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A MILNOR CONDITION FOR NONATOMIC LIPSCHITZ GAMES AND ITS APPLICATIONS*

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One of Milnor's reasonable axioms is generalized to nonatomic games. Several uses of this generalization are discussed. In particular, the general form of values and semivalues on subspaces of pNA_∞ is derived.

Introduction. It was shown in [D-N-W] that every semivalue ψ on pNA has the form

$$(*) \qquad \psi v(S) = \int_0^1 \partial v(t1_I, 1_S)\, d\xi(t),$$

where ξ is a Borel probability measure on $[0,1]$ which is absolutely continuous w.r.t. the Lebesgue measure λ and $d\xi/d\lambda \in L_\infty[0,1]^+$. The unique value on pNA (the Aumann-Shapley value) is obtained by setting $\xi = \lambda$.

Given a class of games in pNA (in particular a single game) one may want to know the general form of values and semivalues defined on the linear symmetric space generated by this class. Such problems have already been discussed in [MI–N] (in the framework of cost allocations) and in [H-N]. The analogous question in the finite case is discussed in [M-2]. Here we restrict our attention to subspaces of games which contain NA. That way we avoid the pathological cases of values which are not even semivalues like, for example, the value which is identically zero on the space of all games $v \in pNA$ for which $v(I) = 0$. Values on subspaces containing NA are not necessarily of the form $(*)$. Consider for example the linear symmetric space Q_v generated by NA and the game $v = \sqrt{\mu_1\mu_2} - \sqrt{\mu_3\mu_4}$, where the μ_i's are mutually singular measures in NA^1. It was proved in [H-N] that there exists a value ψ defined on Q_v for which ψv is not a linear combination of the μ_i's. Therefore ψ is not of the form $(*)$. In this paper we introduce a space for which there is no such example. This space, pNA_∞, is the closure in an appropriate topology of the linear space spanned by the powers of NA^1-measures. We show that every semivalue ψ on a subspace of pNA_∞ (containing NA) has the form $(*)$, with ξ a Borel probability measure on $[0,1]$. In particular, for every differentiable vector measure game v (such games are proved to be in pNA_∞) ψv is a linear combination of the measures for every value ψ on Q_v.

These results are derived from the Main Theorem which states that every semivalue on a subspace of pNA_∞ which contains NA can be extended to a semivalue on pNA_∞. They can be found, together with the Main Theorem, in §2. In §1 we establish some tools needed in the proof of the Main Theorem. We discuss there a generalization to the nonatomic case of one of Milnor's reasonable axioms (see [MIL]): A coalition can't get more than the sum of the maximal marginal contributions of its members and can't get less than the sum of the minimal marginal contributions of its

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members. The results obtained in §1 enable us also to prove a sort of generalization of the diagonality property of continuous values proved in [N]. This generalization, as well as some generalizations of the Main Theorem, are given in §3.

1. Nonatomic Lipschitz games. In this section we define the notion of a Milnor operator and discuss some of its properties. Here, and throughout the discussion we assume the reader is familiar with the terminology and notations of [A-S]. We start with several other needed definitions:

For every u and v in BV write $u \preceq v$ if $(v - u) \in BV^+$. Obviously for λ and μ in FA $\lambda \preceq \mu$ iff $\lambda \leqslant \mu$. A game v in BV is a *Lipschitz game* if there exists $\mu \in NA^+$ s.t. $-\mu \preceq v \preceq \mu$. Note that $-\mu \preceq v \preceq \mu$ iff for every $S \subseteq T \subseteq I$

$$|v(T) - v(S)| \leqslant \mu(T) - \mu(S). \tag{1.1}$$

The space of all Lipschitz games is denoted by AC_∞. Clearly $NA \subseteq AC_\infty \subseteq AC$. It is worth mentioning that AC_∞ is not BV-closed. For example, for every $\mu \in NA^1$, $\sqrt{\mu} \notin AC_\infty$ but $\sqrt{\mu}$ is the (BV-)limit of polynomials in NA^1-measures.

For every $v \in AC_\infty$ define:

$$\|v\|_\infty = \inf \mu(I), \tag{1.2}$$

where the inf ranges over all $\mu \in NA^+$ s.t. $-\mu \preceq v \preceq \mu$. Obviously $\| \; \|_\infty$ is a norm on AC_∞, $\| \; \|_{BV} \leqslant \| \; \|_\infty$ and both norms coincide on NA. Actually, it can be proved that $(AC_\infty, \| \; \|_\infty)$ is a Banach space, but we omit the proof since we have no use for this fact in what follows.

Milnor operators. For every μ_1, μ_2 in NA denote:

$$(\mu_1 \vee \mu_2)(S) = \sup_{S_1 \subseteq S} (\mu_1(S_1) + \mu_2(S \setminus S_1)),$$

$$(\mu_1 \wedge \mu_2)(S) = \inf_{S_1 \subseteq S} (\mu_1(S_1) + \mu_2(S \setminus S_1)).$$

It is known that $\mu_1 \vee \mu_2$ and $\mu_1 \wedge \mu_2$ belong to NA and that they are the least upper bound (l.u.b.) and the greatest lower bound (g.l.b.) of $\{\mu_1, \mu_2\}$ respectively. In fact it is known that NA is a Banach lattice (see [L-T]). By a slight change in the proofs of Theorem III.7.5 and Corollary III.7.6 of [D-S] one gets that NA is also a complete lattice. That is every nonempty bounded from above (respectively, from below) subset of NA has an l.u.b. (respectively, a g.l.b.).

Let $-\mu \preceq v \preceq \mu$, where $v \in AC_\infty$ and $\mu \in NA^+$. Denote $D^v = \{\lambda \in NA: \lambda \succeq v\}$ and $D_v = \{\lambda \in NA: \lambda \preceq v\}$. Clearly $\mu \in D^v$ and $-\mu$ is a lower bound of D^v. Thus, D^v has a g.l.b. Similarly D_v has an l.u.b. We define:

$$v^* = \text{g.l.b.}\, D^v, \quad v_* = \text{l.u.b.}\, D_v.$$

A tedious (but quite trivial) proof shows that

$$v^*(S) = \sup \sum_{i=1}^{n} (v(S_i \cup T_i) - v(T_i)), \quad \text{and} \tag{1.3}$$

$$v_*(S) = \inf \sum_{i=1}^{n} (v(S_i \cup T_i) - v(T_i)), \tag{1.4}$$

where the sup and the inf range over all finite partitions $(S_1, \ldots, S_n)$ of S and over all finite sequences $(T_1, \ldots, T_n)$ s.t. $T_i \subseteq S_i^c$. Let $\psi: Q \to FA$, where $Q \subseteq AC_\infty$. ψ is a *Milnor operator* (or ψ satisfies the *Milnor axiom*) if $\forall v \in Q$

$$v_* \leqslant \psi v \leqslant v^*. \tag{1.5}$$

Note that (1.5) implies that ψv is actually in NA (and not merely in FA).

Let $NA \subseteq Q \subseteq AC_\infty$ be a linear space and let $\psi: Q \to FA$ be a linear operator. By Lemma 1.4 of [M-2] ψ is a Milnor operator iff ψ is a positive projection onto NA. Thus, on subspaces of AC_∞ which contain NA semivalues are precisely the linear symmetric Milnor operators and values are precisely the efficient semivalues. Also:

Lemma 1.6. *Every linear Milnor operator ψ is $\| \|_\infty$-continuous and $\|\psi\|_\infty \leqslant 1$.*

Proof. It is easy to verify that for every $v \in AC_\infty$

$$\|v\|_\infty = \| \, |v_*| \vee |v^*| \, \|, \tag{1.7}$$

where $| \, |$ stands for the usual absolute value of a measure.

For every $v \in Q$, $v_* \leqslant \psi v \leqslant v^*$. Therefore we have:

$$-(|v_*| \vee |v^*|) \leqslant -|v_*| \leqslant v_* \leqslant \psi v \leqslant v^* \leqslant |v^*| \leqslant (|v_*| \vee v^*|).$$

Combine the last inequalities with (1.7) to get the desired result. ∎

It can now be proved:

Theorem 1.8 (The Extension Property of Milnor Operators). *Every linear Milnor operator on a subspace of AC_∞ can be extended to a linear Minor operator on AC_∞.*

Proof. The proof is similar to the analogous proof of Lemma 1.7 in [M-2]. Use Zorn's Lemma instead of the induction process used there. ∎

2. The space pNA_∞. Let pNA_∞ be the $\| \|_\infty$-closed linear space spanned by all powers of NA^1-measures. Obviously every differentiable vector measure game belongs to pNA_∞, but the inclusion $pNA_\infty \subseteq pNA$ is a strict one. For example, for $\mu \in NA^1$, $\sqrt{\mu} \in pNA \setminus pNA_\infty$. It is easy to verify that for every $v \in pNA_\infty$, $S \subseteq I$ and $t \in [0,1]$, $\partial v(t1_I, 1_S)$ exists, $\partial v(t1_I, \cdot) \in NA$, and $\partial v(\cdot, 1_S)$ is continuous on $[0,1]$ (see Chapter IV in [A-S]).

We are now ready to state the next theorem whose proof can be derived from the proofs of the analogous results in [D-N-W] and [M-1]:

Theorem 2.1. *Let $\psi: pNA_\infty \to FA$. Then ψ is a linear symmetric $\| \|_\infty$-continuous operator iff there exists a Borel measure ξ on $[0,1]$ s.t. for every $v \in pNA_\infty$ and $S \subseteq I$*

$$\psi v(S) = \psi_\xi v(S) = \int_0^1 \partial v(t1_I, 1_S)\, d\xi(t). \tag{2.2}$$

Moreover, the correspondence $\xi \leftrightarrow \psi_\xi$ is linear and 1-1 and the following hold:

(1) $\|\psi\|_\infty = \|\xi\|$.
(2) *ψ_ξ is positive iff $\xi \geqslant 0$.*
(3) *ψ_ξ is a projection onto NA iff $\xi[0,1] = 1$.*
(4) *ψ_ξ is efficient iff ξ is the Lebesgue measure λ.*
(5) *ψ_ξ is BV-continuous iff $\xi \ll \lambda$ and $d\xi/d\lambda \in L_\infty[0,1]$.*

Consequently:
(a) ψ_ξ *is a semivalue iff* ξ *is a probability measure.*
(b) *There exists a unique value on* pNA_∞. *It is the A-S value obtained by setting* $\xi = \lambda$ *in* (2.2). ■

MAIN THEOREM. *Every semivalue on a subspace of* pNA_∞ *which contains NA can be extended to a semivalue on* pNA_∞.

Hence every semivalue and every value on a subspace of pNA_∞ *which contains NA has the form* (2.2). ■

We first give a sketch of the proof to clarify how the forthcoming results fit in it.

SKETCH OF PROOF OF MAIN THEOREM. Let $\psi: Q \to FA$ be a semivalue. Denote by $M[0,1]^1$ the compact (in the weak* topology) set of all Borel probability measures on $[0,1]$. We have to show that there exists $\xi \in M[0,1]^1$ s.t. $\psi v = \psi_\xi v$ for every $v \in Q$. Since $M[0,1]^1$ is compact it suffices to show that for every finite number of games $v_1, v_2, \ldots, v_n$ in Q and for every $\epsilon > 0$ there exists $\xi \in M[0,1]^1$ s.t. $\|\psi v_i - \psi_\xi v_i\| < \epsilon$.

Given $v_1, v_2, \ldots, v_n$ in Q one can find $\mu \in NA^1$ for which $v_i \in pNA_\infty(\mu)$ (= the $\| \|_\infty$-closure of the linear space spanned by all powers of NA^1 measures which are absolutely continuous w.r.t. μ). Let $A = (\pi_m)_{m=1}^\infty$ be a μ-admissible sequence of partitions of I which is also a diadic sequence. To each such A we associate a group $H = \bigcup_{m=1}^\infty G_m$ of symmetries of $(I, \mathscr{C})$ s.t. $G_m \subseteq G_{m+1}$, $\#G_m = (\#\pi_m)!$, and G_m is isomorphic to the permutation group of π_m. Also associated to each such A is the space of multilinear games $ML(\mu, A)$, which is the linear space spanned by all games $v = f \circ (\mu^{S_1}, \ldots, \mu^{S_n})$, where f is a multilinear function, $S^i \in \bigcup_{m=1}^\infty \pi_m$, and μ^{S_i} is the normalized restriction of μ to S_i.

In Lemma 2.3 we prove that $ML(\mu, A)$ is $\| \|_\infty$-dense in $pNA_\infty(\mu)$. Lemma 2.6 (combined with Theorem 1.8) shows that ψ can be extended to an H-symmetric linear Milnor operator on pNA_∞. Finally, in Theorem 2.4 we show that every H-symmetric linear Milnor operator on $ML(\mu, A)$ is the restriction to $ML(\mu, A)$ of ψ_ξ for some ξ in $M[0,1]^1$. We therefore proceed as follows: Fix A. Given $\epsilon > 0$ find $u_1, \ldots, u_n$ in $ML(\mu, A)$ s.t. $\|v_i - u_i\|_\infty < \epsilon/2$. Then extend ψ to an H-symmetric linear Milnor operator $\bar{\psi}$ on pNA_∞. By Theorem 2.4 there exists $\xi \in M[0,1]^1$ s.t. $\bar{\psi} = \psi_\xi$ on $ML(\mu, A)$. This implies that $\|\psi v_i - \psi_\xi v_i\| < \epsilon$. ■

Here are a few additional definitions and notations that will be needed in the proof:

A game v is a *multilinear* game if it has the form $v = f \circ (\mu_1, \mu_2, \ldots, \mu_n)$, where f is a multilinear function and $\mu_1, \mu_2, \ldots, \mu_n$ are mutually singular measures in NA^+. The space of all multilinear measures is denoted by ML. Let $\mu \in NA^1$. The space of all measures in NA which are absolutely continuous w.r.t. μ is denoted by $NA(\mu)$. $ML(\mu)$, $pNA_\infty(\mu)$ and $pNA(\mu)$ are defined analogously. Let π be a partition of I and let $A = (\pi_m)_{m=1}^\infty$ be an admissible sequence of partitions of I. The algebra generated by π (respectively, by A) is denoted by Σ_π (respectively, by Σ_A). The linear space spanned by all μ^S, $S \in \pi$ (respectively, by all μ^S, $S \in \bigcup_{m=1}^\infty \pi_m$) is denoted by $NA(\mu, \pi)$ (respectively, by $NA(\mu, A)$). Similarly define $ML(\mu, A)$, $pNA_\infty(\mu, A)$ and $pNA(\mu, A)$. Observe that $ML(\mu, A)$ is a linear space but $ML(\mu)$ and ML are not.

LEMMA 2.3. $ML(\mu, A)$ *is* $\| \|_\infty$*-dense in* $pNA_\infty(\mu)$.

PROOF. The proof can be found in §9 of [M-N]. ■

An admissible sequence of partitions $A = (\pi_m)_{m=1}^\infty$ is *diadic* if every $S \in \pi_m$ is the union of exactly two disjoint sets in π_{m+1}. It is a μ*-sequence* if $\mu(S) = \mu(T)$ for every S and T in π_m, $\forall m \geq 1$. Let G be the group of all symmetries of $(I, \mathscr{C})$ i.e., the group of all 1-1 bi-measurable transformations from I onto I. Let $\pi = \{S_1, \ldots, S_n\}$ be a partition of I. A subgroup H of G is a *π-group* if every $\theta \in H$ acts as a different permutation on π. That is, there exists a group isomorphism $\theta \to \theta'$ from H onto the group of symmetries of $\{1, 2, \ldots, n\}$ s.t. $\theta S_i = S_{\theta' i}$ for every $1 \leqslant i \leqslant n$. A subgroup H of G is a *(μ, A)-group* if $G = \bigcup_{m=1}^{\infty} G_m$, where $G_m \subseteq G_{m+1}$, G_m is a π_m-group and $G_m \subseteq G(\mu)$ (= the group of all $\theta \in G$ for which $\theta\mu = \mu$). By the standardness assumption it is easy to see that for every diadic μ-admissible sequence of partitions A there exists a (μ, A)-group.

THEOREM 2.4. *Let $\mu \in NA^1$, let A be a diadic μ-admissible sequence of partitions, let H be a (μ, A)-group. Then for every linear Milnor H-symmetric operator ψ: $ML(\mu, A) \to NA$ there exists a Borel probability measure ξ s.t. ψ is the restriction to $ML(\mu, A)$ of ψ_ξ.*

PROOF. We will need some results from the theory of finite games.

Let $N = \{1, 2, \ldots, n\}$, and let ϕ be a semivalue on the space $G(N)$ of the games in coalitional form on N. Let (u_A), $A \subseteq N$ and $A \neq \varnothing$ be the standard linear base of $G(N)$. That is, $u_A(B) = 1$, if $A \subseteq B$, and $u_A(B) = 0$ otherwise. As ϕ is a semivalue, there exists a unique sequence $(\beta_a)_{a=1}^{n}$ of real numbers such that $\phi u_A(i) = \beta_a$, if $i \in A$ and $\phi u_A(i) = 0$, if $i \notin A$, where a is the cardinality of A.

We say that the sequence $(p_k)_{k=1}^{\infty}$ is a *semivalue sequence*, if for every $n \geqslant 1$ there exists a semivalue on a space of all games on a finite set of n players whose associated sequence $(\beta_k)_{k=1}^{n}$ is the sequence $(p_k)_{k=1}^{n}$. By the proof of Theorem 1(a) in [D-N-W], (p_k) is a semivalue sequence iff there exists a probability measure $\xi \in M[0,1]^1$ such that for every $k \geqslant 1$

$$p_k = \int_0^1 t^{k-1}\, d\xi(t).$$

We now turn to the proof of the theorem.

Let $m \geqslant 1$. Let $G(\pi_m)$ be the space of all finite games in coalition form built on the players' set π_m. Define $\bar{\psi}$: $G(\pi_m) \to R^{\pi_m}$ as follows:

$$\bar{\psi}(w)(S_i) = \psi\left(F_w \circ \left(\mu^{S_1}, \mu^{S_2}, \ldots, \mu^{S_n}\right)\right)(S_i),$$

where F_w is the multilinear extension of w to $[0,1]^{\pi_m}$ (see [O]). It is easy to verify (see [M-N] for details) that $\bar{\psi}$ is a semivalue on $G(\pi_m)$. If u_A is a member of the standard base of $G(\pi_m)$, then it is easily verified that

$$F_{u_A}\left(\mu^{S_1}, \ldots, \mu^{S_n}\right) = \prod_{S \in A} \mu^S.$$

Therefore there exists a sequence $(p_k^m)_{k=1}^{n_m}$ of nonnegative real numbers such that for every $1 \leqslant k \leqslant n_m$, for every $S_1, S_2, \ldots, S_k \in \pi_m$

$$\psi\left(\prod_{i=1}^{k} \mu^{S_i}\right) = p_k^m\left(\sum_{i=1}^{k} \mu^{S_i}\right) \text{ on } \Sigma_m.$$

Let $S_i = A_i^0 \cup A_i^1$, where $A_i^0, A_i^1 \in \pi_{m+1}$, and let $v = \prod \mu^{S_i}$. Then

$$2^k v = \prod_{i=1}^{k}\left(\mu^{A_i^0} + \mu^{A_i^1}\right) = \sum_{\epsilon} \prod_{i=1}^{k} \mu^{A_i^{\epsilon_i}},$$

where ϵ ranges over all vectors $(\epsilon_1, \epsilon_2, \ldots, \epsilon_k)$ of 0 and 1.

By the linearity of ψ it is easily deduced that $\psi v = p_k^{m+1} \sum \mu^{S_i}$ on Σ_{m+1}. In particular $p_k^m = p_k^{m+1}$. Define p_k to be p_k^m for any m such that $n_m \geqslant k$. Then we have just shown that the sequence $(p_k)_{k=1}^{\infty}$ satisfies: for every $m \geqslant 1$, for every $S_1, S_2, \dots, S_k \in \pi_m$

$$\psi v = p_k \sum_{i=1}^{k} \mu^{S_i} \quad \text{on } \Sigma_A.$$

As Σ_A is dense in the Borel algebra $\mathscr{C}$ of I, then the last equality holds on $\mathscr{C}$ too.

It is obvious that (p_k) is a semivalue sequence and therefore there exists $\xi \in M[0,1]^1$ such that for every $k \geqslant 1$

$$p_k = \int_0^1 t^{k-1}\, d\xi(t).$$

The proof of the theorem follows because for $v = \Pi_{S \in A} \mu^S$ (with $\#A = k$),

$$\partial v(t1_I, T) = t^{k-1} \sum_{S \in A} \mu^S(T). \quad ■$$

LEMMA 2.5. *Let MIL be the set of all linear Milnor operators ψ: $pNA_\infty \to FA$ endowed with the weak topology of operators. That is, $\psi_\alpha \to \psi$ iff $\psi_\alpha v \to \psi v$ (in the weak topology of FA) $\forall v \in pNA_\infty$. Then MIL is compact.*

PROOF. It is obvious that MIL is a closed subset of the space of all linear $\|\,\|_\infty$-continuous operators ψ: $pNA_\infty \to FA$. Therefore it suffices to prove that $\forall v \in pNA_\infty$ $\{\psi v : \psi \in MIL\}$ is a relatively weakly compact subset of FA. This can be proved by showing that there exists $\mu \in NA^+$ s.t. $\{\psi v : \psi \in MIL\}$ is uniformly absolutely continuous w.r.t. μ (see Theorem IV.8.9 in [D-S]). Denote $\mu = |v_*| \vee |v^*|$. Then $-\mu \leqslant \psi v \leqslant \mu$, $\psi \in MIL$. Hence for every $\epsilon > 0$ and for every $S \subseteq I$,

$$\mu(S) < \epsilon \Rightarrow |\psi v(S)| < \epsilon$$

which yields the desired result. ■

A group H is a *locally finite group* (see [R]) if any subgroup of H generated by a finite subset of H is finite. For example, every (μ, A)-group is a locally finite group.

LEMMA 2.6. *Let H be a locally finite subgroup of G. Let K be a nonempty compact convex H-symmetric subset of MIL (that is, $\theta^{-1} \circ \psi \circ \theta \in K$ $\forall \theta \in H$).*

Then there exists $\psi \in K$ s.t. $\theta^{-1} \circ \psi \circ \theta = \psi$ $\forall \theta \in H$.

PROOF. For every finite subset F of H denote

$$K_F = \{\psi \in K : \theta^{-1} \circ \psi \circ \theta = \psi \ \forall \theta \in F\}.$$

We have to show that $\bigcap_F K_F$ is not empty, where F ranges over all finite subsets of H. Since $\bigcap_{i=1}^n K_{F_i} = K_{\bigcup_{i=1}^n F_i}$ it suffices to prove (by the finite intersection property of compact sets) that $K_F \neq \varnothing$ for every finite F. Since every finite subset of H is contained in a finite subgroup of H it suffices to prove that $K_F \neq \varnothing$ for every finite subgroup F of H. Let then F be a finite subgroup of H. Denote

$$\psi = \frac{1}{\#F} \sum_{\theta \in F} (\theta^{-1} \circ \psi \circ \theta).$$

Then clearly $\psi \in K_F$. ■

Proof of the Main Theorem. Let $\psi\colon Q \to FA$ be a semivalue, where $NA \subseteq Q \subseteq pNA_\infty$. We will show that for every $v_1, v_2, \ldots, v_n$ in Q and for every $\epsilon > 0$ there exists a Borel probability measure ξ s.t.

$$\|\psi v_i - \psi_\xi v_i\| < \epsilon. \tag{2.7}$$

This will complete the proof since it implies (by the compactness of $M[0,1]^1$ in the weak* topology) that for every $v_1, v_2, \ldots, v_n$ in Q there exists $\xi \in M[0,1]^1$ s.t.

$$\psi v_i = \psi_\xi v_i, \qquad 1 \leqslant i \leqslant n. \tag{2.8}$$

Now, for every finite subset D of Q denote

$$K_D = \left\{\xi \in M[0,1]^1 \colon \psi v = \psi_\xi v \ \forall v \in D\right\}.$$

Clearly K_D is compact and, for every $D_1, D_2, \ldots, D_n$, $\bigcap_{i=1}^n K_{D_i} = K_{\bigcup_{i=1}^n D_i} \neq \varnothing$ (by (2.8)).

Since $M[0,1]^1$ is compact, $\bigcap_D K_D \neq \varnothing$ which yields the existence of $\xi \in M[0,1]^1$ s.t. $\psi v = \psi_\xi v$ $\forall v \in Q$. Thus, ψ_ξ is an extension of ψ to pNA_∞.

We now prove (2.7): Let $v_1, v_2, \ldots, v_n \in Q$ and $\epsilon > 0$. Since $rv_i \in pNA_\infty$ there exists $\mu \in NA^1$ s.t. $v_i \in pNA_\infty(\mu)$ $\forall 1 \leqslant i \leqslant n$. Let A be a diadic μ-sequence of partitions. By Lemma (2.3) we can find $u_i \in ML(\mu, A)$ s.t. $\|v_i - u_i\|_\infty < \epsilon/2$. Let H be a (μ, A)-group and let K be the set of all $\tilde{\psi} \in MIL$ which extend ψ, $K \neq \varnothing$ by Theorem 1.8. Since ψ is symmetric K is, in particular, H-symmetric. Therefore by Lemma 2.6 there exists $\tilde{\psi} \in K$ which is H-symmetric. By Theorem 2.4 there exists $\xi \in M[0,1]^1$ s.t. (for every $u \in ML(\mu, A)$ and in particular) for every u_i, $1 \leqslant i \leqslant n$, $\tilde{\psi} u_i = \psi_\xi u_i$. Hence:

$$\|\psi v_i - \psi_\xi v_i\| = \|\tilde{\psi} v_i - \psi_\xi v_i\| \leqslant \|\tilde{\psi} v_i - \tilde{\psi} u_i\| + \|\psi_\xi u_i - \psi_\xi v_i\| \leqslant 2\|u_i - v_i\|_\infty < \epsilon. \quad \blacksquare$$

Corollary 2.9. *Let $v = f \circ (\mu_1, \mu_2, \ldots, \mu_n)$, where $\mu_1, \mu_2, \ldots, \mu_n \in NA^1$ and f is a continuously differentiable function defined on the range of $(\mu_1, \mu_2, \ldots, \mu_n)$ with $f(0) = 0$.*

Then, for every value ψ on the linear symmetric space generated by v and NA, ψv is a linear combination of $\mu_1, \mu_2, \ldots, \mu_n$.

Proof. W.l.o.g. (using standard arguments developed in [A-S]) assume that the range of $(\mu_1, \mu_2, \ldots, \mu_n)$ is of a full dimension. Hence:

$$\psi v = \sum_{i=1}^{n} \left(\int_0^1 \frac{\partial f}{\partial x_i}(te)\, d\xi(t) \right) \mu_i,$$

where $\psi = \psi_\xi$ and $e = (1, 1, \ldots, 1)$. $\blacksquare$

Corollary 2.10. *Let $v = \mu_1^{n_1} \mu_2^{n_2}, \ldots, \mu_k^{n_k} \in Q$, where $\mu_i \in NA^1$ and Q is a linear symmetric subspace of BV that contains NA, and let ψ be a value on Q. Then:*

$$\psi v = \sum_{i=1}^{k} \frac{n_i}{n} \mu_i,$$

where $n = \sum_{i=1}^k n_i$. $\blacksquare$

3. Additional applicatio

The diagonality property.
diagonal. Examining the pr
efficiency properties of th
BV-continuous operator is
assumption in the above m
in [N-T] and [T]. Here we p

Theorem 3.1. *Every li*
AC_∞ is diagonal.
Consequently:
Every semivalue and eve
diagonal.

Proof. Let $v \in Q \cap D$
and $M > 0$ s.t. for every n

(3.2)

In addition it was proved i
$n \geqslant 1$ there exists a sequenc

(3.3)

By (3.2), (3.3) and the lineari
$n \geqslant 1$: $\|\psi\|_\infty M \geqslant M_1\sqrt{n}\|\psi v\|$
We now prove (3.2):
Let $\mu_1, \mu_2, \ldots, \mu_k \in NA^1$

$\max_{i,j} |\mu$

Assume also that $-\mu_0 \preceq v \preceq$

Since μ is a nonatomic prob
exists $\theta \in G$ which is a μ-mi
Let $U = \{T \subseteq I\colon \max_{i,j} |\mu_i($
s.t. for every $S \subseteq I$: $\#\{j \geqslant 1$:
be any sequence of ± 1. Then

$\left| \sum_{j=1}^{n} \epsilon_j \theta^j v(T) - \right.$

3. Additional applications.

The diagonality property. In [N] it was proved that every BV-continuous value is diagonal. Examining the proof one can see that no use was made of the positivity or efficiency properties of the value. Thus, we actually have: Every linear symmetric BV-continuous operator is diagonal. Generally, one can't remove the BV-continuity assumption in the above mentioned result. Examples to nondiagonal values are given in [N-T] and [T]. Here we prove:

THEOREM 3.1. *Every linear symmetric $\| \|_\infty$-continuous operator on a subspace of AC_∞ is diagonal.*

Consequently:

Every semivalue and every value on a subspace of AC_∞ which contains NA is diagonal.

PROOF. Let $v \in Q \cap DIAG$. We will prove that there exists a symmetry $\theta \in G$ and $M > 0$ s.t. for every $n \geqslant 1$ and for every sequence $(\epsilon_1, \epsilon_2, \dots, \epsilon_n)$ of ± 1:

$$\left\| \sum_{j=1}^{n} \epsilon_j \theta^j v \right\|_\infty \leqslant M. \tag{3.2}$$

In addition it was proved in [N] that there exists a constant $M_1 > 0$ s.t. for every $n \geqslant 1$ there exists a sequence $(\epsilon_1, \epsilon_2, \dots, \epsilon_n)$ of ± 1 s.t.

$$\left\| \sum_{j=1}^{n} \epsilon_j \theta^j \psi(v) \right\| \geqslant M_1 \sqrt{n} \|\psi(v)\|. \tag{3.3}$$

By (3.2), (3.3) and the linearity, symmetry and $\| \|_\infty$-continuity of ψ we have for every $n \geqslant 1$: $\|\psi\|_\infty M \geqslant M_1 \sqrt{n} \|\psi v\|$ which yields that $\psi v = 0$.

We now prove (3.2):

Let $\mu_1, \mu_2, \dots, \mu_k \in NA^1$ and $\delta > 0$ satisfy:

$$\max_{i,j} |\mu_i(S) - \mu_j(S)| < \delta \Rightarrow v(S) = 0.$$

Assume also that $-\mu_0 \preceq v \preceq \mu_0$, where $\mu_0 \in NA^+$ and $A = \mu_0(I) \neq 0$. Denote:

$$\mu = \frac{1}{k+1} \left(\sum_{i=1}^{k} \mu_i + \frac{1}{A} \mu_0 \right).$$

Since μ is a nonatomic probability measure on a standard measurable space there exists $\theta \in G$ which is a μ-mixing transformation.

Let $U = \{T \subseteq I: \max_{i,j} |\mu_i(T) - \mu_j(T)| < \delta\}$. By [N] there exists a constant $\alpha > 0$ s.t. for every $S \subseteq I$: $\#\{j \geqslant 1: \theta^j S \notin U\} \leqslant \alpha$. Let, then, $n \geqslant 1$ and let $(\epsilon_1, \epsilon_2, \dots, \epsilon_n)$ be any sequence of ± 1. Then for every $S \subseteq T$ we have:

$$\left| \sum_{j=1}^{n} \epsilon_j \theta^j v(T) - \sum_{j=1}^{n} \epsilon_j \theta^j v(S) \right| = \left| \sum_{j \in J} \epsilon_j \big(v(\theta^j T) - v(\theta^j S) \big) \right|,$$

where $J = \{1 \leqslant j \leqslant n: \theta^j T \notin U \text{ or } \theta^j S \notin U\}$. Therefore:

$$\left|\sum_{j=1}^{n} \epsilon_j \theta^j v(T) - \sum_{j=1}^{n} \epsilon_j \theta v(S)\right| \leqslant \sum_{j \in J} |v(\theta^j T) - v(\theta^j S)|$$

$$\leqslant \sum_{j \in J} \left(\mu_0(\theta^j T) - \mu_0(\theta^j S)\right)$$

$$\leqslant A(k+1) \sum_{j \in J} \left(\mu(\theta^j T) - \mu(\theta^j S)\right)$$

$$= A(k+1)(\mu(T) - \mu(S)) \cdot (\#J)$$

$$\leqslant 2A(k+1)\alpha(\mu(T) - \mu(S)).$$

Hence:

$$\left\|\sum_{j=1}^{n} \epsilon_j \theta^j v\right\|_{\infty} \leqslant 2A(k+1)\alpha. \quad \blacksquare$$

Milnor values. Let Q be a linear symmetric subspace of AC_∞. A *Milnor Semivalue* on Q is a linear symmetric Milnor operator $\psi: Q \to FA$. A *Milnor Value* on Q is an efficient Milnor semivalue. Obviously for subspaces $Q \supseteq NA$ the concepts of a semivalue and that of a Milnor semivalue coincide and so do the concepts of values and Milnor values. It can be easily verified that the Main Theorem as well as Corollaries 2.9 and 2.10 and Theorem 3.1 hold for Milnor semivalues and Milnor values without the assumption $Q \supseteq NA$.

REMARKS. As we have already mentioned, the Main Theorem does not hold for pNA. However, in view of the result of [MI-N] we conjecture that the continuous version holds. That is, every BV-continuous semivalue on a subspace of pNA can be extended to a semivalue on pNA.

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