u(S) - u(S \setminus \{s\})) = v(S) \quad \text{for all } S \subseteq N.$$

In the non-atomic case the left-hand side of (**) becomes $\sum_{s \in S} w(s)\partial u(S,\{s\})$. If $\partial u(.,.)$ is linear in the second argument (again, recall (2.1)), this equals $\partial u(S,wS)$, where $(wS)(s) = w(s)1_S(s)$ for all s.

Formally, let $w \in B^+$. We say that w is a *weight function* if it is bounded away from zero. That is, there exists $\beta > 0$ such that $w(s) \geq \beta$ for every $s \in I$. Let $v \in pNA'$. We say that u is a *weighted w-potential* for v if $u \in D$ and

$$\partial u(S,wS) = v(S) \quad \text{for all } S \in C.$$

Denote by POT_w the set of all games in pNA' that have a w-potential.

Lemma 4.1. *Every game $v \in POT_w$ has a unique w-potential $P_w v$ given by*

$$P_w v(S) = \int_0^1 \frac{v(t^w S)}{t} dt \quad \text{for every } S \in C.$$

Proof. Let $v \in POT_w$ and let $u \in D$ be a w-potential for v. By (2.4) there exists a constant $M < \infty$ such that $|\partial u(f,g)| \leq M\|g\|$ for every $f \in B^1$ and $g \in B^+$. Let $S \in C$. For every $0 < t \leq 1$ define $h(t) := u(t^w S)$. By (2.2), (2.3) and the boundedness of w, we obtain $h'(t) = \partial u(t^w S, wt^{w-1}S)$. Therefore h' is continuous on $(0,1]$. We claim that it is integrable on $[0,1]$. Indeed,

$$\int_0^1 |h'(t)|dt \leq \int_0^1 |\partial u(t^w S, wt^{w-1}S)|dt \leq \int_0^1 t^{\beta-1}|\partial u(t^w S, wS)|dt \leq \frac{M\|w\|}{\beta},$$

where β is a positive lower bound of w. Therefore, h is absolutely continuous on $[0,1]$ and $u(S) = \int_0^1 \partial u(t^w S, wt^{w-1}S)dt$. But u is a w-potential of v, therefore $\partial u(f, wf) = v(f)$ for all $f \in B^+$, implying $\partial u(t^w S, wt^{w-1}S) = \frac{v(t^w S)}{t}$ for every $t \in (0,1]$. Hence,

$$u(S) = \int_0^1 \frac{v(t^w S)}{t} dt. \quad \square$$

Corollary 4.2. *POT_w is a linear symmetric restrictable subspace of pNA', and $P_w : POT_w \longrightarrow D$ is a linear positive operator.*

The proof of the next theorem is in the spirit of all previous proofs and therefore we omit it.

Theorem 4.3. *Let w be a weight function in B^+. Every continuously differentiable game has a w-potential, that is $D \subseteq POT_w$, and for every $v \in D$,*

$$\partial(P_w v)(f,g) = \int_0^1 \partial v(t^w f, t^{w-1} g)dt \quad \text{for all } f \in B^1,\ g \in B^+.$$

Moreover, if $v \in D_\infty$, then $P_w v \in D_\infty$ and $\|P_w v\|_\infty \le \frac{1}{\beta}\|v\|_\infty$, where β is a positive lower bound of w. Consequently, $pNA_\infty \subseteq POT_w$, and $P_w v \in pNA_\infty$ for every $v \in pNA_\infty$.

For every $v \in POT_w$ and $S \in C$ define

$$\psi^w v(S) := \partial(P_w v)(I, wS). \tag{4.1}$$

Theorem 4.4. *Let w be a weight function in B^+. The operator $\psi^w : POT_w \longrightarrow NA$ is linear, positive, efficient and satisfies the projection axiom (i.e., $\psi^w \gamma = \gamma$ for every $\gamma \in NA$). Moreover, if $v \in AC_\infty$, then $\|\psi^w v\| \le \|v\|_\infty$ (where the norm $\|\mu\|$ of a measure μ on (I, C) is, as usual, $|\mu|(I)$).*

Proof. Immediate. Note that $\psi^w v \in NA$ by (2.1), and that the "moreover" statement follows from Monderer (1990): any linear positive projection on a subspace of AC_∞ that contains NA has $\| \quad \|_\infty$-norm at most 1. $\square$

We will thus call $\psi^w v$ the *weighted w-value* of v.

5. Asymptotic Weighted Values.

In the previous section the weighted value was obtained by the potential approach. An alternative method is the "asymptotic approach": Approximating the non-atomic games by finite games and considering the limit of the corresponding finite weighted values. As usual, the finite approximations are obtained by partitions of the set of players I into finitely many blocks. The problem is, which weight should one associate to each block Z ? If a "population measure" λ were given, one could take the average of the weights $w(s)$ with respect to λ, that is,

$\frac{1}{\lambda(Z)}\int_Z w(s)d\lambda(s)$. We shall show below that this approach works in the sense that, *independently* of the population measure λ, it yields in the limit precisely the weighted value obtained by the potential in Section 4.

For $\lambda \in NA^+$ we denote by $AC_0(\lambda)$ the set of all games $v \in BV$ such that every null set of λ is a null set of v. That is, $\lambda(I \setminus T) = 0$ implies that T is a carrier of v, i.e. $v(S \cap T) = v(S)$ for every coalition S. Define

$$AC_0 := \cup_{\lambda \in NA^+} AC_0(\lambda) = \cup_{\lambda \in NA^1} AC_0(\lambda).$$

We denote by $AC(\lambda)$ the set of all games in AC that are absolutely continuous with respect to λ, and we denote by $AC_\infty(\lambda)$ the set of all games v for which

$$|v(S) - v(T)| \leq \lambda(S) - \lambda(T) \quad \text{for all } T \subseteq S \in C.$$

Obviously, $AC_\infty(\lambda) \subseteq AC(\lambda) \subseteq AC_0(\lambda)$, and $AC_\infty \subseteq AC \subseteq AC_0$. Also, denote by $pNA_\infty(\lambda)$ the $\| \quad \|_\infty$-closure of the linear space generated by all games of the form μ^n, where $\mu \in NA^1$, $\mu << \lambda$ and $n \geq 1$. Note that

$$pNA_\infty = \cup_{\lambda \in NA^1} pNA_\infty(\lambda).$$

Let Π be a finite measurable partition of I. Denote by v_Π the restriction of v to the field $\mathcal{F}(\Pi)$ generated by Π. Let $\alpha = (\alpha_Z)_{Z \in \Pi}$ be a vector of positive numbers. We denote by $\phi^\alpha v_\Pi$ the weighted Shapley value of v_Π with respect to the weight vector α. $\phi^\alpha v_\Pi$ is considered as an additive game on $\mathcal{F}(\Pi)$.

Let $\mathcal{P} = (\Pi_n)_{n=1}^\infty$ be an admissible sequence of measurable partitions of I. That is, $\mathcal{P}$ is increasing and generates C. Denote by $M(\mathcal{P})$ the set of all $\lambda \in NA^1$ for which $\lambda(Z) > 0$ for every $Z \in \cup_{n=1}^\infty \Pi_n$.

Let w be a weight function and let $v, u \in AC_0(\lambda)$ for some $\lambda \in NA^1$. We say that u is an *asymptotic w-value of v with respect to* λ if, for every admissible sequence of partitions $\mathcal{P}$ with $\lambda \in M(\mathcal{P})$,

$$u(S) = \lim_{n\to\infty} \phi^{\alpha_n} v_{\Pi_n}(S) \quad \text{for all } S \in \Pi_1,$$

where $\alpha_n(Z) := \frac{1}{\lambda(Z)} \int_Z w(s)d\lambda(s)$ for every $Z \in \Pi_n$.

It can be easily verified that if v has an asymptotic w-value w.r.t. λ, then it has a unique such value, which will be denoted $\phi^w_\lambda v$. The set of all games in $AC_0(\lambda)$ that have an asymptotic w-value w.r.t. λ is denoted by $ASYMP_w(\lambda)$. Obviously, $ASYMP_w(\lambda)$ is a linear $\| \quad \|_{BV}$-closed subspace of AC_0, ϕ^w_λ is a linear operator on $ASYMP_w(\lambda)$ and $\phi^w_\lambda v \in FA$ for every $v \in ASYMP_w(\lambda)$.

Denote by $ASYMP_w$ the set of all games $v \in AC_0$ for which there exists $\lambda_0 \in NA^1$ such that

(5.1.1) $$v \in AC_0(\lambda_0);$$

(5.1.2) $$\phi^w_\lambda v \text{ exists for every } \lambda \in NA^1 \text{ with } \lambda_0 << \lambda; \text{ and}$$

(5.1.3) $$\phi^w_\lambda v = \phi^w_{\lambda_0} v \quad \text{for all } \lambda \in NA^1 \text{ with } \lambda_0 << \lambda.$$

Suppose the conditions (5.1) are satisfied by both λ_0 and λ_1. Set $\lambda_2 := \frac{\lambda_0+\lambda_1}{2}$. Then $\lambda_0 << \lambda_2$ and $\lambda_1 << \lambda_2$. Therefore $\phi^w_{\lambda_0} v = \phi^w_{\lambda_2} v = \phi^w_{\lambda_1} v$. Hence, for $v \in ASYMP_w$ we can define $\phi^w v := \phi^w_{\lambda_0} v$ for any probability measure λ_0 in NA^1 satisfying (5.1). $\phi^w v$ is called the *asymptotic w-value* of v. The proof of the following lemma is omitted.

Lemma 5.1. *$ASYMP_w$ is a linear symmetric BV-closed subspace of AC_0 that contains NA, and $\phi^w : ASYMP_w \longrightarrow FA$ is a linear efficient positive operator that satisfies the projection axiom. Moreover (in (3)-(5), $\lambda \in NA^+$)*

(1) *$\|\phi^w v\| \le \|v\|_{BV}$ for every $v \in ASYMP_w$.*

(2) *If* $w = 1_I$ *then* $ASYMP_w = ASYMP \cap AC_0$.

(3) *If* $v \in ASYMP_w \cap AC_0(\lambda)$ *then* $\phi^w v = \phi^w_\lambda v$.

(4) *If* $v \in ASYMP_w \cap AC(\lambda)$ *then* $\phi^w v \in NA$ *and* $\phi^w v << \lambda$.

(5) *If* $v \in ASYMP_w \cap AC_\infty(\lambda)$ *then* $\phi^w v \in AC_\infty(\lambda)$.

We come now to the main result of this section.

Theorem 5.2. *Let* $w \in B^+$ *be a weight function. Then* $pNA_\infty \subseteq ASYMP_w$. *Moreover* $\phi^w = \psi^w$ *on* pNA_∞, *where* ψ^w *is the weighted* w*-value.*

Proof. Let $v \in pNA_\infty$. Since $v \in pNA_\infty = \cup_{\lambda \in NA^1} pNA_\infty(\lambda)$, there exists $\lambda_0 \in NA^1$ such that $v \in pNA_\infty(\lambda_0)$.

Let $\lambda \in NA^1$, $\lambda >> \lambda_0$. Note that $v \in pNA_\infty(\lambda)$. We proceed to prove that the asymptotic w-value w.r.t. λ of v exists and that it equals the w-value of v, i.e. $\phi^w_\lambda v = \psi^w v$. Let $\mathcal{P} = (\Pi_n)_{n=1}^\infty$ be an admissible sequence of partitions with $\lambda \in M(\mathcal{P})$. We have to show that

$$\lim_{n\to\infty} |\phi^{\alpha_n} v_{\Pi_n}(S) - \psi^w v(S)| = 0 \quad \text{for all } S \in \Pi_1,$$

where $\alpha_n(Z) = \frac{1}{\lambda(Z)} \int_Z w(s) d\lambda(s)$ for every $Z \in \Pi_n$.

By Monderer and Neyman (1988), for every $n \geq 1$ there exists a linear positive operator $E^\lambda_n : pNA_\infty(\lambda) \to pNA_\infty(\lambda)$ such that $E^\lambda_n u$ is a multilinear game for every $u \in pNA_\infty(\lambda)$. That is, $E^\lambda_n u = F_n \circ (\lambda_Z)_{Z \in \Pi_n}$, where F_n is a multilinear function on $\times_{Z \in \Pi_n}[0, \lambda(Z)]$. Moreover,

$$\lim_{n\to\infty} \|E^\lambda_n(u) - u\|_\infty = 0 \quad \text{for all } u \in pNA_\infty(\lambda).$$

Let $S \in C$. Set $u_n := E^\lambda_n v$ and

$$\delta_n := |\phi^{\alpha_n} v_{\Pi_n}(S) - \psi^w v(S)|.$$

We have to prove that $\lim_{n\to\infty} \delta_n = 0$. Now

$$\delta_n \le |\phi^{\alpha_n} v_{\Pi_n}(S) - \phi^{\alpha_n}(u_n)_{\Pi_n}(S)| + |\phi^{\alpha_n}(u_n)_{\Pi_n}(S) - \psi^w v(S)|.$$

Therefore

$$\delta_n \le \|\phi^{\alpha_n} v_{\Pi_n} - \phi^{\alpha_n}(u_n)_{\Pi_n}\| + |\phi^{\alpha_n}(u_n)_{\Pi_n}(S) - \psi^w v(S)|. \tag{5.2}$$

Note that ϕ^{α_n} is a linear positive projection on the space of all additive games on Π_n and therefore by Monderer (1988),

$$\|\phi^{\alpha_n} v_{\Pi_n} - \phi^{\alpha_n}(u_n)_{\Pi_n}\| \le \|v_{\Pi_n} - (u_n)_{\Pi_n}\|_\infty.$$

By Monderer and Neyman (1988),

$$\|v_{\Pi_n} - (u_n)_{\Pi_n}\|_\infty \le \|v - u_n\|_\infty,$$

and since

$$\lim_{n\to\infty} \|v - u_n\|_\infty = 0,$$

it remains to prove that

$$\lim_{n\to\infty} |\phi^{\alpha_n}(u_n)_{\Pi_n}(S) - \psi^w v(S)| = 0. \tag{5.3}$$

Let $w_n \in B^+$ be the conditional expectation of w with respect to the field $\mathcal{F}(\Pi_n)$ and the probability measure λ. That is, for $s \in Z \in \Pi_n$,

$$w_n(s) := \alpha_n(Z) = \frac{1}{\lambda(Z)} \int_Z w d\lambda. \tag{5.4}$$

Note that every positive lower bound of w is also a lower bound for w_n for every $n \ge 1$. Therefore, w_n is indeed a weight function. By (5.4) and Owen (1972),

$$\phi^{\alpha_n}(u_n)_{\Pi_n}(S) = \int_0^1 \partial u_n(t^{w_n} I, w_n t^{w_n - 1} S) dt.$$

Therefore (recall (4.1) and Theorem 4.3),

$$\phi^{\alpha_n}(u_n)_{\Pi_n}(S) = \psi^{w_n} u_n(S). \tag{5.5}$$

By Theorem 4.4

$$|\psi^{w_n}(u_n - v)(S)| \leq \|u_n - v\|_\infty \to 0 \quad \text{as } n \to \infty. \tag{5.6}$$

By (5.5) and (5.6), to prove (5.3) we have to show that

$$\lim_{n\to\infty} |\psi^{w_n} v(S) - \psi^w v(S)| = 0. \tag{5.7}$$

As $v \in pNA_\infty(\lambda)$, it can be easily verified (see Aumann and Shapley, Chapter IV, for a proof of a similar result) that $\partial v(.,.)$ is continuous on $B^1 \times B^+$ in the $NA(\lambda)$ topology, where $NA(\lambda) := AC(\lambda) \cap NA$. By the martingale convergence theorem, $w_n \to w$ (λ-a.s.) as $n \to \infty$. Therefore, for every $0 < t \leq 1$, $t^{w_n} I \to t^w I$ and $w_n t^{w_n - 1} S \to w t^{w-1} S$ in the $NA(\lambda)$ topology. By the continuity of $\partial v(.,.)$ in the $NA(\lambda)$ topology, it implies for every $0 < t \leq 1$

$$|\partial v(t^{w_n} I, w_n t^{w_n - 1} S) - \partial v(t^w I, w t^{w-1} S)| \to 0. \tag{5.8}$$

Let β be a positive lower bound for w, thus also for w_n. By (2.4) there exists a bound M such that

$$|\partial v(t^{w_n} I, w_n t^{w_n - 1} S) - \partial v(t^w I, w t^{w-1} S)| \leq 2M\|w\| t^{\beta - 1} \tag{5.9}$$

(note that $\|w_n\| \leq \|w\|$ for every $n \geq 1$). The function $t^{\beta-1}$ is integrable on $[0,1]$, thus (5.8) and (5.9) yield by the Lebesgue dominance convergence theorem,

$$|\psi^{w_n} v(S) - \psi^w v(S)| \leq \int_0^1 |\partial v(t^{w_n} I, w_n t^{w_n - 1} S) - \partial v(t^w I, w t^{w-1} S)| dt \to 0.$$

Hence, (5.7) is proved. □

We conclude by noting that even though the potential approach applies only to a strict subspace of pNA (recall the example at page 11), its use of enable us to obtain asymptotic weighted values for *all* of pNA. Indeed, the existence of the weighted potential and the weighted value on pNA_∞ implies the existence of the asymptotic weighted value on pNA_∞; the latter then naturally extended to pNA, the BV-closure of pNA_∞ by its BV continuity property. Thus we have:

Corollary 5.3. *Let w be a weight function, then $pNA \subseteq ASYMP_w$.*

Proof. By Lemma 5.1 and Theorem 5.2, $ASYMP_w$ is BV-closed and contains pNA_∞, which is BV-dense subset of pNA. $\square$

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