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Asymptotic Properties in Dynamic Programming

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Abstract: In the framework of dynamic programming we provide two results:

- An example where uniform convergence of the T-stage value does not imply equality of the limit and the lower infinite value.
- Generalized Tauberian theorems, that relate uniform convergence of the T-stage value to uniform convergence of values associated with a general distribution on stages.

1 Introduction

Let S be a state space. For each $s\in S$ let $\emptyset\neq\Gamma(s)\subseteq S$, and let f be a real bounded function on S. Consider the dynamic programming problem where the decision maker on day t, at stage s_t, has to choose a new state $s_{t+1}\in\Gamma(s_t)$, and receives a payoff $f(s_t)$. A *play* at $s\in S$ is a sequence $(s_t)_{t=0}^{\infty}$ with $s_0=s$ and $s_{t+1}\in\Gamma(s_t)$ for all $t\geq 0$. One traditionally considers the λ-discounted value $V_\lambda(s)$:

$$V_\lambda(s)=\sup_{(s_t)_{t=0}^{\infty}}(1-\lambda)\sum_{t=0}^{\infty}\lambda^t f(s_t),$$

or the T-stage value $V_T(s)$:

$$V_T(s)=\sup_{(s_t)_{t=0}^{\infty}}\frac{1}{T+1}\sum_{t=0}^{T}f(s_t),$$

where in both cases the supremum ranges over all plays at s.

One can also consider other evaluations: Let $\theta=(\theta(t))_{t=0}^{\infty}$ be a probability on the set of non-negative integers and define:

$$V_\theta(s)=\sup_{(s_t)_{t=0}^{\infty}}\sum_{t=0}^{\infty}\theta(t)f(s_t).$$

Lehrer and Sorin (1992) proved that if either one of the limits $\lim_{\lambda\to 1}V_\lambda(s)$, or $\lim_{T\to\infty}V_T(s)$ exists uniformly in $s\in S$, then the other limit also exists uniformly, and the limit functions coincide.

In Section 3 we give sufficient conditions on linearly ordered families $(\Theta, <)$ of probabilities on the integers to get analogous results for $(V_\theta)_{\theta\in\Theta}$ and $(V_T)_{T\geq 0}$.

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There are other natural ways of evaluating streams of payoffs in dynamic programming (except for those discussed above):

The lower (long-run average) value,

$$\underline{V}(s) = \sup_{(s_t)_{t=0}^{\infty}} \liminf_{T\to\infty} \frac{1}{T+1} \sum_{t=0}^{T} f(s_t),$$

and the upper (long-run average) value,

$$\overline{V}(s) = \sup_{(s_t)_{t=0}^{\infty}} \limsup_{T\to\infty} \frac{1}{T+1} \sum_{t=0}^{T} f(s_t),$$

where, again, the supremum is taken on all plays at s.

Lehrer and Monderer (1989) proved that uniform convergence of $(V_\lambda)_{\lambda\in(0,1)}$ to some V implies $V = \overline{V}$, and showed in an example that it does not imply the equality $V = \underline{V}$. If one allows the decision maker to use mixed strategies, i.e., to choose a play in random, and then defines the payoff of each state as the expectation, one obtains new evaluations. It is clear that the evaluations V_λ, V_T, V_θ, and $\overline{V}$ will not change by allowing mixed strategies, but $\underline{V}$ will change in general. Let

$$\underline{U}(s) = \sup_{\mu\in\Delta} \liminf_{T\to\infty} E_\mu \left(\frac{1}{T+1} \sum_{t=0}^{T} f(s_t)\right),$$

where Δ is the set of all probabilities on the set of plays, endowed with the cylinder σ-field, and E_μ stands for the expectation operator with respect to μ.

Obviously $\underline{U} \geq \underline{V}$. As for the relationship between $\underline{U}$ and the limit V of the discounted value functions, Mertens and Neyman (1981) provided sufficient conditions, stronger than the uniform convergence of $(V_\lambda)_{\lambda\in(0,1)}$ (and satisfied in every finite setup), that ensure the equality $\underline{U} = V$ (even for stochastic games). In Section 2 we show that uniform convergence alone is not sufficient by providing a counter example. See Mertens (1987) for related conjectures, hints, and comments. Other type of necessary conditions, for specific types of dynamic programming problems, are discussed in Dutta (1991).

2 The Counter Example

Every rooted directed tree without terminal nodes naturally defines a dynamic programming problem when we attach payoffs to the nodes. Our dynamic programming problem will be defined as a tree, constructed inductively in the spirit of Lehrer and Monderer (1989).

Given two decreasing vanishing sequences $(\varepsilon_n)_{n=1}^{\infty}$ and $(\delta_n)_{n=1}^{\infty}$, define for every real number x the tree $T(x)$ as follows:

Every node of $T(x)$ except for the root has an outdegree one, and the root itself has countably many branches. On the n^{th} branch of the root the payoff, $g(s)$, is 0 until node $[\varepsilon_n n]+1$, it then equals $x-\delta_n$ until node n, and from then on it equals 0. Define a valuation φ at each node s different from the root as follows: $\varphi(s)=x-\delta_n$ for every s in the n^{th} branch appearing before the n^{th} node in this branch, and $\varphi(s)=0$ for every node thereafter. Set $T_1=T(1)$. T_2 is obtained from T_1 by attaching the tree $T(\varphi(s))$ to each node s of T_1, different from the root, and keeping the old payoff of s (i.e., its payoff in T_1). One can continue naturally and define inductively the trees T_3, T_4, ... and finally define $T=\bigcup_{n=1}^{\infty} T_n$. Denote the root of T by s_0, and the payoff function by g.

Note that although g is bounded from above by 1, it is not necessarily bounded from below. Therefore we replace g with a new bounded payoff function f, defined by: $f(s)=\max(g(s),0)$ for every node s of T.

It is clear that $\lim_{T\to\infty} V_T(s)=\varphi(s)$ uniformly on all nodes s of T. In particular $V(s_0)=1$.

We will show that for a specific choice of the sequences $(\varepsilon_n)_{n=1}^{\infty}$ and $(\delta_n)_{n=1}^{\infty}$, $\underline{U}(s_0)=0$.

Let then $\alpha>0$ and let us prove that $\underline{U}(s_0)<\alpha$. Assume in negation that there exists $\mu\in\Delta$ such that for some integer M, $T\geq M$ implies

$$E_\mu\left(\frac{1}{T+1}\sum_{t=0}^{T} f(s_t)\right)\geq\alpha. \tag{2.1}$$

We remark that we can assume that all plays in the support of μ belong to the following set Ω:

If a play in Ω is on the n^{th} branch of some $T(.)$, it remains in this branch until exactly node n. In fact, if some play leaves the branch before node $[\varepsilon_n n]+1$, the decision maker will increase his payoff by leaving the branch at its root, and if a play leaves the n^{th} branch after node n, it is better for the decision maker to leave it at precisely node n. In particular, a play in Ω never remains in a branch of some $T(.)$ and is thus characterized by a sequence $(m_i)_{i=1}^{\infty}$ of integers inducing the path: Branch m_1 of $T(1)$ until the m_1th node, s_{m_1}, branch m_2 of $T(\varphi(s_{m_1}))$ until node m_2 of this branch (with the valuation $1-\delta_{m_1}-\delta_{m_2}$), etc. ... Finally, for every play in Ω,

$$\sum_{i=1}^{\infty}\delta_{m_i}\leq 1. \tag{2.2}$$

Having done the above reduction, we can now replace any strictly positive payoff on any play in Ω by 1.

The basic idea of the proof is to choose a sequence $(\delta_n)_{n=1}^{\infty}$ converging very slowly to zero, implying by (2.2), that for every play in Ω, for a set of integers i with positive density, $\varepsilon_{m_i} m_i$ is much larger than $\sum_{k<i} m_k$. Hence, every play in Ω has "many" large blocks of zeros.

More precisely, let $M_1=2$, and define inductively $n_i=\sum_{k\leq i} M_k$ and $M_{i+1}=n_i^4$ for every $i\geq 1$. Define $\delta_{n_i}=\frac{1}{i}$ for all i and extend δ by monotonicity to all other integers. Choose $\varepsilon_n = \frac{1}{\sqrt{n}}$. We say that a play w is good in the i^{th} block $I_i = [n_{i-1}, n_i]$ if a sequence of ones starts in this block. That is, if w is determined by $m_1, m_2, \ldots$, there exists m_k adapted to I_i in the sense that

$$\sum_{j<k} m_j + \varepsilon_{m_k} m_k \in I_i . \tag{2.3}$$

Set $S_n(w) = \frac{1}{n} \sum_{k=1}^{n} w_k$. We claim that there exists i_0 such that for every $i > i_0$ and for every $w \in \Omega$, if $S_{n_i}(w) \geq \alpha$, then w is good in the i^{th} block. Otherwise, denote by k the largest integer such that the k^{th} sequence of ones in w starts before the i^{th} block. Then $\varepsilon_{m_k} m_k \leq n_{i-1}$, and hence $m_k \leq n_{i-1}^2 = \frac{1}{n_{i-1}^2} M_i$. This implies that this sequence of ones ends very early in the i^{th} block, and that $w_t = 0$ for $\left(1 - \frac{1}{n_{i-1}^2}\right) M_i$ t's in this block. As $\frac{n_{i-1}}{M_i} \to 0$ as $i \to \infty$, then $S_{n_i}(w)$ must be very small contradicting our assumption.

Define $J_i(w)$ to be one if $S_{n_i}(w) \geq \alpha$ and 0 otherwise. If $J_i(w) = 1$, one can by the above claim, define $k(w, i)$ as the smallest k that satisfy (2.3). Denote $\theta_i(w) = \delta_{k(w,i)}$ if $J_i(w) = 1$ and 0 otherwise.

Using the monotone convergence theorem we have:

$$1 \geq E_\mu\left(\sum_{i \geq i_0} J_i(w)\theta_i(w)\right) \geq \sum_{i \geq i_0} E_\mu(J_i(w)\theta_i(w) \geq \sum_{i \geq i_0} E_\mu(J_i(w))\delta_{n_i}.$$

Since (2.1) at n_i implies $E_\mu(J_i(w)) \geq \alpha$, we obtain, recalling that $\delta_{n_i} = \frac{1}{i}$,

$$1 \geq \left(\sum_{i \geq i_0} \frac{1}{i}\right) \alpha,$$

a contradiction. ■

3 Uniform Convergence

We first establish a few notations. Let D denote the set of all probability distributions θ on the set $N = \{0, 1, 2, \ldots\}$ of non-negative integers, that are non-increasing. That is,

$$\theta(t+1) \leq \theta(t) \quad \text{for all } t \in N. \tag{A}$$

For real numbers $\alpha \leq \beta$ and for a distribution θ,

$$\theta[\alpha, \beta] = \sum_{\alpha \le t \le \beta} \theta(t).$$

For $\theta \in D$, define $\hat{\theta}$ on N as follows:

$$\hat{\theta}(t) = (\theta(t) - \theta(t+1))(t+1) \quad \text{for all } t \in N. \tag{3.1}$$

Note that

$$\sum_{t=0}^{T} \hat{\theta}(t) = \sum_{t=0}^{T} \theta(t) - (T+1)\,\theta(T+1) \quad \text{for all } T \ge 0. \tag{3.2}$$

Because of (A), $\lim_{t \to \infty} t\theta(t) = 0$, and therefore $\hat{\theta}$ is a probability distribution on N.

Let $a = (a_t)_{t=0}^{\infty}$ be a bounded sequence. For every $T \ge 0$, denote

$$S_T(a) = \frac{1}{T+1} \sum_{t=0}^{T} a_t,$$

and denote $S(a) = (S_t(a))_{t=0}^{\infty}$. For every probability θ, set,

$$S_\theta(a) = \sum_{t=0}^{\infty} \theta(t) a_t.$$

Observe that by (3.1), similarly to the way (3.2) was obtained, we have $S_\theta(a) = S_{\hat{\theta}}(S(a))$ for all sequences a and probabilities θ, that is,

$$\sum_{t=0}^{\infty} \theta(t) a_t = \sum_{t=0}^{\infty} \hat{\theta}(t) S_t(a). \tag{3.3}$$

We consider linearly ordered families $(\Theta, >)$, where $\Theta \subseteq D$, and "$>$" is a linear (complete) order on Θ, satisfying:

$$\forall \varepsilon > 0,\ \forall N \ge 0,\ \exists \theta_0 \in \Theta, \text{ such that } \forall \theta > \theta_0,\ \sum_{t=0}^{N} \theta(t) < \varepsilon, \tag{B}$$

which is obviously equivalent to:

$$\forall \varepsilon > 0,\ \exists \theta_0 \in \Theta, \text{ such that } \forall \theta > \theta_0,\ \theta(0) < \varepsilon. \tag{B^*}$$

Note that Condition (B) implies that for every $\theta \in \Theta$, there exists $\bar{\theta} \in \Theta$, with $\theta < \bar{\theta}$. Therefore, the notions of lim, lim inf, lim sup, etc. ... are naturally defined for real-valued function on Θ. An increasing sequence $(\theta_n)_{n=0}^{\infty}$ in Θ, is *increasing to* ∞, if for every $\theta \in \Theta$, there exists an integer N such that $\theta_n > \theta$ for all $n \ge N$. For the equivalence results we will need the next properties:

(C) $\exists \varepsilon_0 > 0$ and $\varphi : (0, \varepsilon_0) \to (0, 1)$ such that $\forall \varepsilon < \varepsilon_0$, $\exists J(\varepsilon)$, and a sequence $(\theta_{n,\varepsilon})_{n=J(\varepsilon)}^{\infty}$, that increases to ∞ and satisfies:

$$\hat{\theta}_{n,\varepsilon}[(1-\varepsilon)n,\, n] > \varphi(\varepsilon) \quad \text{for all } n \ge J(\varepsilon).$$

(D) There exists a sequence $(\bar{\theta}_n)_{n=0}^{\infty}$, that increases to ∞, and $\exists \varepsilon_0 > 0$ and $\psi : (0, \varepsilon_0) \to (0, 1)$ such that $\forall \varepsilon < \varepsilon_0$, $\exists I(\varepsilon)$,

$$\hat{\bar{\theta}}_n[\psi(\varepsilon)n,\, n] \ge 1 - \varepsilon \quad \text{for all } n \ge I(\varepsilon).$$

3.1 Preliminary Results

We will assume without loss of generality that the payoff function in our dynamic programming satisfies $0 \le f \le 1$.

Lemma 3.1. $\forall \varepsilon > 0$, $\forall N$, $\exists \theta_0$ *such that* $\forall \theta > \theta_0$, $\forall s_0 \in S$, $\exists n \ge N$ *satisfying* $V_n(s_0) \ge V_\theta(s_0) - \varepsilon$.

Proof: By condition (B) and by (3.2), there exists θ_0, such that $\sum_{t=0}^{N} \hat{\theta}(t) < \frac{\varepsilon}{2}$ for all $\theta > \theta_0$. Let $\theta > \theta_0$, and let $s_0 \in S$. Let $s = (s_t)_{t=0}^{\infty}$ be an $\frac{\varepsilon}{2}$-optimal play for θ in s_0. Then by (3.3),

$$\sum_{t=N+1}^{\infty} \hat{\theta}(t) S_t(f(s)) \ge V_\theta(s_0) - \varepsilon,$$

where $f(s) = (f(s_t))_{t=0}^{\infty}$.

As $\sum_{t=N+1}^{\infty} \hat{\theta}(t) \le 1$, the above inequality implies that a convex combination of $\{S_t(f(s)) \mid t \ge N+1\}$ is greater or equals $V_\theta(s_0) - \varepsilon$. Therefore there exists $t \ge N+1$ with $S_t(f(s)) \ge V_\theta(s_0)$, implying $V_t(s_0) \ge V_\theta(s_0) - \varepsilon$. ■

Corollary 3.2.

$$\limsup_{n \to \infty} V_n \ge \limsup_{\theta \to \infty} V_\theta.$$

Lemma 3.3. $\limsup V_\theta$ *is non-increasing in plays. That is,*

$$\limsup V_\theta(s_0) \ge \limsup V_\theta(s_1) \quad \textit{for every } s_1 \in \Gamma(s_0).$$

Proof: Note that if $(s_t)_{t=1}^{\infty}$ is ε-optimal in s_1 for θ, then $s = (s_t)_{t=0}^{\infty}$ is a play in s_0. Hence, it suffices to prove that for every $\varepsilon > 0$, for sufficiently large θ,

$$\sum_{t=0}^{\infty} \theta(t) f(s_{t+1}) - f(s_t) < \varepsilon.$$

By rearranging terms and by (3.3), the last inequality can be proved by showing that

$$\sum_{t=0}^{\infty} \hat{\theta}(t) \frac{f(s_{t+1}) - f(s_0)}{t+1} < \varepsilon.$$

Hence, it suffices to prove that for every $\varepsilon > 0$, for sufficiently large θ,

$$\sum_{t=0}^{\infty} \hat{\theta}(t) \frac{1}{t+1} < \varepsilon,$$

which follows easily from Condition (*B*). ■

Lemma 3.4 (Lehrer and Sorin (1992)). $\forall \varepsilon > 0$, $\forall n > \frac{2}{\varepsilon}$, *and* $\forall s_0 \in S$, *there exist a play* $s = (s_t)_{t=0}^{\infty}$ *and a stage L such that*

$$\frac{1}{T+1} \sum_{t=0}^{T} f(s_{L+t}) \ge V_n(s_0) - \varepsilon \quad \textit{for every } 0 \le T \le \frac{\varepsilon}{2} n.$$

3.2 From V_θ to V_n

Proposition 1. Assume $\lim_{\theta \to \infty} V_\theta = V$, *uniformly.*

$$\forall \varepsilon > 0 \ \exists N, \textit{ such that } \forall n \ge N, \ V_n \le V + \varepsilon.$$

Proof: Set $\varepsilon_1 = \frac{\varepsilon}{3}$. By the uniform convergence assumption, there exists θ_0, such that

$$|V_\theta(s_0) - V(s_0)| < \varepsilon_1 \quad \text{for all } s_0 \in S. \tag{3.4}$$

Let M be an integer satisfying

$$\sum_{t=0}^{M} \hat{\theta}_0(t) > 1 - \varepsilon_1, \tag{3.5}$$

and let N be an integer satisfying $N > \frac{2}{\varepsilon_1}$. We now show that N satisfies the assertion of the proposition. Indeed, let $n \ge N$, and let $s_0 \in S$. By Lemma 3.4, there exists a play $s = (s_t)_{t=0}^{\infty}$ and an integer L that satisfy the assertion of Lemma 3.4 for ε_1. By (3.3) and (3.5), this implies, $V_{\theta_0}(s_L) \ge V_n(s_0) - 2\varepsilon_1$. Therefore $V(s_L) \ge V_n(s_0) - 3\varepsilon_1$, by (3.4). Hence, by Lemma 3.3, and because $3\varepsilon_1 = \varepsilon$,

$$V(s_0) \geq V_n(s_0) - \varepsilon. \qquad \blacksquare$$

Proposition 2. Assume $(\Theta, >)$ *satisfies Condition* (C)*, and uniform convergence of* $(V_\theta)_{\theta \in \Theta}$ *to* V.

$$\forall \varepsilon > 0,\ \exists N,\ \text{such that } \forall n \geq N,\ V_n \geq V - \varepsilon.$$

Proof: Otherwise, there exists $\varepsilon > 0$ such that for every N, there exists $n \geq N$ and $s_0 \in S$ with $V_n(s_0) < V(s_0) - \varepsilon$. We now choose a particular integer N as follows: set $\varepsilon_1 = \varepsilon_2 = \dfrac{\varepsilon}{2}$, and choose ε_3, ε_4, ε_5 in a way that will be described later. Choose an integer K satisfying the following 4 properties.

(1) K is large enough such that at every play $s = (s_t)_{t=0}^{\infty}$, $\forall n \geq K$, if $V_n(s_0) < V(s_0) - \varepsilon$, then

$$S_T(f(s)) \leq V(s_0) - \varepsilon_1 \quad \text{for all } (1 - \varepsilon_1)n \leq T \leq n.$$

(2) Let $J(\varepsilon_2)$ and the sequence $(\theta_{n,\varepsilon_2})_{n \geq J(\varepsilon_2)}$ satisfy the property stated in Condition (C). Choose $K \geq J(\varepsilon_2)$. That is,

$$\hat{\theta}_n[(1 - \varepsilon_2)n, n] > \varphi(\varepsilon_2) \quad \text{for every } n \geq K,$$

where $\theta_n = \theta_{n,\varepsilon_2}$.

(3) As $(\theta_n)_{n=k}^{\infty}$ is increasing to ∞, and $V_\theta \to V$, we can choose K large enough such that

$$-\varepsilon_4 \leq V_{\theta_n} - V \leq \varepsilon_4 \quad \text{for all } n \geq K.$$

(4) By Proposition 1, we can choose K large enough, such that for every $n \geq K$,

$$V_n \leq V + \varepsilon_3 \quad \text{for all } n \geq K.$$

Finally, choose $N > K$ satisfying

$$\sum_{t=0}^{K} \hat{\theta}_n(t) < \varepsilon_5 \quad \text{for al } n \geq N.$$

By our initial assumption there exists $n \geq N$ and s_0 with $V_n(s_0) < V(s_0) - \varepsilon$. Let $s = (s_t)_{t=0}^{\infty}$ be any play at s_0. Set $a_t = \hat{\theta}_n(t) S_t(f(s))$. Then

$$S_{\theta_n}(f(s)) = \sum_{t=0}^{K} a_t + \sum_{K < t < (1-\varepsilon_2)n} a_t + \sum_{(1-\varepsilon_2)n \leq t \leq n} a_t + \sum_{t > n} a_t.$$

Therefore, by the way we chose N,

$$S_{\theta_n}(f(s)) \le V(s_0) + \Delta,$$

where

$$\Delta = \varepsilon_3 + \varepsilon_5 - \varphi(\varepsilon_2)\varepsilon_1 .$$

As the last inequality holds for every play at s_0, then

$$V_{\theta_n}(s_0) \le V(s_0) + \Delta .$$

Hence, by property (3), satisfied by K and hence by N, and recalling that $\varepsilon_1 = \varepsilon_2 = \frac{\varepsilon}{2}$, we have

$$\varphi\left(\frac{\varepsilon}{2}\right)\frac{\varepsilon}{2} \le \varepsilon_3 + \varepsilon_4 + \varepsilon_5 .$$

Thus we can have a contradiction by choosing ε_i, $i = 3, 4, 5$, to be less than $\frac{1}{3}\varphi\left(\frac{\varepsilon}{2}\right)\frac{\varepsilon}{2}$. ■

3.3 From V_n to V_θ

Proposition 3. Assume $\lim_{n\to\infty} V_n = W$ *uniformly.*

$$\forall \varepsilon > 0,\ \exists \theta_0, \text{ such that } \forall \theta > \theta_0,\ V_\theta \le W + \varepsilon .$$

Proof: The proof is an immediate consequence of Lemma 3.1. ■

Lemma 3.5 (Lehrer and Sorin (1992)). *Assume* $\lim_{n\to\infty} V_n = W$ *uniformly. Then for every* ε *small enough, there exists an integer* N*, such that for every* $n \ge N$ *and* $s_0 \in S$*, there is a play* $s = (s_t)_{t=0}^{\infty}$ *at* s_0 *satisfying:*

$$\frac{1}{T+1}\sum_{t=0}^{T} f(s_t) \ge W(s_0) - \varepsilon \quad \text{for every } \varepsilon n \le T \le (1-\varepsilon)n .$$

Proposition 4. Assume $(\Theta, >)$ *satisfies condition (D), and* $\lim_{n\to\infty} V_n = W$ *uniformly.*

$$\forall \varepsilon > 0,\ \exists N, \text{ such that } \forall n \ge N,\ V_{\bar\theta_n} \ge W - \varepsilon,$$

where $(\bar\theta_n)_{n=0}^{\infty}$ *is defined in Condition (D).*

Proof: Let $\varepsilon>0$. Let $\delta>0$ satisfies $\frac{\delta}{1-\delta}<\min(\psi(\varepsilon),\varepsilon)$. Then by Lemma 3.5 there exists N such that for every $n\geq N$ and $s_0\in S$, there is a play $s=(s_t)_{t=0}^{\infty}$ at s_0 satisfying:

$$\frac{1}{T+1}\sum_{t=0}^{T} f(s_t)\geq W(s_0)-\delta \quad \text{for every } \delta n\leq T\leq(1-\delta)n.$$

Without loss of generality we can choose $N\geq I(\varepsilon)$. Note that if $m\geq N$ (assuming that N was chosen large enough), there exists $n\geq N$, with

$$[\psi(\varepsilon)m,\ m]\subseteq[\delta n,\ (1-\delta)n].$$

Hence, $\hat{\theta}_m[\psi(\varepsilon)m,\ m]\geq 1-\varepsilon$, and $S_T(f(s))\geq 1-\delta\geq 1-\varepsilon$, for $T\in[\psi(\varepsilon)m,\ m]$. Therefore,

$$V_{\theta_m}(s_0)\geq W(s_0)-2\varepsilon \quad \text{for all } m\geq N \text{ and all } s_0\in S.$$

■

Remark 1.

If the sequence $(\bar{\theta}_n)_{n=0}^{\infty}$, given in Condition (D) is dense in $(\Theta,>)$ (in the sense that its uniform convergence implies the uniform convergence of $(V_\theta)_{\theta\in\Theta}$), then under conditions (C) and (D), uniform convergence of $(V_n)_{n=0}^{\infty}$ implies uniform convergence of $(V_\theta)_{\theta\in\Theta}$ to the same limit function. As it was proved in Lehrer and Sorin (1992), such is the case when $\Theta=\{\theta_\lambda:\ \lambda\in[0,1)\}$, where $\theta_\lambda(t)=(1-\lambda)\lambda^t$, and "$>$" is the natural order on real numbers.

Remark 2.

Let $(\Theta,>)$ be a linearly ordered set of distributions on N satisfying (B), (C^*), and (D^*), where (C^*) and (D^*) are obtained from (C) and (D) respectively, by replacing $\hat{\theta}$ with θ everywhere. Define,

$$U_\theta(s_0)=\sup_{(s_t)_{t=0}^{\infty}}\sum_{t=0}^{\infty}\theta(t)S_t(f(S)).$$

It is obvious that our proofs yield the equivalence theorem for this solution concept as well. E.g., for every $0<\lambda<1$ define

$$U_\lambda(s_0)=\sup_{(s_t)_{t=0}^{\infty}}(1-\lambda)\sum_{t=0}^{\infty}\lambda^t S_t(f(s)).$$

Then (U_λ) converges uniformly if and only if (V_n) converges uniformly, and both share the same limit function.

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1 Introduction

Let S be a state space. For each $s \in S$ let $\emptyset \neq \Gamma(s) \subseteq S$, and let f be a real bounded function on S. Consider the dynamic programming problem where the decision maker on day t, at stage s_t, has to choose a new state $s_{t+1} \in \Gamma(s_t)$, and receives a payoff $f(s_t)$. A *play* at $s \in S$ is a sequence $(s_t)_{t=0}^{\infty}$ with $s_0 = s$ and $s_{t+1} \in \Gamma(s_t)$ for all $t \geq 0$. One traditionally considers the λ-discounted value $V_\lambda(s)$:

$$V_\lambda(s) = \sup_{(s_t)_{t=0}^{\infty}} (1-\lambda) \sum_{t=0}^{\infty} \lambda^t f(s_t),$$

or the T-stage value $V_T(s)$:

$$V_T(s) = \sup_{(s_t)_{t=0}^{\infty}} \frac{1}{T+1} \sum_{t=0}^{T} f(s_t),$$

where in both cases the supremum ranges over all plays at s.

One can also consider other evaluations: Let $\theta = (\theta(t))_{t=0}^{\infty}$ be a probability on the set of non-negative integers and define:

$$V_\theta(s) = \sup_{(s_t)_{t=0}^{\infty}} \sum_{t=0}^{\infty} \theta(t) f(s_t).$$

Lehrer and Sorin (1992) proved that if either one of the limits $\lim_{\lambda \to 1} V_\lambda(s)$, or $\lim_{T \to \infty} V_T(s)$ exists uniformly in $s \in S$, then the other limit also exists uniformly, and the limit functions coincide.

In Section 3 we give sufficient conditions on linearly ordered families $(\Theta, <)$ of probabilities on the integers to get analogous results for $(V_\theta)_{\theta \in \Theta}$ and $(V_T)_{T \geq 0}$.

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There are other natural ways of evaluating streams of payoffs in dynamic programming (except for those discussed above):

The lower (long-run average) value,

$$\underline{V}(s) = \sup_{(s_t)_{t=0}^{\infty}} \liminf_{T\to\infty} \frac{1}{T+1} \sum_{t=0}^{T} f(s_t),$$

and the upper (long-run average) value,

$$\overline{V}(s) = \sup_{(s_t)_{t=0}^{\infty}} \limsup_{T\to\infty} \frac{1}{T+1} \sum_{t=0}^{T} f(s_t),$$

where, again, the supremum is taken on all plays at s.

Lehrer and Monderer (1989) proved that uniform convergence of $(V_\lambda)_{\lambda\in[0,1)}$ to some V implies $V=\overline{V}$, and showed in an example that it does not imply the equality $V=\underline{V}$. If one allows the decision maker to use mixed strategies, i.e., to choose a play in random, and then defines the payoff of each state as the expectation, one obtains new evaluations. It is clear that the evaluations V_λ, V_T, V_θ, and $\overline{V}$ will not change by allowing mixed strategies, but $\underline{V}$ will change in general. Let

$$\underline{U}(s) = \sup_{\mu\in\Delta} \liminf_{T\to\infty} E_\mu\left(\frac{1}{T+1} \sum_{t=0}^{T} f(s_t)\right),$$

where Δ is the set of all probabilities on the set of plays, endowed with the cylinder σ-field, and E_μ stands for the expectation operator with respect to μ.

Obviously $\underline{U}\geq\underline{V}$. As for the relationship between $\underline{U}$ and the limit V of the discounted value functions, Mertens and Neyman (1981) provided sufficient conditions, stronger than the uniform convergence of $(V_\lambda)_{\lambda\in[0,1)}$ (and satisfied in every finite setup), that ensure the equality $\underline{U}=V$ (even for stochastic games). In Section 2 we show that uniform convergence alone is not sufficient by providing a counter example. See Mertens (1987) for related conjectures, hints, and comments. Other type of necessary conditions, for specific types of dynamic programming problems, are discussed in Dutta (1991).

2 The Counter Example

Every rooted directed tree without terminal nodes naturally defines a dynamic programming problem when we attach payoffs to the nodes. Our dynamic programming problem will be defined as a tree, constructed inductively in the spirit of Lehrer and Monderer (1989).

Given two decreasing vanishing sequences $(\varepsilon_n)_{n=1}^{\infty}$ and $(\delta_n)_{n=1}^{\infty}$, define for every real number x the tree $T(x)$ as follows:

Every node of $T(x)$ except for the root has an outdegree one, and the root itself has countably many branches. On the n^{th} branch of the root the payoff, $g(s)$, is 0 until node $[\varepsilon_n n]+1$, it then equals $x-\delta_n$ until node n, and from then on it equals 0. Define a valuation φ at each node s different from the root as follows: $\varphi(s)=x-\delta_n$ for every s in the n^{th} branch appearing before the n^{th} node in this branch, and $\varphi(s)=0$ for every node thereafter. Set $T_1=T(1)$. T_2 is obtained from T_1 by attaching the tree $T(\varphi(s))$ to each node s of T_1, different from the root, and keeping the old payoff of s (i.e., its payoff in T_1). One can continue naturally and define inductively the trees T_3, T_4, ... and finally define $T=\bigcup_{n=1}^{\infty} T_n$. Denote the root of T by s_0, and the payoff function by g.

Note that although g is bounded from above by 1, it is not necessarily bounded from below. Therefore we replace g with a new bounded payoff function f, defined by: $f(s)=\max(g(s), 0)$ for every node s of T.

It is clear that $\lim_{T\to\infty} V_T(s)=\varphi(s)$ uniformly on all nodes s of T. In particular $V(s_0)=1$.

We will show that for a specific choice of the sequences $(\varepsilon_n)_{n=1}^{\infty}$ and $(\delta_n)_{n=1}^{\infty}$, $\underline{U}(s_0)=0$.

Let then $\alpha>0$ and let us prove that $\underline{U}(s_0)<\alpha$. Assume in negation that there exists $\mu\in\Delta$ such that for some integer M, $T\geq M$ implies

$$E_\mu\left(\frac{1}{T+1}\sum_{t=0}^{T} f(s_t)\right)\geq\alpha. \tag{2.1}$$

We remark that we can assume that all plays in the support of μ belong to the following set Ω:

If a play in Ω is on the n^{th} branch of some $T(.)$, it remains in this branch until exactly node n. In fact, if some play leaves the branch before node $[\varepsilon_n n]+1$, the decision maker will increase his payoff by leaving the branch at its root, and if a play leaves the n^{th} branch after node n, it is better for the decision maker to leave it at precisely node n. In particular, a play in Ω never remains in a branch of some $T(.)$ and is thus characterized by a sequence $(m_i)_{i=1}^{\infty}$ of integers inducing the path: Branch m_1 of $T(1)$ until the m_1th node, s_{m_1}, branch m_2 of $T(\varphi(s_{m_1}))$ until node m_2 of this branch (with the valuation $1-\delta_{m_1}-\delta_{m_2}$), etc. ... Finally, for every play in Ω,

$$\sum_{i=1}^{\infty}\delta_{m_i}\leq 1. \tag{2.2}$$

Having done the above reduction, we can now replace any strictly positive payoff on any play in Ω by 1.

The basic idea of the proof is to choose a sequence $(\delta_n)_{n=1}^{\infty}$ converging very slowly to zero, implying by (2.2), that for every play in Ω, for a set of integers i with positive density, $\varepsilon_{m_i}m_i$ is much larger than $\sum_{k<i}m_k$. Hence, every play in Ω has "many" large blocks of zeros.

More precisely, let $M_1=2$, and define inductively $n_i=\sum_{k\leq i}M_k$ and $M_{i+1}=n_i^4$ for every $i\geq 1$. Define $\delta_{n_i}=\frac{1}{i}$ for all i and extend δ by monotonicity to all other integers. Choose $\varepsilon_n = \frac{1}{\sqrt{n}}$. We say that a play w is good in the i^{th} block $I_i = [n_{i-1}, n_i]$ if a sequence of ones starts in this block. That is, if w is determined by $m_1, m_2, \ldots$, there exists m_k adapted to I_i in the sense that

$$\sum_{j<k} m_j + \varepsilon_{m_k} m_k \in I_i. \tag{2.3}$$

Set $S_n(w) = \frac{1}{n} \sum_{k=1}^{n} w_k$. We claim that there exists i_0 such that for every $i > i_0$ and for every $w \in \Omega$, if $S_{n_i}(w) \geq \alpha$, then w is good in the i^{th} block. Otherwise, denote by k the largest integer such that the k^{th} sequence of ones in w starts before the i^{th} block. Then $\varepsilon_{m_k} m_k \leq n_{i-1}$, and hence $m_k \leq n_{i-1}^2 = \frac{1}{n_{i-1}^2} M_i$. This implies that this sequence of ones ends very early in the i^{th} block, and that $w_t = 0$ for $\left(1 - \frac{1}{n_{i-1}^2}\right) M_i$ t's in this block. As $\frac{n_{i-1}}{M_i} \to 0$ as $i \to \infty$, then $S_{n_i}(w)$ must be very small contradicting our assumption.

Define $J_i(w)$ to be one if $S_{n_i}(w) \geq \alpha$ and 0 otherwise. If $J_i(w) = 1$, one can by the above claim, define $k(w, i)$ as the smallest k that satisfy (2.3). Denote $\theta_i(w) = \delta_{k(w,i)}$ if $J_i(w) = 1$ and 0 otherwise.

Using the monotone convergence theorem we have:

$$1 \geq E_\mu\left(\sum_{i \geq i_0} J_i(w)\theta_i(w)\right) \geq \sum_{i \geq i_0} E_\mu(J_i(w)\theta_i(w) \geq \sum_{i \geq i_0} E_\mu(J_i(w))\delta_{n_i}.$$

Since (2.1) at n_i implies $E_\mu(J_i(w)) \geq \alpha$, we obtain, recalling that $\delta_{n_i} = \frac{1}{i}$,

$$1 \geq \left(\sum_{i \geq i_0} \frac{1}{i}\right)\alpha,$$

a contradiction. ■

3 Uniform Convergence

We first establish a few notations. Let D denote the set of all probability distributions θ on the set $N = \{0, 1, 2, \ldots\}$ of non-negative integers, that are non-increasing. That is,

$$\theta(t+1) \leq \theta(t) \quad \text{for all } t \in N. \tag{A}$$

For real numbers $\alpha \leq \beta$ and for a distribution θ,

$$\theta[\alpha, \beta] = \sum_{\alpha \le t \le \beta} \theta(t).$$

For $\theta \in D$, define $\hat{\theta}$ on N as follows:

$$\hat{\theta}(t) = (\theta(t) - \theta(t+1))(t+1) \quad \text{for all } t \in N. \tag{3.1}$$

Note that

$$\sum_{t=0}^{T} \hat{\theta}(t) = \sum_{t=0}^{T} \theta(t) - (T+1)\,\theta(T+1) \quad \text{for all } T \ge 0. \tag{3.2}$$

Because of (A), $\lim_{t \to \infty} t\theta(t) = 0$, and therefore $\hat{\theta}$ is a probability distribution on N.

Let $a = (a_t)_{t=0}^{\infty}$ be a bounded sequence. For every $T \ge 0$, denote

$$S_T(a) = \frac{1}{T+1} \sum_{t=0}^{T} a_t,$$

and denote $S(a) = (S_t(a))_{t=0}^{\infty}$. For every probability θ, set,

$$S_\theta(a) = \sum_{t=0}^{\infty} \theta(t) a_t.$$

Observe that by (3.1), similarly to the way (3.2) was obtained, we have $S_\theta(a) = S_{\hat{\theta}}(S(a))$ for all sequences a and probabilities θ, that is,

$$\sum_{t=0}^{\infty} \theta(t) a_t = \sum_{t=0}^{\infty} \hat{\theta}(t) S_t(a). \tag{3.3}$$

We consider linearly ordered families $(\Theta, >)$, where $\Theta \subseteq D$, and "$>$" is a linear (complete) order on Θ, satisfying:

$$\forall \varepsilon > 0,\ \forall N \ge 0,\ \exists \theta_0 \in \Theta, \text{ such that } \forall \theta > \theta_0,\ \sum_{t=0}^{N} \theta(t) < \varepsilon, \tag{B}$$

which is obviously equivalent to:

$$\forall \varepsilon > 0,\ \exists \theta_0 \in \Theta, \text{ such that } \forall \theta > \theta_0,\ \theta(0) < \varepsilon. \tag{B^*}$$

Note that Condition (B) implies that for every $\theta \in \Theta$, there exists $\bar{\theta} \in \Theta$, with $\theta < \bar{\theta}$. Therefore, the notions of lim, lim inf, lim sup, etc. ... are naturally defined for real-valued function on Θ. An increasing sequence $(\theta_n)_{n=0}^{\infty}$ in Θ, is *increasing to* ∞, if for every $\theta \in \Theta$, there exists an integer N such that $\theta_n > \theta$ for all $n \ge N$. For the equivalence results we will need the next properties:

(C) $\exists \varepsilon_0 > 0$ and $\varphi : (0, \varepsilon_0) \to (0, 1)$ such that $\forall \varepsilon < \varepsilon_0$, $\exists J(\varepsilon)$, and a sequence $(\theta_{n,\varepsilon})_{n=J(\varepsilon)}^{\infty}$, that increases to ∞ and satisfies:

$$\hat{\theta}_{n,\varepsilon}[(1-\varepsilon)n, n] > \varphi(\varepsilon) \quad \text{for all } n \geq J(\varepsilon).$$

(D) There exists a sequence $(\bar{\theta}_n)_{n=0}^{\infty}$, that increases to ∞, and $\exists \varepsilon_0 > 0$ and $\psi: (0, \varepsilon_0) \to (0, 1)$ such that $\forall \varepsilon < \varepsilon_0$, $\exists I(\varepsilon)$,

$$\hat{\bar{\theta}}_n[\psi(\varepsilon)n, n] \geq 1 - \varepsilon \quad \text{for all } n \geq I(\varepsilon).$$

3.1 Preliminary Results

We will assume without loss of generality that the payoff function in our dynamic programming satisfies $0 \leq f \leq 1$.

Lemma 3.1. $\forall \varepsilon > 0$, $\forall N$, $\exists \theta_0$ *such that* $\forall \theta > \theta_0$, $\forall s_0 \in S$, $\exists n \geq N$ *satisfying* $V_n(s_0) \geq V_\theta(s_0) - \varepsilon$.

Proof: By condition (B) and by (3.2), there exists θ_0, such that $\sum_{t=0}^{N} \hat{\theta}(t) < \frac{\varepsilon}{2}$ for all $\theta > \theta_0$. Let $\theta > \theta_0$, and let $s_0 \in S$. Let $s = (s_t)_{t=0}^{\infty}$ be an $\frac{\varepsilon}{2}$-optimal play for θ in s_0. Then by (3.3),

$$\sum_{t=N+1}^{\infty} \hat{\theta}(t) S_t(f(s)) \geq V_\theta(s_0) - \varepsilon,$$

where $f(s) = (f(s_t))_{t=0}^{\infty}$.

As $\sum_{t=N+1}^{\infty} \hat{\theta}(t) \leq 1$, the above inequality implies that a convex combination of $\{S_t(f(s)) \mid t \geq N+1\}$ is greater or equals $V_\theta(s_0) - \varepsilon$. Therefore there exists $t \geq N+1$ with $S_t(f(s)) \geq V_\theta(s_0)$, implying $V_t(s_0) \geq V_\theta(s_0) - \varepsilon$. ■

Corollary 3.2.

$$\limsup_{n \to \infty} V_n \geq \limsup_{\theta \to \infty} V_\theta.$$

Lemma 3.3. $\limsup V_\theta$ *is non-increasing in plays. That is,*

$$\limsup V_\theta(s_0) \geq \limsup V_\theta(s_1) \quad \textit{for every } s_1 \in \Gamma(s_0).$$

Proof: Note that if $(s_t)_{t=1}^{\infty}$ is ε-optimal in s_1 for θ, then $s = (s_t)_{t=0}^{\infty}$ is a play in s_0. Hence, it suffices to prove that for every $\varepsilon > 0$, for sufficiently large θ,

$$\sum_{t=0}^{\infty} \theta(t) f(s_{t+1}) - f(s_t) < \varepsilon.$$

By rearranging terms and by (3.3), the last inequality can be proved by showing that

$$\sum_{t=0}^{\infty} \hat{\theta}(t) \frac{f(s_{t+1}) - f(s_0)}{t+1} < \varepsilon.$$

Hence, it suffices to prove that for every $\varepsilon > 0$, for sufficiently large θ,

$$\sum_{t=0}^{\infty} \hat{\theta}(t) \frac{1}{t+1} < \varepsilon,$$

which follows easily from Condition (B). ■

Lemma 3.4 (Lehrer and Sorin (1992)). $\forall \varepsilon > 0$, $\forall n > \frac{2}{\varepsilon}$, *and* $\forall s_0 \in S$, *there exist a play* $s = (s_t)_{t=0}^{\infty}$ *and a stage* L *such that*

$$\frac{1}{T+1} \sum_{t=0}^{T} f(s_{L+t}) \geq V_n(s_0) - \varepsilon \quad \textit{for every } 0 \leq T \leq \frac{\varepsilon}{2} n.$$

3.2 From V_θ to V_n

Proposition 1. Assume $\lim_{\theta \to \infty} V_\theta = V$, *uniformly.*

$\forall \varepsilon > 0 \; \exists N$, *such that* $\forall n \geq N$, $V_n \leq V + \varepsilon$.

Proof: Set $\varepsilon_1 = \frac{\varepsilon}{3}$. By the uniform convergence assumption, there exists θ_0, such that

$$|V_\theta(s_0) - V(s_0)| < \varepsilon_1 \quad \text{for all } s_0 \in S. \tag{3.4}$$

Let M be an integer satisfying

$$\sum_{t=0}^{M} \hat{\theta}_0(t) > 1 - \varepsilon_1, \tag{3.5}$$

and let N be an integer satisfying $N > \frac{2}{\varepsilon_1}$. We now show that N satisfies the assertion of the proposition. Indeed, let $n \geq N$, and let $s_0 \in S$. By Lemma 3.4, there exists a play $s = (s_t)_{t=0}^{\infty}$ and an integer L that satisfy the assertion of Lemma 3.4 for ε_1. By (3.3) and (3.5), this implies, $V_{\theta_0}(s_L) \geq V_n(s_0) - 2\varepsilon_1$. Therefore $V(s_L) \geq V_n(s_0) - 3\varepsilon_1$, by (3.4). Hence, by Lemma 3.3, and because $3\varepsilon_1 = \varepsilon$,

$$V(s_0) \geq V_n(s_0) - \varepsilon.$$ ■

Proposition 2. Assume $(\Theta, >)$ *satisfies Condition* (*C*), *and uniform convergence of* $(V_\theta)_{\theta \in \Theta}$ *to* V.

$$\forall \varepsilon > 0,\ \exists N,\ \text{such that}\ \forall n \geq N,\ V_n \geq V - \varepsilon.$$

Proof: Otherwise, there exists $\varepsilon > 0$ such that for every N, there exists $n \geq N$ and $s_0 \in S$ with $V_n(s_0) < V(s_0) - \varepsilon$. We now choose a particular integer N as follows: set $\varepsilon_1 = \varepsilon_2 = \dfrac{\varepsilon}{2}$, and choose ε_3, ε_4, ε_5 in a way that will be described later. Choose an integer K satisfying the following 4 properties.

(1) K is large enough such that at every play $s = (s_t)_{t=0}^\infty$, $\forall n \geq K$, if $V_n(s_0) < V(s_0) - \varepsilon$, then

$$S_T(f(s)) \leq V(s_0) - \varepsilon_1 \quad \text{for all } (1 - \varepsilon_1)n \leq T \leq n.$$

(2) Let $J(\varepsilon_2)$ and the sequence $(\theta_{n,\varepsilon_2})_{n \geq J(\varepsilon_2)}$ satisfy the property stated in Condition (*C*). Choose $K \geq J(\varepsilon_2)$. That is,

$$\hat{\theta}_n[(1 - \varepsilon_2)n, n] > \varphi(\varepsilon_2) \quad \text{for every } n \geq K,$$

where $\theta_n = \theta_{n,\varepsilon_2}$.

(3) As $(\theta_n)_{n=k}^\infty$ is increasing to ∞, and $V_\theta \to V$, we can choose K large enough such that

$$-\varepsilon_4 \leq V_{\theta_n} - V \leq \varepsilon_4 \quad \text{for all } n \geq K.$$

(4) By Proposition 1, we can choose K large enough, such that for every $n \geq K$,

$$V_n \leq V + \varepsilon_3 \quad \text{for all } n \geq K.$$

Finally, choose $N > K$ satisfying

$$\sum_{t=0}^{K} \hat{\theta}_n(t) < \varepsilon_5 \quad \text{for al } n \geq N.$$

By our initial assumption there exists $n \geq N$ and s_0 with $V_n(s_0) < V(s_0) - \varepsilon$. Let $s = (s_t)_{t=0}^\infty$ be any play at s_0. Set $a_t = \hat{\theta}_n(t) S_t(f(s))$. Then

$$S_{\theta_n}(f(s)) = \sum_{t=0}^{K} a_t + \sum_{K < t < (1-\varepsilon_2)n} a_t + \sum_{(1-\varepsilon_2)n \leq t \leq n} a_t + \sum_{t > n} a_t.$$

Therefore, by the way we chose N,

$$S_{\theta_n}(f(s)) \leq V(s_0) + \Delta,$$

where

$$\Delta = \varepsilon_3 + \varepsilon_5 - \varphi(\varepsilon_2)\varepsilon_1.$$

As the last inequality holds for every play at s_0, then

$$V_{\theta_n}(s_0) \leq V(s_0) + \Delta.$$

Hence, by property (3), satisfied by K and hence by N, and recalling that $\varepsilon_1 = \varepsilon_2 = \frac{\varepsilon}{2}$, we have

$$\varphi\left(\frac{\varepsilon}{2}\right)\frac{\varepsilon}{2} \leq \varepsilon_3 + \varepsilon_4 + \varepsilon_5.$$

Thus we can have a contradiction by choosing ε_i, $i = 3, 4, 5$, to be less than $\frac{1}{3}\varphi\left(\frac{\varepsilon}{2}\right)\frac{\varepsilon}{2}$. ■

3.3 From V_n to V_θ

Proposition 3. Assume $\lim_{n\to\infty} V_n = W$ *uniformly.*

$$\forall \varepsilon > 0,\ \exists \theta_0, \text{ such that } \forall \theta > \theta_0,\ V_\theta \leq W + \varepsilon.$$

Proof: The proof is an immediate consequence of Lemma 3.1. ■

Lemma 3.5 (Lehrer and Sorin (1992)). *Assume* $\lim_{n\to\infty} V_n = W$ *uniformly. Then for every* ε *small enough, there exists an integer* N*, such that for every* $n \geq N$ *and* $s_0 \in S$*, there is a play* $s = (s_t)_{t=0}^{\infty}$ *at* s_0 *satisfying:*

$$\frac{1}{T+1}\sum_{t=0}^{T} f(s_t) \geq W(s_0) - \varepsilon \quad \text{for every } \varepsilon n \leq T \leq (1-\varepsilon)n.$$

Proposition 4. Assume $(\Theta, >)$ *satisfies condition (D), and* $\lim_{n\to\infty} V_n = W$ *uniformly.*

$$\forall \varepsilon > 0,\ \exists N, \text{ such that } \forall n \geq N,\ V_{\bar{\theta}_n} \geq W - \varepsilon,$$

where $(\bar{\theta}_n)_{n=0}^{\infty}$ *is defined in Condition (D).*

Proof: Let $\varepsilon > 0$. Let $\delta > 0$ satisfies $\dfrac{\delta}{1-\delta} < \min(\psi(\varepsilon), \varepsilon)$. Then by Lemma 3.5 there exists N such that for every $n \geq N$ and $s_0 \in S$, there is a play $s = (s_t)_{t=0}^{\infty}$ at s_0 satisfying:

$$\frac{1}{T+1} \sum_{t=0}^{T} f(s_t) \geq W(s_0) - \delta \quad \text{for every } \delta n \leq T \leq (1-\delta) n.$$

Without loss of generality we can choose $N \geq I(\varepsilon)$. Note that if $m \geq N$ (assuming that N was chosen large enough), there exists $n \geq N$, with

$$[\psi(\varepsilon) m, m] \subseteq [\delta n, (1-\delta) n].$$

Hence, $\hat{\hat{\theta}}_m [\psi(\varepsilon) m, m] \geq 1 - \varepsilon$, and $S_T(f(s)) \geq 1 - \delta \geq 1 - \varepsilon$, for $T \in [\psi(\varepsilon) m, m]$. Therefore,

$$V_{\tilde{\theta}_m}(s_0) \geq W(s_0) - 2\varepsilon \quad \text{for all } m \geq N \text{ and all } s_0 \in S. \qquad \blacksquare$$

Remark 1.

If the sequence $(\tilde{\theta}_n)_{n=0}^{\infty}$, given in Condition (D) is dense in $(\Theta, >)$ (in the sense that its uniform convergence implies the uniform convergence of $(V_\theta)_{\theta \in \Theta}$), then under conditions (C) and (D), uniform convergence of $(V_n)_{n=0}^{\infty}$ implies uniform convergence of $(V_\theta)_{\theta \in \Theta}$ to the same limit function. As it was proved in Lehrer and Sorin (1992), such is the case when $\Theta = \{\theta_\lambda : \lambda \in [0, 1)\}$, where $\theta_\lambda(t) = (1-\lambda)\lambda^t$, and "$>$" is the natural order on real numbers.

Remark 2.

Let $(\Theta, >)$ be a linearly ordered set of distributions on N satisfying (B), (C^*), and (D^*), where (C^*) and (D^*) are obtained from (C) and (D) respectively, by replacing $\hat{\theta}$ with θ everywhere. Define,

$$U_\theta(s_0) = \sup_{(s_t)_{t=0}^{\infty}} \sum_{t=0}^{\infty} \theta(t) S_t(f(S)).$$

It is obvious that our proofs yield the equivalence theorem for this solution concept as well. E.g., for every $0 < \lambda < 1$ define

$$U_\lambda(s_0) = \sup_{(s_t)_{t=0}^{\infty}} (1-\lambda) \sum_{t=0}^{\infty} \lambda^t S_t(f(s)).$$

Then (U_λ) converges uniformly if and only if (V_n) converges uniformly, and both share the same limit function.

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