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Weighted Majority Games Have Many μ-values[1]

By D. Monderer[2]

1 Introduction

Measure-based values been introduced by Aumann and Kurz ([1], 1977) and have been discussed by Hart ([3], 1980), Monderer ([4], 1986), and Monderer and Neyman ([5], 1988).

Let μ a non-atomic probability measure. It was proved in [4] that the μ-symmetry axiom is sufficient (together with the linearity and the efficiency axioms) to determine the Aumann and Shapley value on the space $pNA(\mu)$ of smooth non-atomic games which are absolutely continuous with respect to μ. A shorter proof of this fact was given in [5].

In this paper we show that this is not the case when we move away from smooth games. It is proved that there are many μ-values on the space $bv'NA(\mu)$, which is the closed linear space generated by the smooth games, and by all weighted majority games that are continuous at $\emptyset$ and at I (the set of players) with weights which are absolutely continuous with respect to μ.

Since every μ-value on $bv'NA(\mu)$ is continuous with norm 1 and satisfies the projection axiom, and because by [6] $bv'NA(\mu)$ is a subspace of $ASYMP$, we provide an example of a μ-value of norm 1 which is not a partition μ-value, where the notion of a partition μ-value is defined in analogy to the notion of a partition value defined in [7]. It is not known whether there is a value with norm 1 which is not a partition value.

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2 Dov Monderer, MEDS, Kellogg Graduate School of Management, Northwestern University, Evanston, IL 60208, USA.

2 μ-Values on $bv'NA(\mu)$

We shall use the same notations as in [2] and [4] and also the following:

$bv'NA(\mu)$ will denote the closed linear subspace of BV generated by all games of the form $v = f \circ \lambda$, where $\lambda \in NA^1(\mu)$ (i.e., λ is absolutely continuous with respect to μ) and $f \in bv'$ (i.e., f is a function of bounded variation on the interval [0, 1], which is continuous at 0 and 1 and satisfies $f(0) = 0$). $s'NA(\mu)$ will denote the closed linear subspace of $bv'NA(\mu)$ generated by all games of the form $v = f \circ \lambda$, where $\lambda \in NA^1(\mu)$ and $f \in s'$ (i.e., f is a singular function in bv', that is $f'(x) = 0$ a.e. w.r.t. the Lebesque measure). Finally, let Q be the dense subspace of $s'NA(\mu)$ consisting of all games of the form $v = \Sigma_{i=1}^{n} f_i \circ \lambda_i$, *where* $f_i \in s'$ and $\lambda_i \in NA^1(\mu)$.

Theorem A: There exist continuous μ-values with norm 1 on $bv'NA(\mu)$ which satisfy the projection axiom and are different from the restriction to $bv'NA(\mu)$ of the Aumann-Shapley value on $bv'NA$. ■

Before proving Theorem A we state one of its corollaries whose proof is given in the introduction.

Corollary 1: There exists a continuous μ-value with norm 1 which satisfies the projection axiom and which is not a partition value. ■

The proof of Theorem A will be given through Lemmas 2–4.

Lemma 2: There exists a function $\tau : NA^1(\mu) \to NA^1(\mu)$, different from the identity map, such that:

(1) τ is μ-symmetric. i.e.,

$$\tau(\theta_* \circ \lambda) = \theta_* \circ \tau(\lambda) \text{ for every } \lambda \in NA^1(\mu) \text{ and every } \theta \in G(\mu),$$

where $G(\mu)$ is the group of all automorphisms of the players set $(I, \mathscr{C})$ which preserve μ.

(2) τ satisfies the dummy property. i.e.,

$$\tau(\lambda) \text{ is absolutely continuous w.r.t. } \lambda$$

for every $\lambda \in NA^1(\mu)$.

Proof: Let T be any measureable subset of I for which $0 < \mu(T) < 1$. Define measures μ_1, μ_2 and μ_0 as follows:

$$\mu_1(S) = \frac{\mu(S \cap T)}{\mu(T)}\,;\ \mu_2(S) = \frac{\mu(S \cap T^c)}{\mu(T^c)}\,;$$

$\mu_0 = \alpha\mu_1 + (1 - \alpha)\mu_2$, where $0 < \alpha < 1$ is any number such that $\alpha \neq \mu(T)$, and $T^c = I \setminus T$.

Denote $F = \{\theta_* \circ \mu_0 : \theta \in G(\mu)\}$ and define:

$$\tau(\lambda) = \begin{cases} \lambda, \text{ for } \lambda \notin F \\ \mu, \text{ for } \lambda \in F. \end{cases}$$

It is clear that τ is not identity map, and that it satisfies properties (1) and (2). ■

Lemma 3: Let $\tau : NA^1(\mu) \to NA^1(\mu)$ satisfy properties (1) and (2) of Lemma 2. Then the operator $\psi : Q \to NA$ defined by

$$\psi(\sum_{i=1}^{n} f_i \circ \lambda_i) = \sum_{i=1}^{n} f_i(1)\tau(\lambda_i)$$

is a continuous μ-value on Q with norm 1, and it can be uniquley extended to a μ-value $\phi : s'NA(\mu) \to NA$ with norm 1.

Proof: Let $v = \Sigma_i f_i \circ \lambda_i$. Without loss of generality we can assume that $\lambda_j \neq \lambda_i$ for $i \neq j$. We will show that

$$\| \sum_{i=1}^{n} f_i \circ \lambda_i) \| \geq \| \sum_{i=1}^{n} f_i(1)\tau(\lambda_i) \|,$$

which will simultaneously prove that ψ is well defined and that $\|\psi(v)\| \leq \|v\|$ for all $v \in Q$ (that is, $\|\psi\| \leq 1$). indeed, by Lemma 8.17 in [2],

$$\| \sum_{i=1}^{n} f_i \circ \lambda_i) \| = \sum_{i=1}^{n} \|f_i\| \geq \sum_{i=1}^{n} |f_i(1)| =$$

$$\sum_{i=1}^{n} |f_i(1)| \|\tau(\lambda_i)\| \geq \| \sum_{i=1}^{n} f_i(1)\tau(\lambda_i) \|.$$

Let f be any monotone function in s' with $f(1) = 1$. Then

$$\|\psi(f \circ \mu)\| = \|\mu\| = 1 = \|f\| = \|f \circ \mu\|,$$

which yields that actually $\|\psi\| = 1$. Obviously ψ is linear, μ-symmetric and efficient and since $\|\psi\| \leq 1$ we get from Proposition 4.6 in [2] that ψ is a positive operator. In order to complete the proof we have to show that ψ satisfies the dummy axiom, but this is an obvious consequence of the dummy property of τ.

Since Q is dense in $s'NA(\mu)$, ψ can be uniquely extended to a norm one operator from $s'NA(\mu)$ to NA, Which we will denote by ϕ. Clearly ϕ is efficient μ-symmetric and positive. We now prove that ϕ is a μ-value by showing that it satisfies the dummy axiom. Let $v \in s'NA(\mu)$ and let S_0 be a null set of v (i.e., $v(S) = v(S \cap S_0^c)$ for all $S \subseteq I$). Let (u^n) be a sequence in Q which converges to v.

For every $w \in BV$ and every $T \subseteq I$, $\|w_T\| \leq \|w\|$ (where $w_T(S) = w(T \cap S)$ for all $S \subseteq I$). Thus $u^n_{S_0^c} \to v_{S_0^c} = v$. As obviously $u^n_{S_0^c} \in Q$ for all $n \geq 1$, we get by the continuity of ϕ that $\phi(u^n_{S_0^c}) \to \phi(v)$. As S_0 is a null set of $u^n_{S_0^c}$ for all $n \geq 1$, and $\phi = \psi$ satisfies the dummy axiom on Q, S_0 is a null set of $\phi(u^n_{S_0^c})$ and thus of $\phi(v)$ too. ■

Lemma 4:

$$bv'NA(\mu) = pNA(\mu) \oplus s'NA(\mu),$$

where $\oplus$ stands for direct sum.

Moreover, for every $u \in pNA(\mu)$ and $w \in s'NA(\mu)$,

$$\|u+w\| = \|u\| + \|w\|.$$

Proof: The proof follows from corollaries 8.21 and 8.22, and equality 8.24 in (2]. ■

Proof of Theorem A: Let $\tau : NA^1(\mu) \to NA^1(\mu)$ be a map satisfying (1) and (2) of

Lemma 2: let $\phi : s'NA(\mu) \to NA$ be the μ-value derived from τ. Denote by φ the unique value (i.e., the Aumann-Shapley value) on $pNA(\mu)$ and define $\overline{\varphi} : bv'NA(\mu) \to NA$ as follows:

$$\overline{\varphi}(u+w) = \varphi(u) + \phi(w),$$

where $u \in pNA(\mu)$ and $w \in s'NA(\mu)$. By Lemmas 2–4 $\overline{\varphi}$ is a μ-value with norm 1 on $bv'NA(\mu)$ which satisfies the projection axiom and is different from the Aumann-Shapley value. ■

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