

# Multipotential Games <sup>\*</sup>

Dov Monderer

Technion–Israel Institute of Technology

E-mail: dov@ie.technion.ac.il

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## Abstract

We introduce and analyze  $q$ -potential games and  $q$ -congestion games, where  $q$  is a positive integer. A 1-potential (congestion) game is a potential (congestion) game. We show that a game is a  $q$ -potential game if and only if it is ( up to an isomorphism) a  $q$ -congestion game. As a corollary we derive the result that every game in strategic form is a  $q$ -congestion game for some  $q$ . It is further shown that every  $q$ -congestion game is isomorphic to a  $q$ -network game, where the network environment is defined by a directed graph with one origin and one destination. In addition we discuss the issue of representing  $q$ -congestion games with non-negative cost functions by congestion models with non-negative and monotonic facility cost functions.

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# 1 Introduction

**NOTE:** In this paper we consider games in strategic form in which the data consists of cost functions and not of payoff functions.

Models of congestion come with many real-life stories and in various mathematical forms. They seem to originate at transportation engineering Wardrop [1952], and they have been analyzed by several researchers from various additional fields, in particular computer science, communication networks, and economics–game theory.<sup>1</sup> Every congestion model gives rise to a game in strategic form (normal form). This paper deals mainly with the existence and properties of representations of games in strategic form (normal form) by congestion models. A representative existence theorem is the one of Monderer and Shapley [1996], stating that every potential game is isomorphic to a congestion game.<sup>2</sup> Our starting point is the model of Rosenthal [1973].

A congestion form is defined by a finite set of players, each of which holding one unit of goods, a finite set of facilities, and per-unit cost functions associated with the facilities. Each player must use a subset of facilities in order to make its unit of goods valuable. The non-empty set of feasible subsets of facilities is player-specific. When a player chooses a subset of facilities her per-facility cost depends on the number of other players that

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<sup>1</sup>See Altman and Wynter [2004] for a commendable attempt to unify the research.

<sup>2</sup>Some of the congestion models are represented by graphs. Other graphical representations of games have been analyzed in the literature of computers science and artificial intelligence. In some of these representations the focus is on dependencies among players' utility functions (see, e.g., Kearns *et al.* [2001]; Koller and Milch [2001]; Mura [2000]; Vickrey and Koller [2002]). Other types of representations focus on actions' dependencies– see Leyton-Brown and Tennenholtz [2003]; N. Bhat and K. Leyton-Brown [2004].

decide to use the facility, and her total cost is the sum of costs of the facilities in this subset.

Each congestion form  $F$  defines a game in strategic form,  $\Gamma_F$ , which is called a congestion game. In  $\Gamma_F$  the strategy set of a player is her set of feasible subsets of facilities, and her cost function is as described above. The distinction between congestion forms and congestion games is important. The form, which is also a sort of game, contains more information than its associated game in strategic form. Two distinct forms may induce isomorphic games. This distinction resembles the distinction between an extensive-form game and its associated strategic form game. For example, a congestion game rarely reveals the structure of the facility cost functions. Indeed, one can think on various natural solution concepts for a congestion form that have no sense in its associated game. For example, the players can choose the facilities sequentially, which gives rise to a multistage game in which we can deal with subgame perfect equilibria and other natural game theoretic solution concepts in multistage games. On the other hand, many interesting conceptual or computational concepts for congestion forms require as input only the game. For example, Nash equilibrium, strong equilibrium, correlated equilibrium, social surplus, and price of anarchy. Two congestion forms are said to be equivalent if they generate isomorphic games.<sup>3</sup>

We proceed to discuss a natural generalization of congestion forms (games), in which the facility cost functions are player-specific. In many situations it is natural to assume that the cost functions are player-specific. Forms with player-specific cost functions are called PS-congestion forms, and their associated games in strategic form are called PS-congestion games. We define

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<sup>3</sup>Unfortunately, there is no explicit distinction in the CS/AI literature between forms and games. Both are referred to as congestion games.

and discuss PS-congestion forms (games) of type  $q$ , or in short,  $q$ -congestion forms (games), where  $q$  is a positive integer. A PS-congestion form is of type  $q$  if the set of vectors of facility cost functions contains at most  $q$  distinct vectors, that is there exist at most  $q$  types of players. A 1-congestion form (game) is a congestion form (game). Of course,  $n$ -congestion games cover all  $n$ -person PS-congestion games. However, using a generalization of potential games described below, we actually show that  $n$ -congestion games cover all  $n$ -player games in strategic form.

We introduce a new type of games:  $q$ -potential games. A 1-potential game is simply a potential game in the sense of Monderer and Shapley [1996]. Roughly speaking, a game is a  $q$ -potential game if the set of players can be partitioned into  $q$  non-empty and mutually disjoint subsets  $N_j$ ,  $1 \leq j \leq q$ , in such a way that if we fix the strategies of all players outside  $N_j$ , the remaining sub-game is a potential game. We show that the proof of equivalence between potential games and congestion games can be extended. That is, every  $q$ -congestion game is a  $q$ -potential game, and every finite  $q$ -potential game is isomorphic to a  $q$ -congestion game. We notice that every game is an  $n$ -potential game, where  $n$  is the number of players. Therefore, as claimed above, every game in strategic form is isomorphic to an  $n$ -congestion game. Hence, we get the somewhat surprising result that every game in strategic form is isomorphic to a PS-congestion game.<sup>4</sup>

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<sup>4</sup>In some contexts it is useful to consider congestion forms, which generalize PS-congestion forms. In such forms the facility cost functions are not only player-specific, but also depend on the identity of the users of the facility. We denote such forms by ID-congestion forms. Classical ID-congestion forms are congestion forms with weights. Relationships between congestion games, ID-congestion games, and the Shapley value were given for example in Monderer and Shapley [1996]; Ui [2000]; Monderer [2006]. Another type of generalization of congestion forms is local-effect games Leyton-Brown and

Next we discuss the efficiency of representation of games by PS-congestion forms. As we said, a PS-congestion form contains more information than its associated game. However, as was noticed, e.g., in Papadimitriou [2005], the form has, in general, a more compact representation. For example, in a PS-congestion form of type  $q$ , with  $m$  facilities and  $n$  players the facility cost functions are described by  $qmn$  numbers. If in addition the size of each strategy set is  $s$ , then each strategy set can be described by at most  $sm$  numbers (assuming each facility is described by a number), and hence the description of the form requires at most  $qmn + smn$  numbers, which is bounded above by  $mn^2 + smn$ . The standard description of the associated congestion game requires  $ns^n$  numbers. Hence, if  $m$  is not too big the representation of the form is more succinct than the standard representation of its associated game.

One may worry that computation of equilibrium is not efficient for succinct representations of the data. However, in Papadimitriou [2005] it was shown that computation of correlated equilibrium in many classes of forms with succinct representation, including the class of congestion forms is efficient. It can be easily seen that the proof that congestion forms satisfy the polynomial expectation property in Papadimitriou [2005] actually shows that PS-congestion forms have this property. Hence, by Theorem 4.1 in Papadimitriou [2005] every PS-congestion form has a polynomial correlated equilibrium scheme.

We next proceed to discuss the issue of faithful representations. Given Tennenholtz [2003], in which nodes in a graph represent actions, cost functions are associated with nodes, and every such cost function depends on the number of users of the neighboring nodes. These type of games were generalized in N. Bhat and K. Leyton-Brown [2004] to action-graph games, which were proved to have full expressive power.

a PS-congestion form with non-negative cost functions, its associated game also has non-negative cost functions. On the other hand, given a  $q$ -potential game with non-negative cost functions, the representing  $q$ -congestion form generated in the proof of our representation theorem may have facility cost functions with negative values. It is an open question for us whether every  $q$ -potential game with non-negative cost functions can be represented by a  $q$ -congestion form with non-negative facility cost functions. However, we present some partial results.

Many applications of congestion forms/games naturally come from networks. Hence, one may wish to consider a special type of congestion forms/games, in which facilities are edges in graphs and feasible subset of facilities are routes. It is natural to call such congestion forms network forms, and their associated congestion games network games. A natural question is how much we lose when we restrict attention to network forms. There are many modeling choices to make. We have decided to take the seemingly most restrictive definition: A network form is defined by a directed graph with only one origin and one destination. Nevertheless, we prove that dealing with networks does not restrict the generality. We prove that every congestion form is equivalent to a network form, or equivalently, that every congestion game is isomorphic to a network game. Hence, all other potential candidates for the term network games are isomorphic to network games in our sense, because they are in particular congestion games. The proof is based on a transformation that transforms every congestion form to an equivalent network form. The number of facilities in the network form is twice the number of facilities in the congestion form. Hence we do not lose computational efficiency. At this point it is worth mentioning that the transformation mentioned above, transforming congestion forms to equivalent network forms, preserves the

properties of both non-negativity and monotonicity. We actually prove the stronger result” For every  $q \geq 1$ , every  $q$ -congestion form is equivalent to a  $q$ -network form.

## 2 $q$ -Potential Games and $q$ -Congestion games.

### 2.1 Isomorphic games

A *game in strategic form* is a tuple  $\Gamma = (N, (X_i)_{i \in N}, (C_i)_{i \in N})$ , where  $N$  is a finite set of players which, whenever convenient, we take to be  $\{1, \dots, n\}$ ;  $X_i$  is a set of strategies for  $i$ ; and  $C_i : X \rightarrow R$  is the cost function of  $i$ , where  $X = \times_{i \in N} X_i$ .  $\Gamma$  is called a *finite game* if the sets of strategies are finite sets. We say that two games  $\Gamma^1$  and  $\Gamma^2$  are *isomorphic* if each of them is obtained from the other by changing the names of the players and the names of the strategies. That is, there exist bijection functions (i.e., functions which are both one-to-one and onto)  $\tau : N^1 \rightarrow N^2$  and  $\alpha_i : X_i^1 \rightarrow X_{\tau(i)}^2$ ,  $i \in N^1$ , such that:

for every  $i \in N^1$  and for every  $(x_j^1)_{j \in N^1} \in X^1$ ,

$$C_{\tau(i)}^2((\alpha_j(x_j^1))_{j \in N^1}) = C_i^1((x_j^1)_{j \in N^1}).$$

### 2.2 Potential games

Let  $\Gamma = (N, (X_i)_{i \in N}, (C_i)_{i \in N})$  be a game in strategic form. Let  $X_{-i}$  denotes the set of strategy profiles of all players but  $i$ . A function  $P : X \rightarrow R$  is a

potential function<sup>5</sup> for  $i$  if for every  $x_i, y_i \in X_i$ , and for every  $x_{-i} \in X_{-i}$ ,

$$C_i(x_i, x_{-i}) - C_i(y_i, x_{-i}) = P(x_i, x_{-i}) - P(y_i, x_{-i}).$$

Following Monderer and Shapley [1996],  $\Gamma$  is a potential game<sup>6</sup> if there exists a function  $P$  which is a potential for every player  $i$ .

## 2.3 Congestion forms and congestion games

For the basic model of congestion forms and congestion games we follow Rosenthal [1973] and Monderer and Shapley [1996].

A Congestion Form is a tuple  $F = (M, N, (\Sigma_i)_{i \in N}, (c_a)_{a \in M})$ , where  $M$  is a finite set consisting of  $m$  elements, which are called facilities,  $N$  is a finite set consisting of  $n$  elements, which are called players; For every  $i \in N$ ,

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<sup>5</sup>In Monderer and Shapley [1996], four types of potential functions were defined: exact potential, weighted potential, ordinal potential, and generalized ordinal potential. The term "potential" was used interchangeably with the term "exact potential". In recent literature, some other types of potentials have been considered. For example, best-response potential Voorneveld [2000], various types of strong potentials Holzman and Law-yone (Lev-tov) [1997], potential functions for mechanism design Jehiel *et al.* [2004], several types of potential functions that represent various acyclicity properties Kukushkin [2002, 2004], and generalized potential functions Moriss and Ui [2004]. In some other works, the term potential is used for ordinal or generalized ordinal potentials. In this paper we keep the terminology of Monderer and Shapley [1996].

<sup>6</sup>"Potential game" were defined in Monderer and Shapley [1996]. Potential games in the differentiable setup and discrete 2-person potential games were previously discussed in Thépot [1980, 1981], where they are called centralizable games. However, potential functions for various types of games have been used in the literature in several research fields much earlier. See e.g., Wardrop [1952]; Beckmann *et al.* [1956]. Additional references can be found in Monderer and Shapley [1996]. Non-atomic potential games were defined in Sandholm [2001].

$\Sigma_i \subseteq 2^M \setminus \{\emptyset\}$  is a non empty set of subsets of facilities, which is called the feasible set of  $i$ , and for every  $a \in M$   $c_a : [0, \infty) \rightarrow \mathfrak{R}$  is the per-unit facility cost function associated with  $a \in M$ ; If  $k$  of the users choose  $a$ , each of them pays  $c_a(k)$ .

Every congestion form  $F = (M, N, (\Sigma_i)_{i \in N}, (c_a)_{a \in M})$  defines a game in strategic form  $\Gamma_F$ , in which the set of players is  $N$ ,  $\Sigma_i$  is the set of strategies of  $i$ , and for every  $i \in N$  the cost function of player  $i$  is defined on  $\Sigma = \times_{i \in N} \Sigma_i$  as follows:

$$C_i(A) = C_i(A_1, \dots, A_n) = \sum_{a \in A_i} c_a(n_a(A)),$$

where  $n_a(A) = |\{j \in N : a \in A_j\}|$ .<sup>7</sup> A game  $\Gamma$  in strategic form is called a congestion game if  $\Gamma = \Gamma_F$  for some congestion form  $F$ . Two congestion forms are equivalent if they generate isomorphic congestion games.

## 2.4 Player-specific facility cost functions

When the cost functions associated with the facilities are player-specific we get a congestion form with player-specific facility cost functions or, in short a PS-congestion form. Formally: A PS-Congestion Form is a tuple  $F = (M, N, (\Sigma_i)_{i \in N}, ((c_a^i)_{a \in M})_{i \in N})$  such that all components except for the cost functions are defined as in a congestion form, and  $c_a^i : [0, \infty) \rightarrow \mathfrak{R}$  is the  $i$ -per-unit facility cost function associated with  $a \in M$ ; If  $k$  of the users choose  $a$ , agent  $i$  pays  $c_a^i(k)$ . A PS-congestion form is facility-symmetric if  $\Sigma_i = \Sigma_j$  for every  $i, j \in N$ . A PS-congestion form is simple if  $\Sigma_i$  contains only singletons for every player  $i$ . Every PS-congestion form  $F =$

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<sup>7</sup>Hence, only the values of  $c_a$  on the set of integers  $\{1, \dots, n\}$  are relevant. However, it will be useful later, and it does not restrict the generality, to define  $c_a$  on the whole interval  $[0, \infty)$ .

$(M, N, (\Sigma_i)_{i \in N}, ((c_a^i)_{a \in M})_{i \in N})$  uniquely defines a game in strategic form  $\Gamma_F$ , in which the set of players is  $N$ ,  $\Sigma_i$  is the set of strategies of  $i$ , and for every  $i \in N$  the cost function of player  $i$  is defined as follows:

$$c_i(A) = c_i(A_1, \dots, A_n) = \sum_{a \in A_i} c_a^i(n_a(A)).$$

A game  $\Gamma$  in strategic form is called a PS-congestion game if  $\Gamma = \Gamma_F$  for some PS-congestion form  $F$ . Thus, every congestion form is a PS-congestion form, and every congestion game is a PS-congestion game.<sup>8</sup>

## 2.5 $q$ -Congestion forms and $q$ -congestion games

Roughly speaking, a PS-congestion form is of type  $q$  if the players can be partitioned into  $q$  types, where two players are of the same type if they share the same facility cost functions. Formally, Let  $F = (M, N, (\Sigma_i)_{i \in N}, ((c_a^i)_{a \in M})_{i \in N})$  be a PS-congestion form. A finite set  $K \subseteq (\mathfrak{R}^{[0, \infty)})^M$  is a cover for  $F$  if for every player  $i$  there exists  $c = (c_a)_{a \in M} \in K$  such that for every  $a \in M$   $c_a^i = c_a$ .

Let  $q$  be a positive integer. We say that  $F$  is a  $q$ -congestion form if it has a cover  $H$  with  $|H| \leq q$ . Obviously, if  $F$  is a  $q$ -congestion form,  $F$  is a  $(q + 1)$ -congestion form. Because  $K = \{(c_a^1)_{a \in M}, \dots, (c_a^n)_{a \in M}\}$  is a cover for  $F$ , every  $n$ -person PS-congestion form is an  $n$ -congestion form. Let  $1 \leq q(F) \leq n$  be the index of  $F$ , defined as the minimal cardinality of a cover for  $F$ . Obviously,  $F$  is a congestion form if and only if  $q(F) = 1$ . If  $F$  is a  $q$ -congestion form,  $\Gamma_F$  is called a  $q$ -congestion game.

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<sup>8</sup>Simple and facility-symmetric PS-congestion forms/games were discussed, e.g., in Quint and Shubic [1994]; Milchtaich [1996]; Voorneveld *et al.* [1999]; Konishi *et al.* [1997].

## 2.6 $q$ -Potential games

Let  $\Gamma = (N, (X_i)_{i \in N}, (C_i)_{i \in N})$  be a game in strategic form. Let  $H$  be a set of real-valued functions defined on  $X$ . We say that  $H$  is a cover of  $\Gamma$  if for every  $i \in N$  there exists  $P \in H$ , which is a potential function for  $i$ . Let  $q$  be a positive integer. We say that  $\Gamma$  is a  $q$ -potential game if it has a cover  $H$  with  $|H| \leq q$ . Obviously, if  $\Gamma$  is a  $q$ -potential game,  $\Gamma$  is a  $(q+1)$ -potential game. Because  $C_i$  itself is a potential for  $i$ , every  $n$ -person game is an  $n$ -potential game. Let  $1 \leq q(\Gamma) \leq n$  be the potential index of  $\Gamma$  defined as the minimal cardinality of a cover for  $\Gamma$ . Obviously,  $\Gamma$  is a potential game if and only if  $\Gamma$  is a 1-potential game if and only if  $q(\Gamma) = 1$ .

Let  $\Gamma$  be a  $q$ -potential game. Let  $H = \{P_1, \dots, P_q\}$ ,  $|H| = q$ , be a cover of  $\Gamma$ , and let  $H_s$  be the set of all players  $i$  such that  $P_s$  is a potential function for  $i$ . A partition of the player set  $N$  to  $q$  nonempty and mutually disjoint subsets  $\pi = (N_s)_{s=1}^q$ , is consistent with  $H$  if  $P_s$  is a potential function for every  $i \in N_s$ , that is,  $N_s \subseteq H_s$  for every  $1 \leq s \leq q$ . In a potential game with a potential function  $P$ , all players behaves as if there exists one player whose goal is to minimize  $P$  over  $X$ . In a  $q$ -potential game with a cover  $H = \{P_1, \dots, P_q\}$  and an  $H$ -consistent partition  $\pi = (N_s)_{s=1}^q$ , the players behave as if there are  $q$  players,  $I_s$ ,  $1 \leq s \leq q$ , playing a  $q$ -person game with the set of strategies  $X_{[s]}$  for player  $I_s$ , where  $X_{[s]} = \times_{i \in N_s} X_i$ . For  $x = (x_{[s]})_{1 \leq s \leq q}$ , the cost function of  $I_s$  is  $P_s$ . Note that every equilibrium  $x$  in the associated  $q$ -person game is also an equilibrium in the original game. Similarly, every correlated equilibrium corresponds to a correlated equilibrium.

Unfortunately, as is shown in the next example, a cover with a minimal cardinality may have more than one consistent partition. In particular, the partition of the players' set to  $q$  subsets in a  $q$ -potential game with an index  $q$  is not uniquely determined by the game. Bellow is an example for a 3-player

game with a potential index 2 in which both partitions 12, 3 and 1, 23 are consistent with the same cover  $H$ , where  $|H| = 2$ .

**Example 1** We construct a game  $\Gamma$ . The strategy set of every player  $1 \leq i \leq 3$  is  $\{0, 1\}$ . The cost functions are:  $C_1(x) = C_1(x_1, x_2, x_3) = x_2x_3$ ,  $C_2(x) = x_1x_3$ ,  $C_3(x) = x_1x_3 + x_1x_2$ . We first show that this is not a 1-potential game. Indeed, by Monderer and Shapley [1996] it suffices to show that there exists a closed path of strategy profiles in  $X$ ,  $\gamma = x(0), x(1), x(2), x(3), x(4)$  with  $x(0) = x(4)$ , such that  $x(t+1)$  is obtained from  $x(t)$  by changing the strategy of exactly one player,  $i_t$ ,  $0 \leq t \leq 3$ , and such that  $I(\Gamma) \neq 0$ , where

$$I(\gamma) = \sum_{t=0}^3 [C_{i_t}(x(t+1)) - C_{i_t}(x(t))].$$

Indeed, for the path  $\gamma = (0, 1, 0), (1, 1, 0), (1, 1, 1), (0, 1, 1), (0, 1, 0)$ ,  $I(\gamma) = -1$ . Observe that  $P(x) = 0$  for every  $x \in X$  is a potential function for both player 1 and player 2, and that  $C_2$  is a potential function for both player 2 and player 3. Hence,  $H = \{0, C_2\}$  is a cover for  $\Gamma$ , and both partitions 12, 3 and 1, 23 are consistent with  $H$ . ■

*Detecting potential functions is easier in a differentiable setup. The next lemma is useful:*

**Lemma 1** Let  $\Gamma = (N, (X_i)_{i \in N}, (C_i)_{i \in N})$  be a game in strategic form such that  $X_i$  is a convex set of full dimension in some Euclidean space,  $\mathfrak{R}^{m_i}$ . Assume in addition that all payoff functions are twice continuously differentiable. Let  $N_1 \subseteq N$ .

1.  $P : X \rightarrow \mathfrak{R}$  is a potential function for every player  $i \in N_1$  if and only if  $P(x_{N_1}, x_{-N_1})$  is twice continuously differentiable in  $x_{N_1}$ , and  $\frac{\partial P}{\partial x_{i_j}} = \frac{\partial C_i}{\partial x_{i_j}}$  for every  $i \in N_1$ , and for every  $1 \leq j \leq m_i$ .

2. There exists a joint potential function for all players in  $N_1$  if and only if for every  $i, k \in N_1$  and for every  $1 \leq j \leq m_i, 1 \leq s \leq m_k$ ,

$$\frac{\partial^2 C_i}{\partial x_{ij} \partial x_{ks}} = \frac{\partial^2 C_k}{\partial x_{ij} \partial x_{ks}}. \quad < 1 >$$

3.  $\Gamma$  is a  $q$ -potential game if and only if there exists a partition of the set of players to  $q$  mutually disjoint non-empty subsets  $(N_t)_{t=1}^q$  such that  $< 1 >$  holds for every  $1 \leq t \leq q$ , for every two players  $i, k \in N_t$ , and for every  $1 \leq j \leq m_i, 1 \leq s \leq m_k$ .

**Proof:** As observed in Monderer and Shapley [1996], the assertion holds for the case  $q = m_1 = \dots = m_n = 1$ . The proof of the general case is obvious as well, and therefore it is omitted. ■

## 2.7 An example to a multipotential game

Consider an oligopoly multi-product quantity competition. There are  $q$  products and  $n$  firms. The set of firms is partitioned to  $q$  nonempty and mutually disjoint subsets  $N_1, \dots, N_q$ . Each firm in  $N_q$  produces product  $j$ . The inverse demand function for product  $j$  is  $F_j(Q_1, \dots, Q_q)$ , where  $Q_k$  is the total supply of product  $k$ . Each firm  $i$  decides about the level of production  $x_i \geq 0$  of the product it produces. The cost of production  $x_i$  units for firm  $i$  is  $c_i(x_i)$ . The profit of firm  $i \in N_j$  is:

$$\pi_i(x_1, \dots, x_n) = x_i F_j \left( \sum_{t \in N_1} x_t, \dots, \sum_{t \in N_q} x_t \right) - c_i(x_i).$$

Assume all inverse demand functions are affine, and the cost functions are continuously differentiable. It is obvious that for every  $j$  and for every two players  $i, k \in N_j$ ,

$$\frac{\partial^2 \pi_i}{\partial x_i \partial x_k} = \frac{\partial^2 C \pi_k}{\partial x_i \partial x_k}.$$

Hence, the competition generates a natural  $q$ -potential game by applying the payoff version of Lemma 1.

## 2.8 Representation of $q$ -potential games by $q$ -congestion forms

It was proved in Rosenthal [1973] that every congestion game is a potential game. It was proved in Monderer and Shapley [1996] that every finite potential game is isomorphic to a congestion game. The two theorems are extended in this section.

### **Theorem 1**

- (1) Every  $q$ -congestion game is a  $q$ -potential game.
- (2) Every finite  $q$ -potential game is isomorphic to a  $q$ -congestion game.

**Proof of Theorem 1 (long version):**

- (1) Let  $F = (M, N, (\Sigma_i)_{i \in N}, ((c_a^i)_{a \in M})_{i \in N})$  be a  $q$ -congestion form. Let  $K = \{(c_a^{[s]})_{a \in M} : 1 \leq s \leq q\}$  be cover for  $F$ .

Let  $s$  be an integer,  $1 \leq s \leq q$ . Let  $F^{[s]} = (M, N, (\Sigma_i)_{i \in N}, (c_a^{[s]})_{a \in M})$  be the congestion form obtained from  $F$  by replacing all cost functions in  $F$  with  $(c_a^{[s]})_{a \in M}$ .

Let  $P_s$  be the potential function associated by Rosenthal [1973] with the congestion form  $F^{[s]}$ . That is,

$$P_s(A) = P_s(A_1, \dots, A_n) = \sum_{a \in \cup_{i=1}^n A_i} \sum_{k=1}^{n_A(a)} c_a^{[s]}(k).$$

Let  $i \in N$ , and let  $s$  satisfy  $c_a^i = c_a^{[s]}$  for every  $a \in M$ . By Rosenthal [1973]  $P_s$  is a potential function for player  $i$  in the game  $\Gamma_{F^{[s]}}$ . Hence,  $\{P_1, \dots, P_q\}$  is a cover for  $\Gamma_F$ . That is,  $\Gamma_F$  is a  $q$ -potential game.

- (2) We will basically use the proof of the case  $q = 1$  given in Monderer and Shapley [1996], cleaning it a little bit.

Let  $\Gamma = (N, (X_i)_{i \in N}, (C_i)_{i \in N})$  be a  $q$ -potential game in strategic form. Without lose of generality we can assume that the potential index of  $\Gamma$  equals  $q$ . Let  $H = \{P_s : 1 \leq s \leq q\}$  be a cover for  $\Gamma$ , and let  $\pi = (N_s)_{s=1}^q$  be a partition consistent with  $H$ . Then for every  $i$  there exists a unique integer  $s_i$  such that  $i \in N_{s_i}$ . Because  $P_{s_i}$  is a potential function for  $i$ ,  $C_i(x_i, x_{-i}) - P_{s_i}(x_i, x_{-i})$  does not depend on  $x_i$ . Therefore, there exists a function  $f_i : X_{-i} \rightarrow \mathfrak{R}$  such that  $C_i(x) = P_{s_i}(x) + f_i(x_{-i})$  for every  $x \in X$ .

We proceed to define a PS-congestion form. Let  $M = 2^{X_1} \times \dots \times 2^{X_n}$  be the set of facilities.<sup>9</sup> A generic element of  $M$  will be denoted by  $T = (T_1, \dots, T_n)$ . Let

$$M_1 = \{T \in M : |T_i| = 1 \text{ for every } i \in N\},$$

and

$$M_2 = \{T \in M : \exists i \in N \text{ such that } T_i = X_i \text{ and } |T_j| = |X_j| - 1 \text{ for all } j \neq i\}.$$

For every  $i \in N$  and for every  $x_i \in X_i$  let  $A_{x_i} \subseteq M$  be defined as follows:

$$A_{x_i} = \{T \in M_1 \cup M_2 : x_i \in T_i\},$$

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<sup>9</sup>This set seems huge, however we are not actually using all facilities.

and let

$$\Sigma_i = \{A_{x_i} : x_i \in X_i\}.$$

Let  $1 \leq s \leq q$ . For every  $T \in M$  we define  $c_T^{[s]}$  on  $\{1, \dots, n\}$  (and extend arbitrarily to the interval  $[0, \infty)$ ) as follows:

- For every  $T \notin M_1 \cup M_2$ ,  $c_T^{[s]}$  is identically zero.
- for  $T \in M_1 \cup M_2$ ,  $c_T^{[s]}(k) = 0$ , for  $1 < k < n$ .
- For  $T \in M_1$ ,  $T = (\{x_1\}, \dots, \{x_n\})$ ,  $c_T^{[s]}(1) = 0$ , and

$$c_T^{[s]}(n) = P_s(x_1, \dots, x_n).$$

- For every  $T \in M_2$ ,  $c_T^{[s]}(n) = 0$ . For such  $T$  there exists a unique  $i \in N$ , and a unique  $x_{-i} \in X_{-i}$  such that  $T_i = X_i$ , and  $T_j = X_j \setminus \{x_j\}$  for every  $j \neq i$ . If  $i \notin N_s$ ,  $c_T^{[s]}(1) = 0$ . If  $i \in N_s$ ,  $c_T^{[s]}(1) = f_i(x_{-i})$ .

Let  $c_T^i = c_T^{[s_i]}$ . It is now clear that the PS-congestion form  $F = (N, M, (\Sigma_i)_{i \in N}, ((c_T^i)_{T \in M})_{i \in N})$  is of type  $q$ , and  $\Gamma_F$  is isomorphic to  $\Gamma$ .

■

We end with a somewhat surprising corollary:

**Corollary 1** *Every game in strategic form is isomorphic to a PS-congestion game.*

**Proof:** *As we noticed above, every game in strategic form is an  $n$ -potential game. By Theorem 1 every  $n$ -potential game is isomorphic to an  $n$ -congestion game.* ■

## 2.9 Adding weights

Let  $F = (M, N, (\Sigma_i)_{i \in N}, (c_a)_{a \in M})$  be a congestion form, and let  $w = (w_i)_{i \in N}$  be a vector of positive weights. Consider the following interpretation. There exists a standard unit of goods. However, every player uses her own unit such that one unit of  $i$  equals  $w_i$  standard units. Since the cost functions refer to standard units,  $F$  and  $w$  give rise to the game  $\Gamma_{F,w}$  in which the set of players is  $N$ ,  $\Sigma_i$  is the set of strategies of  $i$ , and for every  $i \in N$ , the cost function of player  $i$  is defined as follows:

$$C_i(A) = c_i(A_1, \dots, A_n) = w_i \sum_{a \in A_i} c_a(w_a(A)),$$

Where  $w_a(A) = \sum_{j: a \in A_j} w_j$ . The game  $\Gamma_{F,w}$  is called a congestion game with weights.<sup>10, 11</sup>

Let  $q(w)$  be the cardinality of the set of distinct weights.

**Theorem 2** Let  $F$  be a congestion form, and let  $w$  be a vector of positive weights. If  $q(w) = q$ ,  $\Gamma_{F,w}$  is a  $q$ -potential game.

**Proof:** Assume  $w_{[s]}$ ,  $1 \leq s \leq q$  are the distinct weights. For every  $s$ , let  $N_s = \{i \in N : w_i = w_{[s]}\}$ . Fix an  $s$ . Let  $y = (A_j)_{j \in N \setminus N_s}$  be a fixed strategy profile of the players outside  $N_s$ . We will show that  $\Gamma_{F,w}(y)$  is a potential game. The assertion will follow from Assertion 3 in Lemma ??.

Indeed, for every  $a \in M$  let  $w^y(a) = \sum_{j \in N \setminus N_s: a \in A_j} w_j$ . Define  $c_a^{[s]} : [0, \infty) \rightarrow \Re$  as follows:

$$c_a^{[s]}(x) = w_{[s]} c_a(w_{[s]}x + w^y(a)), \quad 0 \leq x < \infty.$$

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<sup>10</sup>We do not wish to call it, as it should have been called, a weighted congestion game because this term was used in Monderer and Shapley [1996] for another method of incorporating weights into congestion games.

<sup>11</sup>An example in Libman and Orda [2001] shows a congestion form with non negative and increasing cost functions that does not possess a pure-strategy equilibrium.

Obviously  $\Gamma(y)$  is isomorphic to the congestion game generated by the congestion form  $F(y) = (M, N_s, (\Sigma_i)_{i \in N_s}, (c_a^{[s]})_{a \in M})$ . Therefore  $\Gamma(y)$  is a potential game. ■

### 3 Faithful representations

In many real-life applications it is natural to assume that the facility cost functions of a PS-congestion form have a special structure. In particular, it is natural to assume that the facility cost functions are nonnegative and in addition are either non-decreasing or non-increasing, depending on the context. It is easy to check that the particular representation method described in the proof of Theorem 1 may represent a finite  $q$ -potential game with nonnegative costs by a  $q$ -congestion form in which some of the facility cost functions take negative values. Actually, the representation method depends on the choice of the potential functions. However, it can be seen that there exists a finite 1-potential game such that for every choice of a potential function the representation method yields a 1-congestion form with some facility cost functions that take negative values. This suggests three questions:

**Question 1:** *Can every finite  $q$ -potential game with non-negative cost functions be represented (up to an isomorphism) by a  $q$ -congestion form with non-negative facility cost functions?*

**Question 2:** *Can every finite  $q$ -potential game with non-negative cost functions be represented (up to an isomorphism) by a  $q$ -congestion form with non-negative and non-decreasing cost functions?*

**Question 3:** *Can every finite  $q$ -potential game with non-negative costs be represented (up to an isomorphism) by a  $q$ -congestion form with non-negative and non-increasing cost functions?*

We show that the answer to Question 2 is negative by the next example:

**Example 2** Consider the following parametric game with  $z > 0$ .

$$\Gamma_z =$$

		$x_2$	$y_2$
$x_1$	0	0	$z$
$y_1$	$z$	0	0

$\Gamma_z$  is a congestion game because it is a potential game with a potential function  $P_z$ , where

$$P_z =$$

		$x_2$	$y_2$
$x_1$	0	$z$	
$y_1$	$z$	$z$	

Let  $N = \{1, 2\}$ . Assume in negation that  $\Gamma_z$  is isomorphic to  $\Gamma_F$ , where  $F = (M, N, (\sum_{i=1}^2 (c_a)_{a \in M}))$  is a congestion form in which the cost functions are non-negative and non-decreasing. In particular, for every facility  $a$

$$0 \leq c_a(1) \leq c_a(2).$$

Because  $\Gamma_F$  is isomorphic to  $\Gamma_z$ , we may assume that the feasible sets in  $F$  are parameterized as follows:  $\Sigma_1 = \{A_{x_1}, A_{y_1}\}$ , and  $\Sigma_2 = \{A_{x_2}, A_{y_2}\}$ . Also, without loss of generality we can assume that  $M = A_{x_1} \cup A_{y_1} \cup A_{x_2} \cup A_{y_2}$ . Note that

$$\sum_{a \in A_{y_2} \setminus A_{y_1}} c_a(1) + \sum_{a \in A_{y_2} \cap A_{y_1}} c_a(2) = C_2(A_{y_1}, A_{y_2}) = 0.$$

Therefore,  $c_a(1) = 0$  for every  $a \in A_{y_2} \setminus A_{y_1}$ , and  $c_a(2) = 0$  for every  $a \in A_{y_2} \cap A_{y_1}$ . Since  $c_a(1) \leq c_a(2)$ , we conclude that  $c_a(1) = 0$  for every  $a \in A_{y_2}$ . It follows that

$$\sum_{a \in A_{y_2} \cap A_{x_1}} c_a(2) = C_2(A_{x_1}, A_{y_2}) = z > 0. \quad < 3 >$$

On the other hand,

$$\sum_{a \in A_{x_1} \setminus A_{y_2}} c_a(1) + \sum_{a \in A_{x_1} \cap A_{y_2}} c_a(2) = C_1(A_{x_1}, A_{y_2}) = 0,$$

and therefore  $\sum_{a \in A_{y_2} \cap A_{x_1}} c_a(2) = 0$ , contradicting  $< 3 >$ . ■

*Note, however, that  $\Gamma_Z$  in Example 2 can be represented by a simple congestion game with non-negative and non-increasing cost functions. Consider the congestion form  $F_z$  with two resources  $a, b$ .  $c_a(1) = c_a(2) = 0$ .  $c_b(1) = z$ , and  $c_b(2) = 0$ . It is easily verified that  $\Gamma_{F_z}$  is isomorphic to  $\Gamma_z$ . Hence, Question 1 and 3 can still have positive answers.*

*If we are less ambitious, and we allow representing  $q$ -potential games with  $r$ -potential games with  $r > q$ , we show that the answer to the modified version of Question 1 and 2 is positive.*

**Lemma 2** *Every game in strategic form is isomorphic to a PS-congestion game derived from a PS-congestion form with non-negative and non-decreasing facility cost functions.*

**Proof:** Let  $\Gamma = (N, (X_i)_{i \in N}, (C_i)_{i \in N})$  be a game in strategic form with non-negative cost functions. We define a PS-congestion form as follows: Let the set of facilities be  $X$ . For every  $i \in N$  and  $x_i \in X_i$  let  $A_{x_i} = \{x_i\} \times X_{-i} \subseteq X$ . Let  $\Sigma_i = \{A_{x_i} \mid x_i \in X_i\}$ . Finally, for every  $x \in X$  let  $c_x^i(k) = 0$  if  $1 \leq k < n$ , and  $c_x^i(n) = C_i(x)$ , and extend  $c_x^i$  to the whole interval  $[0, \infty)$  arbitrarily. Obviously the PS-congestion form just defined,  $F$  generates an PS-congestion game, which is isomorphic to  $\Gamma$ . Moreover the facility cost functions in  $F$  are non-negative and non-decreasing. ■

In the following example we show two congestion forms with the same combinatorial structure (same set of facilities and same feasible sets) and with positive cost functions that represent the same game. However, the facility cost functions in one of them are decreasing and in the other, increasing.

**Example 3** The two congestion forms bellow represent the following potential game:

$$\Gamma = \begin{array}{cc} & \begin{array}{cc} x_2 & y_2 \end{array} \\ \begin{array}{c} x_1 \\ y_1 \end{array} & \begin{array}{|c|c|} \hline 11 & 7 \\ \hline 11 & 12 \\ \hline 12 & 8 \\ \hline 7 & 8 \\ \hline \end{array} \end{array}$$

Consider the parametric congestion forms  $F$ , in which  $M = \{a, b, c, d\}$ ,  $\Sigma_1 = \{\{a, b\}, \{c, d\}\}$ ,  $\Sigma_2 = \{\{a, c\}, \{d, d\}\}$ . The cost functions are

$$c_z(1) = x_z, \text{ and } c_z(2) = y_z \text{ for every } z \in \{a, b, c, d\}.$$

Hence,  $\Gamma$  can be represented by this particular congestion form if there exists a solution to the following linear system with 8 equations and 8 variables:

$$\Gamma = \begin{array}{cc} & \begin{array}{cc} ac & bd \end{array} \\ \begin{array}{c} ab \\ cd \end{array} & \begin{array}{|cc|} \hline y_a + x_b & y_b + x_a \\ \hline y_a + x_c & y_b + x_d \\ \hline y_c + x_d & y_d + x_c \\ \hline y_c + x_a & y_d + x_b \\ \hline \end{array} \end{array}$$

It is easy to see that this system has solutions depending on two parameters  $s, t$ :

$$\begin{aligned} x_b = x_c = t, \quad y_b = y_c = s, \\ y_a = 11 - t, \quad y_d = 8 - t, \quad x_a = 7 - s, \quad x_d = 12 - s. \end{aligned}$$

Obviously  $t = 1, s = 6$  gives a congestion form with positive and increasing cost functions, while  $t = 6, s = 1$  give a congestion form with positive and decreasing cost functions. ■

## 4 Network Forms and Network Games

*Much of the literature about congestion forms has been motivated by transportation systems, and by digital networks. In such models, facilities are edges in a graph, and feasible sets of facilities are routes. In this paper the terms network form and network game are defined in a specific way. All other graphical models fall under the category of congestion forms (games).*

Consider a loop-free directed graph  $GR$  with a finite set of vertices  $V = V_{GR}$ , and a set of edges  $E = E_{GR}$ . Every feasible subset of facilities for  $i$  represents a feasible route (a path with distinct vertices) in the graph. For every  $o, d \in V, o \neq d$  we denote by  $R(o, d)$  the set of all routes that connect  $o$  to  $d$ . A PS-network form is a PS-congestion form  $F = (M, N, (\Sigma_i)_{i \in N}, ((c_a^i)_{a \in M})_{i \in N})$  for which there exists a directed graph, and two distinct vertices in this graph  $o$  and  $d$ , with  $R(o, d) \neq \emptyset$  such that  $M \subseteq E$  and  $\Sigma_i \subseteq R(o, d)$  for every agent  $i$ . A game  $\Gamma$  in strategic form is called a PS-network game if  $\Gamma = \Gamma_F$  for some PS-network form  $F$ . Naturally, a  $q$ -network form is a PS-network form of type  $q$ , and a  $q$ -network game is a PS-network game derived from a  $q$ -network form. A 1-network form is also called a network form, and a 1-network game is also called a network game.

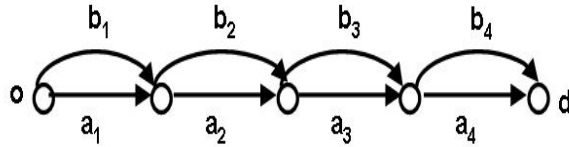
**Theorem 3** Every  $q$ -congestion form is equivalent to a  $q$ -network form.

**Proof:** Let  $F = (M, N, (\Sigma_i)_{i \in N}, ((c_a^i)_{a \in M})_{i \in N})$  be a  $q$ -congestion form. Assume  $M = \{a_1, \dots, a_m\}$ , and let  $K = \{b_1, \dots, b_m\}$  be an arbitrary finite copy of  $M$  that does not intersect with  $M$ . We construct a graph  $GR$  as follows: The set of vertices is  $V = \{1, \dots, m+1\}$ . For every  $1 \leq j \leq m$  we connect  $j$  to  $j+1$  with two edges,  $a_j, b_j$ . That is,  $j$  is the tail of both,  $a_j$  and  $b_j$ , and  $j+1$  is their head. We denote  $o = 1$  and  $d = m+1$ . For every  $A_i \in \Sigma_i$  we associate a route  $\alpha_{A_i}$  as follows:  $\alpha_{A_i} = z_1, z_2, \dots, z_m$ , where  $z_j = a_j$  if  $a_j \in A_i$ , and  $z_j = b_j$  if  $a_j \notin A_i$ . With the edges  $a_j$  we associate the cost function  $(c_{a_j}^i)_{i \in N}$ , and with the edges  $b_j$  we associate the cost functions which are constantly zero. It is obvious that we constructed a PS-network form of type  $q$ , and that the  $q$ -network game derived from this form is isomorphic to  $\Gamma_F$ . ■

The following example together with its associated figure illustrates the proof.

**Example 4** Consider the congestion form  $F$ , in which  $N = \{1, 2\}$ ,  $M = \{a_1, a_2, a_3, a_4\}$ ,  $\Sigma_1 = \{\{a_1, a_2\}, \{a_3, a_4\}\}$ , and  $\Sigma_2 = \{\{a_1, a_3\}, \{a_2, a_4\}\}$ .  $F$  is transformed to the network form shown in Figure 1 in which  $\Sigma_1^* = \{a_1 a_2 b_3 b_4, b_1 b_2 a_3 a_4\}$ , and  $\Sigma_2^* = \{a_1 b_2 a_3 b_4, b_1 a_2 b_3 a_4\}$ . The cost functions on the  $a_j$ -links,  $j = 1, 2, 3, 4$  are the original cost functions, and the cost functions on the  $b_j$ -links are constantly zero.

Figure 1:



**Corollary 2** Every finite game in strategic form is isomorphic to a PS-network game.

**Proof:** The proof follows from combining Corollary 1 with Theorem 3. ■

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