

Low Discounting and the Upper Long-Run Average Value in Dynamic Programming

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Consider a dynamic programming problem, where the discounted value functions converge to a limit function as the discount factor tends to 1. It is proved that the limit function must be the Markov upper long-run average value function, if the convergence holds in the weak topology on the space of all bounded measurable functions on the state space. Necessary and sufficient conditions for the existence of the weak limit are given. The results are applied to compact continuous dynamic programming problems used extensively in economics. *Journal of Economic Literature* Classification Numbers: C72, C73. © 1994 Academic Press, Inc.

I. INTRODUCTION

Relationships between low discounting and averaging have been studied in several contexts. Mertens and Neyman (1981) showed that in a finite zero-sum stochastic game the discounted value functions converge to the value of the undiscounted game, when the discount factor tends to 1. They also showed that the above property holds for infinite stochastic games, if the discounted value functions of these games satisfy a sort of bounded variation condition. Sorin (1986) showed that the sets of dis-

counted Nash equilibrium payoffs of a finite (nonzero-sum) stochastic game do not necessarily converge to the set of equilibrium payoffs of the undiscounted game. Fudenberg and Maskin (1986) studied the subgame perfect equilibrium payoffs' sets in a low-discounted n -player repeated game. They showed that these sets converge to the individually rational and feasible payoff sets, as the discount factor goes to 1. Fudenberg and Levine (1989) investigated a subset of the equilibrium payoffs in an n -player repeated game with imperfect monitoring.

Lehrer and Monderer (1989) looked at a deterministic *dynamic programming* (from here on denoted DP) problem. They showed that if the discounted value functions converge uniformly on the set of states, then the limit must be the upper long-run average value.

In this paper, we study stochastic DP problems and a weak convergence notion. We provide necessary and sufficient conditions for the existence of both the weak and the uniform limits of the discounted value functions. Moreover, we prove that any limit point of the discounted value functions in either topology must be the Markov upper long-run average value function.

When we use the upper long-run average value, we assume an "optimistic" decision maker who considers the best average he is about to experience infinitely many times. Our results show that, typically, the best present value that a patient decision maker (who does not discount future value by a great amount) can ensure himself is close to the Markov upper long-run average value. The lower long-run average value (which corresponds to a "pessimistic" decision maker who takes into account the worst periods) has been used more often than the upper one. However, unless strong conditions are imposed on the problem (for instance, those in Mertens and Neyman (1981)), our results show that in most cases low-discounting is equivalent (in the sense of the value) to using the upper value. An example of a (deterministic) DP problem, where the lower and upper value functions do not coincide (even though the uniform limit of the discounted value functions exists) is given by Lehrer and Monderer (1989). Lately, Dutta (1991) investigated convergence of optimal policies in stochastic DP.

The paper is organized as follows: In Section 2, we provide necessary preliminaries. In Section 3, we describe stochastic DP problems and state our main results, Theorem A and its corollaries. In Section 4, we describe deterministic DP problems and prove that it suffices to prove Theorem A for the deterministic case. In Section 5, we define simple deterministic DP problems and show that it suffices to prove Theorem A for the simple model. In Section 6, we prove Theorem A for the simple model. Section 7 is devoted to an application to economic theory.

2. PRELIMINARIES

Throughout this paper, we deal with several measurable spaces. Subsets under discussion of measurable sets are assumed to be measurable. For example, the phrase "for all $T \subset S$," where S is a given measurable set, should be understood as "for all measurable subsets T of S ." The set product $\prod_{i \in I} S_i$ of a given family $((S_i, C_i))_{i \in I}$ of measurable spaces will automatically be endowed with the product σ -field $\prod_{i \in I} C_i$. Let (S, C) be a measurable space. We denote by $\Delta(S, C)$ the set of all probability measures on (S, C) . $\Delta(S)$ will stand for $\Delta(S, C)$, when the σ -field C is clear from the context.

The norm $\|\gamma\|$ of a bounded signed measure γ is defined as

$$\|\gamma\| = \sup_{T \subset S} (|\gamma(T)| + |\gamma(T^c)|),$$

where T^c denotes the complementary set of T . The Banach space of all real-valued bounded measurable functions on S (w.r.t. the supremum norm) is denoted by $B(S, C)$. $B(S, C)$ is not shortened to $B(S)$ because the latter will stand for the space of all bounded measurable functions on S (without any measurability restrictions). That is, $B(S) = B(S, 2^S)$, where 2^S is the set of all subsets of S . The topology on $B(S, C)$ is assumed to be the topology induced by the norm unless we specify otherwise. The dual space of a Banach space X is denoted by X^* . The weak topology on X is the smallest topology in which every functional in X^* is continuous. A subset of X is *relatively compact*, if its closure is compact. It is *relatively weakly compact*, if its weak closure is weakly compact. We need the following theorem, whose proof is given in the proof of Theorem V.6.1 in Dunford and Schwartz (1988).

THEOREM 2.1 (Eberlein–Smulian). *Let $M \subseteq X$, where X is a Banach space. Then M is relatively weakly compact iff every sequence $(X_n)_{n=1}^\infty$ in M has a subsequence that converges to an element in X .*

3. THE MODEL AND STATEMENT OF RESULTS

A stochastic DP problem is defined by four objects:

- a measurable space (S, C) , the set of states;
- a measurable space (A, F) , the set of actions;
- a function $q: S \times A \rightarrow \Delta(S)$, the transition function; and
- a function $f: S \times A \rightarrow R$, the payoff function.

The stochastic problems discussed in this paper are assumed to satisfy Properties 3.1–3.3.

Property 3.1. For every $T \subseteq S$, the function

$$(s, a) \rightarrow q(s, a)(T)$$

is measurable on $S \times A$.

Property 3.2. f is measurable and bounded.

The statement of Property 3.3 requires additional notation and is done later.

$$\text{Set } H_n = (S \times A)^{n-1} \times S \text{ for all } n \geq 1.$$

A *strategy* for the decision maker is a sequence $\sigma = (\sigma_1, \sigma_2, \dots)$ of functions

$$\sigma_n: H_n \rightarrow \Delta(A)$$

satisfying that for every $B \subseteq A$ the function

$$h \rightarrow \sigma_n(h)(B)$$

is measurable on H_n for all $n \geq 1$. The set of all strategies is denoted by Σ .

A strategy σ is *Markov*, if it is time and state-dependent and history independent; that is, if there exists a sequence $(f_n)_{n \geq 1}$ of functions $f_n: S \rightarrow \Delta(A)$ such that for all $n \geq 1$, $\sigma_n(h, s) = f_n(s)$ for all $(h, s) \in H_n$. If σ is a Markov strategy, we identify σ_n with f_n . That is, we consider σ_n as a function defined on S only. A Markov strategy σ is *stationary*, if it is time independent; i.e., $\sigma_n = \sigma_1$ for all $n \geq 1$.

Set $H = (S \times A)^N$, where N denotes the set of non-negative integers. The coordinate variables on H will be denoted by $Z_0, A_0, Z_1, A_1, \dots$. Any initial probability measure $P \in \Delta(S)$ and a strategy σ naturally induce a probability measure on H . The expected value operator w.r.t. this measure is denoted by $E_{\sigma, P}$. For every $s \in S$, let $\delta_s \in \Delta(S)$ be the measure concentrated on s only. $E_{\sigma, s}$ stands for E_{σ, δ_s} . Let $X_t, t \geq 0$, be the random variable on H specifying the payoff at time t . That is,

$$X_t = f(Z_t, A_t).$$

Let $0 \leq \lambda < 1$. The λ -discounted value $U_\lambda(P)$ of $P \in \Delta(S)$ is defined by

$$U_\lambda(P) = \sup_{\sigma \in \Sigma} (1 - \lambda) \sum_{t=0}^{\infty} \lambda^t E_{\sigma, P}(X_t).$$

For every $s \in S$ set

$$V_\lambda(s) = U_\lambda(\delta_s).$$

V_λ is called the λ -discounted value function. A strategy σ is ε -optimal for λ , if

$$(1 - \lambda) \sum_{t=0}^{\infty} \lambda^t E_{\sigma, s}(X_t) \geq V_\lambda(s) - \varepsilon \quad \text{for all } s \in S.$$

We are now able to state Property 3.3.

Property 3.3 For all $0 \leq \lambda < 1$ and for all $\varepsilon > 0$ there exists an ε -optimal strategy for λ .

Almost every non-pathological DP problem satisfies Properties 3.1–3.3. See, e.g., Blackwell (1965), Bhattachayara and Majumdar (1989a, 1989b), and Stokey and Lucas with Prescott (1989). We now state without proofs three of the basic properties of the discounted value functions. The proofs are omitted because they can be easily deduced from Properties 3.1–3.3.

LEMMA 3.1. V_λ is measurable and bounded. That is, $V_\lambda \in B(S, C)$.

LEMMA 3.2. For all $P \in \Delta(S)$,

$$U_\lambda(P) = \int_S V_\lambda(s) dP(s).$$

LEMMA 3.3. For all $P, Q \in \Delta(S)$,

$$|U_\lambda(P) - U_\lambda(Q)| \leq M \|P - Q\|,$$

where

$$M = \sup_{S \times A} |f(s, a)|.$$

Let $P \in \Delta(S)$. The upper long-run average value $\bar{U}(P)$ is defined by

$$\bar{U}(P) = \sup_{\sigma \in \Sigma} \limsup_{T \rightarrow \infty} \left(\frac{1}{T+1} \sum_{t=0}^T E_{\sigma, P}(X_t) \right).$$

The Markov upper long-run average value $\bar{U}_M(P)$ is defined by

$$\bar{U}_M(P) = \sup_{\sigma \in \Sigma_M} \limsup_{T \rightarrow \infty} \left(\frac{1}{T+1} \sum_{t=0}^T E_{\sigma, P}(X_t) \right),$$

where Σ_M denotes the set of Markov strategies. Obviously $\bar{U}_M(P) \leq \bar{U}(P)$. Various conditions on a DP problem may be imposed to ensure equality. The issue of equality between \bar{U} and \bar{U}_M is beyond the scope of this paper. As before, $\bar{V}(s)$ and $\bar{V}_M(s)$ stand for $\bar{U}(\delta_s)$ and $\bar{U}_M(\delta_s)$, respectively.

\bar{V} (respectively, \bar{V}_M) is called the (respectively, Markov) upper long-run average value function. We now state our main theorems.

THEOREM A. *Let $\lambda_n \rightarrow 1$, as $n \rightarrow \infty$, and suppose $\lim_{n \rightarrow \infty} V_{\lambda_n} = V$ in the weak topology of $B(S, C)$. Then $V = \bar{V}_M$.*

COROLLARY B. *Let $\lambda_n \rightarrow 1$, as $n \rightarrow \infty$, and suppose $\lim_{n \rightarrow \infty} V_{\lambda_n} = V$ in the norm topology of $B(S, C)$. Then $V = \bar{V}_M$.*

COROLLARY C. *Suppose $\lim_{\lambda \rightarrow 1} V_\lambda = V$ in the weak topology of $B(S, C)$. Then $V = \bar{V}_M$.*

COROLLARY D. *Suppose $\lim_{\lambda \rightarrow 1} V_\lambda = V$ in the norm topology of $B(S, C)$. Then $V = \bar{V}_M$.*

COROLLARY E. *The limit $\lim_{\lambda \rightarrow 1} V_\lambda$ exists in the weak topology of $B(S, C)$, iff the set $\{V_\lambda : 0 \leq \lambda < 1\}$ is relatively weakly compact subset of $B(S, C)$.*

COROLLARY F. *The limit $\lim_{\lambda \rightarrow 1} V_\lambda$ exists in the norm topology of $B(S, C)$, iff the set $\{V_\lambda : 0 \leq \lambda < 1\}$ is a relatively compact subset of $B(S, C)$.*

We claim that Theorem A implies each of the Corollaries B to F. Indeed, the implications $A \Rightarrow B$, $A \Rightarrow C$, $B \Rightarrow D$, and $B \Rightarrow F$ are self-evident. The implication $A \Rightarrow E$ becomes clear also in view of Theorem 2.1.

4. THE DETERMINISTIC CASE

A deterministic DP problem is a stochastic problem in which the set of states and the set of actions are not endowed with measurable structures. In particular, the transition function and the strategies are assumed to be

deterministic. More precisely, a deterministic DP problem is defined by four objects:

- a set S , the set of states;
- a set A , the set of actions;
- a function $q: S \times A \rightarrow S$, the transition function; and
- a bounded function $f: S \times A \rightarrow R$.

Unlike the stochastic case, we do not impose additional requirements on our deterministic model. A *strategy* is a sequence $\sigma = (\sigma_1, \sigma_2, \dots)$ of functions

$$\sigma_n: (S \times A)^{n-1} \times S \rightarrow A.$$

We do not allow mixed strategies. Every strategy σ and an initial state $s \in S$ naturally determine a play

$$(s_0, a_0, s_1, a_1, \dots) = (s_0(\sigma, s), a_0(\sigma, s), s_1(\sigma, s), a_1(\sigma, s), \dots),$$

where $s_0 = s$, $a_0 = \sigma_1(s)$, and for every $t \geq 1$, $s_t = q(s_{t-1}, a_{t-1})$, and

$$a_t = \sigma_t(s_0, a_0, \dots, s_{t-1}, a_{t-1}, s_t).$$

Let $0 \leq \lambda < 1$. Define

$$V_\lambda(s) = \sup_\sigma \sum_{t=0}^{\infty} \lambda^t f(s_t, a_t),$$

where σ ranges over all strategies, and $(s_t, a_t) = (s_t(\sigma, s), a_t(\sigma, s))$ for all $t \geq 0$.

Note that Property 3.3 (i.e., the existence of ε -optimal strategies for every λ) that was taken as an assumption in the stochastic model is automatically satisfied by every deterministic DP problem.

Denote

$$V(s) = \sup_\sigma \limsup_{T \rightarrow \infty} \left(\frac{1}{T+1} \sum_{t=0}^T f(s_t, a_t) \right).$$

Note that the Markov upper long-run average value coincides, in the deterministic model, with the upper long-run average value.

We now state Theorem A of Section 3 for the deterministic model.

LEMMA A1. *In a deterministic DP problem, suppose $\lambda_n \rightarrow 1$, as $n \rightarrow \infty$, and that $V_{\lambda_n} \rightarrow V$ weakly in $B(S)$. Then $V = \bar{V}$.*

PROPOSITION 4.1. *Lemma A1 implies Theorem A.*

Proof of Proposition 4.1. Let (S, C) , (A, F) , q , and f determine a stochastic DP problem satisfying properties 3.1–3.3, as described in Section 3. Let $\lambda_n \rightarrow 1$ and assume $V_{\lambda_n} \rightarrow V$ weakly in $B(S, C)$. We show that $V = \bar{V}_M$.

Construct a deterministic DP problem \hat{S} , \hat{A} , \hat{q} , and \hat{f} as follows:

- $\hat{S} = \Delta(S)$.
- \hat{A} is the set of all stationary strategies $\sigma: S \rightarrow \Delta(A)$.
- $\hat{q}: \hat{S} \times \hat{A} \rightarrow \hat{S}$ is defined as

$$\hat{q}(P, \sigma)(T) = \int_S \int_A q(s, a)(T) d\sigma(a) dP(s)$$

for all $P \in \hat{S}$, $\sigma \in \hat{A}$, and $T \subseteq S$.

- $\hat{f}: \hat{S} \times \hat{A} \rightarrow R$ is defined by

$$\hat{f}(P, \sigma) = \int_S \int_A f(s, a) d\sigma(a) dP(s).$$

To distinguish between the various value functions that are involved in our proof, we use the following notation: U_λ , V_λ , \bar{U}_M , and \bar{V}_M are the value functions associated with the stochastic model, as described in Section 3. V_λ^d and \bar{V}^d are the λ -discounted value function and the upper long-run average value function, respectively, of the deterministic model.

LEMMA 4.1. $\bar{V}^d = \bar{U}_M$, and $V_\lambda^d = U_\lambda$.

Proof of Lemma 4.1. The first equality is obvious.

The second equality is a consequence of the following result of Blackwell (1965): If there is an ε -optimal strategy for λ for every $\varepsilon > 0$, then for every $\varepsilon > 0$ there is an ε -optimal strategy which is a stationary strategy. ■

LEMMA 4.2. *Suppose $\lambda_n \rightarrow 1$ and $V_{\lambda_n} \rightarrow V$ weakly in $B(S, C)$. Then $U_{\lambda_n} \rightarrow U$ weakly in $B(\Delta(S))$, where*

$$U(P) = \int_S V(s) dP(s).$$

Proof of Lemma 4.2. Define $\psi: B(S, C) \rightarrow B(\Delta(S))$ as follows:

$$(\psi g)(P) = \int_S g(s) dP(s).$$

Note that ψ is linear and $\|\psi g\| \leq \|g\|$ for all $g \in B(S, C)$. Note also that $\psi V = U$, and that by Lemma 3.2 $\psi V_\lambda = U_\lambda$ for all $0 \leq \lambda < 1$.

The operator ψ allows us to transform functionals on $B(\Delta(S))$ to functionals on $B(S, C)$. We have to show that $U_{\lambda_n} \rightarrow U$ weakly in $B(\Delta(S))$. For that matter, it suffices to show that for every finite number of functionals $\phi_1, \phi_2, \dots, \phi_m \in B(\Delta(S))^*$ and for every $\varepsilon > 0$ there exists $N \geq 1$ such that for every $n \geq N$,

$$|\phi_i(U_{\lambda_n}) - \phi_i(U)| < \varepsilon \quad \text{for all } 1 \leq i \leq m. \tag{4.1}$$

Because $\phi_1 \circ \psi, \phi_2 \circ \psi, \dots, \phi_m \circ \psi \in B(S, C)^*$, and because $V_{\lambda_n} \rightarrow V$ weakly in $B(S, C)$, there exists $N \geq 1$ s.t. for all $n \geq N$,

$$|(\phi_i \circ \psi)(V_{\lambda_n}) - (\phi_i \circ \psi)(V)| < \varepsilon \quad \text{for all } 1 \leq i \leq m.$$

As $(\phi_i \circ \psi)(V_{\lambda_n}) = \phi_i(U_{\lambda_n})$, and $(\phi_i \circ \psi)(V) = \phi_i(U)$, we get (4.1). ■

We continue to prove Proposition 4.1: As $V_{\lambda_n} \rightarrow V$ weakly in $B(S, C)$, then by Lemma 4.3 $U_{\lambda_n} \rightarrow U$ weakly in $B(\Delta(S))$, where $U(P) = \int V(s) dP(s)$ for all $P \in \Delta(S)$. By Lemma 4.1, $V_{\lambda_n}^d \rightarrow U$ weakly in $B(\Delta(S))$ and therefore, by Lemma A1, $U = \bar{V}^d$. Hence, $U = \bar{U}_M$ by Lemma 4.1. Because weak convergence implies pointwise convergence, we get

$$U_{\lambda_n}(\delta s) \rightarrow \bar{U}_M(\delta s) \quad \text{for all } s \in S.$$

Therefore

$$V_{\lambda_n}(s) \rightarrow \bar{V}_M(s) \quad \text{for all } s \in S,$$

and hence $V(s) = \bar{V}_M(s)$ for all $s \in S$. ■

In the next section, we reduce the problem one step further. We show that it suffices to prove Lemma A1 for the simple deterministic model.

5. THE SIMPLE DETERMINISTIC MODEL

A simple deterministic DP problem is a deterministic model in which the payoff function depends on the state only. More precisely, a simple DP problem is defined by three objects, S, Γ , and f , where S is the set of

states; $\Gamma(s)$ is a nonempty subset of S for all $s \in S$, the set of states that are reachable from s ; and $f: S \rightarrow R$ is the bounded payoff function.

A play is a sequence $\tau = (s_t)_{t=0}^\infty$ of states s.t. $s_{t+1} \in \Gamma(s_t)$ for all $t \geq 0$. Let $0 \leq \lambda < 1$. Denote

$$V_\lambda(s) = \sup_\tau (1 - \lambda) \sum_{t=0}^\infty \lambda^t f(s_t),$$

where τ ranges over all plays $(s_t)_{t=0}^\infty$ with $s_0 = s$. We also define

$$\bar{V}(s) = \sup_\tau \limsup_{T \rightarrow \infty} \left(\frac{1}{T+1} \sum_{t=0}^T f(s_t) \right).$$

LEMMA A2. *In a simple deterministic DP problem, if $\lambda_n \rightarrow 1$ and $V_{\lambda_n} \rightarrow V$ weakly in $B(S)$, then $V = \bar{V}$.*

PROPOSITION 5.1. *Lemma A2 implies Lemma A1.*

Proof of Proposition 5.1. Let S, A, q, f be the components of a deterministic DP problem as defined in Section 4, and suppose $\lambda_n \rightarrow 1$ and $V_{\lambda_n} \rightarrow V$ weakly in $B(S)$. Construct a simple DP problem \hat{S}, Γ , and \hat{f} as follows:

$$\hat{S} = S \cup (S \times A).$$

For every $z \in \hat{S}$ define $\Gamma(z)$ as follows: for $s \in S$,

$$\Gamma(s) = \{(s, a) : a \in A\},$$

and for $(s, a) \in S \times A$

$$\Gamma(s, a) = \{(q(s, a), b) : b \in A\}.$$

Finally, define $\hat{f}: \hat{S} \rightarrow R$ by

$$\hat{f}(s) = 0 \quad \text{for all } s \in S$$

and

$$\hat{f}(s, a) = f(s, a) \quad \text{for all } (s, a) \in S \times A.$$

To distinguish the simple model from the model under discussion, we

denote by \tilde{V}_λ and \bar{V} the λ -discounted value function and the upper long-run average value function respectively in the simple model.

The proof of Proposition 5.1 follows from the following three lemmas, whose proofs are straightforward and therefore are omitted.

LEMMA 5.1.

$$\tilde{V}_\lambda(s) = \lambda V_\lambda(s) \quad \text{for all } s \in S,$$

and

$$\tilde{V}_\lambda(s, a) = (1 - \lambda)f(s, a) + \lambda V_\lambda(q(s, a)) \quad \text{for all } (s, a) \in S \times A.$$

LEMMA 5.2.

$$\tilde{V}(s) = \bar{V}(s) \quad \text{for all } s \in S,$$

and

$$\tilde{V}(s, a) = \bar{V}(q(s, a)) \quad \text{for all } (s, a) \in S \times A.$$

LEMMA 5.3. Let $\lambda_n \rightarrow 1$ and suppose $V_{\lambda_n} \rightarrow V$ weakly in $B(S)$. Then $\tilde{V}_{\lambda_n} \rightarrow W$ weakly in $B(\hat{S})$, where W is defined by

$$W(s) = V(s) \quad \text{for all } s \in S,$$

and

$$W(s, a) = V(q(s, a)) \quad \text{for all } (s, a) \in S \times A.$$

In the next section we prove Lemma A2.

6. THE PROOF OF LEMMA A2

Before proving Lemma A2, we need more notation and lemmas.

Let $(a_r)_{r \geq 0}$ be a bounded sequence. For every $n \geq 0$ denote

$$S_n(a) = \frac{1}{n+1} \sum_{r=0}^n a_r,$$

and for $k \geq 0$, $S_n(a^k) = S_n(a_k, a_{k+1}, \dots)$. That is,

$$S_n(a^k) = \frac{1}{n+1} \sum_{t=0}^n a_{k+t}.$$

Let $0 \leq \lambda < 1$. Denote

$$S_\lambda(a) = (1 - \lambda) \sum_{t=0}^\infty \lambda^t a_t,$$

and for $k \geq 0$,

$$S_\lambda(a^k) = (1 - \lambda) \sum_{t=0}^\infty \lambda^t a_{k+t}.$$

It is well-known that the following formula holds for $S_\lambda(a)$:

$$S_\lambda(a) = (1 - \lambda)^2 \sum_{t=0}^\infty \lambda^t (t + 1) S_t(a). \tag{6.1}$$

LEMMA 6.1. *Let $\varepsilon > 0$ and let $K \geq 1$. Then there exists $n_0 > K$ such that for every $n \geq n_0$, for every sequence $a = (a_t)_{t=0}^n$ of numbers in $[0, 1]$, there exists $1 \leq L \leq n - K$ such that*

$$S_t(a^L) \geq S_n(a) - \varepsilon \quad \text{for all } 0 \leq t \leq K.$$

Proof. The proof can be found in Lehrer and Sorin (1992). ■

LEMMA 6.2. *Let $\varepsilon > 0$. Then there exists a nondecreasing integer-valued function $M^\varepsilon(\lambda)$, defined for $0 < \lambda < 1$ such that $\lim_{\lambda \rightarrow 1} M^\varepsilon(\lambda) = \infty$, and such that for every sequence $a = (a_t)_{t=0}^\infty$ in $[0, 1]$,*

$$S_\lambda(a) \geq \min \{S_t(a) : 0 \leq t \leq M^\varepsilon(\lambda)\} - \varepsilon.$$

Proof. Note that $(1 - \lambda)^2 \sum_{t=0}^\infty \lambda^t (t + 1) = 1$. Let $M^\varepsilon(\lambda)$ be the smallest integer K for which

$$(1 - \lambda)^2 \sum_{t=0}^K \lambda^t (t + 1) \geq 1 - \varepsilon.$$

It can be easily verified (using (6.1)) that $M^\varepsilon(\lambda)$ satisfies the desired properties. ■

LEMMA 6.3. *Let $a = (a_t)_{t=0}^\infty$ be a sequence in $[0, 1]$, and denote*

$$A = \limsup_{T \rightarrow \infty} S_T(a).$$

Then for every $0 < \lambda_1, \lambda_2, \dots, \lambda_m < 1$ and for every $\varepsilon > 0$, there exists an integer $L \geq 1$ such that

$$S_{\lambda_i}(a^L) \geq A - \varepsilon \quad \text{for all } 1 \leq i \leq m.$$

Proof. Set $\delta = \varepsilon/3$, and $K = \max_{i=1}^m M^\delta(\lambda_i)$, where $M^\delta(\lambda_i)$ is defined in Lemma 6.2. Let n_0 be the integer derived from Lemma 6.1 w.r.t. δ and K . Let $n \geq n_0$ be an integer satisfying

$$S_n(a) \geq A - \delta. \quad (6.2)$$

By Lemma 6.1 there exists $L \geq 1$ such that

$$S_t(a^L) \geq S_n(a) - \delta \quad \text{for all } 0 \leq t \leq K. \quad (6.3)$$

Since $K \geq M^\delta(\lambda_i)$ for all $1 \leq i \leq m$, and by Lemma 6.2, (6.3) implies

$$S_{\lambda_i}(a^L) \geq S_n(a) - 2\delta \quad \text{for all } 1 \leq i \leq m. \quad (6.4)$$

Combine (6.2) and (6.4) to get

$$S_{\lambda_i}(a^L) \geq A - 3\delta \quad \text{for all } 1 \leq i \leq m.$$

Hence,

$$S_{\lambda_i}(a^L) \geq A - \varepsilon \quad \text{for all } 1 \leq i \leq m. \quad \blacksquare$$

LEMMA 6.4 (Main Lemma) *Let $\varepsilon > 0$ and let $K \geq 1$. Then there exists $n_0 > K$ such that for every $n \geq n_0$ and for every sequence $a = (a_i)_{i=0}^n$ in $[0, 1]$ there exists $1 \leq L \leq n - K$ such that the following are satisfied:*

$$S_{L-1}(a) \geq S_n(a) - \varepsilon \quad (6.5)$$

$$S_t(a^L) \geq S_n(a) - \varepsilon \quad \text{for all } 0 \leq t \leq K. \quad (6.6)$$

Proof. By Lemma 6.1, there exists $n_1 > K$ that ensures that for every $n \geq n_1$ there exists $1 \leq L \leq n - K$ such that (6.6) is satisfied.

Let $n_0 > n_1$ satisfy $n_1/n_0 < \varepsilon$. Let $n \geq n_0$ and let $1 \leq L \leq n - K$ be

the maximal integer satisfying (6.6) We proceed to show that (6.5) is also satisfied. Suppose (6.5) is not satisfied. Then

$$S_{L-1}(a) < S_n(a) - \varepsilon. \tag{6.7}$$

Since

$$S_n(a) = \frac{L}{n+1} S_{L-1}(a) + \frac{n+1-L}{n+1} S_{n-L}(a^L), \tag{6.8}$$

then (6.7) implies

$$S_{n-L}(a^L) > S_n(a).$$

Therefore, $n - L < n_0$, as otherwise one can find an integer $L^1 > L$ for which (6.6) is satisfied in contradiction to the maximality of L . As $n_1/n_0 < \varepsilon$, then $n - L < n_0$, $n > n_1$, and (6.8) imply

$$S_{L-1}(a) > S_n(a) - \varepsilon,$$

in contradiction to (6.7). Therefore (6.5) is satisfied also. ■

LEMMA 6.5. *Let $0 < \lambda_1, \lambda_2, \dots, \lambda_m < 1$ and let $\varepsilon > 0$. Then there exists $0 < \lambda_0 < 1$ such that for $\lambda \geq \lambda_0$ and for every sequence $a = (a_i)_{i=0}^\infty$ in $[0, 1]$ there exists an integer $L \geq 1$ such that*

$$S_{L-1}(a) \geq S_\lambda(a) - \varepsilon,$$

and

$$S_{\lambda_i}(a^L) \geq S_\lambda(a) - \varepsilon \quad \text{for all } 1 \leq i \leq m.$$

Proof. The proof is similar to the proofs of the previous lemmas, and does not involve new ideas. Therefore we omit it. ■

The following lemma is known as the Mazur Theorem. Its proof can be found in Dunford and Schwartz (1988).

LEMMA 6.6 (Mazur Theorem). *Let X be a Banach space. Let $x_n \rightarrow x$ weakly in X . Then for every $\varepsilon > 0$ and for every integer N , there exists a finite number of integers $N \leq n_1 < n_2 < \dots < n_m$, and a sequence $\alpha_1, \alpha_2, \dots, \alpha_m$ of non-negative numbers with $\sum_{i=1}^m \alpha_i = 1$ such that*

$$\|x - \sum_{i=1}^m \alpha_i x_{\xi_i}\| < \varepsilon.$$

Proof of Lemma A2. Let $S, \Gamma,$ and f define a simple deterministic DP problem as described in Section 5. Let $\xi_n \rightarrow 1$ and suppose $V_{\xi_n} \rightarrow V$ weakly in $B(S)$. We proceed to prove that $V = \bar{V}$. We need the following lemma.

LEMMA 6.7. *Let $(s_t)_{t=0}^\infty$ be a play (that is, $s_{t+1} \in \Gamma(s_t)$ for all $t \geq 0$). Then $V(s_t) \geq V(s_{t+1})$ for all $t \geq 0$.*

Proof of Lemma 6.7. Obviously for every $0 \leq \lambda < 1,$

$$V_\lambda(s_t) \geq (1 - \lambda)f(s_t) + \lambda V_\lambda(s_{t+1}). \tag{6.9}$$

Substituting $\lambda = \xi_n$ in (6.9) and letting $n \rightarrow \infty$ yields $V(s_t) \geq V(s_{t+1})$.

Proof of $V \geq \bar{V}$. Let $s_0 \in S$ and let $\varepsilon > 0$. As $V_{\xi_n} \rightarrow V$ weakly in $B(S)$, then by Lemma 6.6 there exists $0 < \lambda_1, \lambda_2, \dots, \lambda_m < 1$ and a convex coefficient $\alpha_1, \alpha_2, \dots, \alpha_m$ such that

$$\|V - \sum_{i=1}^m \alpha_i V_{\lambda_i}\| < \varepsilon. \tag{6.10}$$

Let $(s_t)_{t=0}^\infty$ be a play such that

$$\limsup_{t \rightarrow \infty} S_t(f(s)) \geq \bar{V}(s_0) - \varepsilon, \tag{6.11}$$

where

$$f(s) = (f(s_t))_{t=0}^\infty.$$

By Lemma 6.3 there exists $L \geq 1$ such that

$$S_{\lambda_i}(f(s^L)) \geq \limsup_{t \rightarrow \infty} S_t(f(s)) - \varepsilon \quad \text{for all } 1 \leq i \leq m. \tag{6.12}$$

By (6.11) and (6.12),

$$\sum_{i=1}^m \alpha_i V_{\lambda_i}(s^L) \geq \sum_{i=1}^m \alpha_i S_{\lambda_i}(f(s^L)) \geq \bar{V}(s_0) - 2\varepsilon.$$

Hence

$$V(s^L) \geq \bar{V}(s_0) - 3\varepsilon$$

by (6.10).

Therefore by Lemma 6.7,

$$V(s_0) \geq \bar{V}(s_0) - 3\varepsilon. \tag{6.13}$$

As (6.13) holds for every $\varepsilon > 0$, then $V(s_0) \geq \bar{V}(s_0)$.

To prove the second part of Lemma A2, we need the following:

LEMMA 6.8. *Let $s_0 \in S$ and let $\varepsilon > 0$. Then there exists a finite play $s_0, s_1, \dots, s_L, L \geq 1$, such that*

$$S_{L-1}(f(s)) \geq V(s_0) - \varepsilon$$

and

$$V(s_L) \geq V(s_0) - \varepsilon.$$

Proof of Lemma 6.8. Set $\delta = \varepsilon/4$. By Lemma 6.6, there exist $0 < \lambda_1, \lambda_2, \dots, \lambda_m < 1$ and convex coefficients $\alpha_1, \alpha_2, \dots, \alpha_m$ such that

$$\|V - \sum_{k=1}^m \alpha_k V_{\lambda_k}\| < \delta. \tag{6.14}$$

Let λ_0 be derived from Lemma 6.5 w.r.t. δ and $\lambda_1, \lambda_2, \dots, \lambda_m$.

Apply Lemma 6.6 again to get a finite sequence $\lambda_0 < \eta_1, \eta_2, \dots, \eta_p < 1$ and convex coefficients $\beta_1, \beta_2, \dots, \beta_p$ such that

$$\|V - \sum_{i=1}^p \beta_i V_{\eta_i}\| < \delta.$$

In particular,

$$\sum_{i=1}^p \beta_i V_{\eta_i}(s_0) > V(s_0) - \delta.$$

Therefore there exists at least one index $1 \leq j \leq p$ such that

$$V_{\eta_j}(s_0) > V(s_0) - \delta. \quad (6.15)$$

Set $\lambda = \eta_j$. Hence $\lambda > \lambda_0$ and

$$V_\lambda(s_0) > V(s_0) - \delta. \quad (6.16)$$

Let $(s_t)_{t=0}^\infty$ be a play such that

$$S_\lambda(f(s)) \geq V_\lambda(s_0) - \delta.$$

Therefore, by (6.16),

$$S_\lambda(f(s)) \geq V(s_0) - 2\delta. \quad (6.17)$$

By Lemma 6.5 (recall that $\lambda > \lambda_0$), there exists $L \geq 1$ such that

$$S_{L-1}(f(s)) \geq S_\lambda(f(s)) - \delta \quad (6.18)$$

and

$$S_{\lambda_i}(f(s^L)) \geq S_\lambda(f(s)) - \delta \quad \text{for all } 1 \leq i \leq m. \quad (6.19)$$

By (6.19), $V_{\lambda_i}(s_L) \geq S_\lambda(f(s)) - \delta$. Therefore

$$\sum_{i=1}^m \alpha_i V_{\lambda_i}(s_L) \geq S_\lambda(f(s)) - \delta. \quad (6.20)$$

Combine (6.17) and (6.7) to get

$$S_{L-1}(f(s)) \geq V(s_0) - 3\delta,$$

and combine (6.14), (6.20), and (6.17) to get

$$V(s_L) \geq V(s_0) - 4\delta.$$

Since $4\delta \leq \varepsilon$, and ε is arbitrary, then $V(s_L) \geq V(s_0)$.

Proof of $\bar{V} \geq V$. Let $s_0 \in S$ and let $\varepsilon > 0$. Let $\varepsilon_K > 0$, $K \geq 1$, satisfy $\sum_{K=1}^\infty \varepsilon_K < \varepsilon$.

We construct inductively a play $(s_t)_{t=0}^\infty$ with

$$\limsup_{t \rightarrow \infty} S_t(f(s)) \geq V(s_0) - \varepsilon.$$

Apply Lemma 6.8 to s_0 and ε_1 to get a finite play s_0, s_1, \dots, s_{L_0} , $L_0 \geq 1$, such that

$$\frac{f(s_0) + f(s_1) + \dots + f(s_{L_0-1})}{L_0} \geq V(s_0) - \varepsilon_1,$$

and

$$V(s_{L_0}) \geq V(s_0) - \varepsilon_1.$$

Apply Lemma 6.8 to s_{L_0} and ε_2 to get a finite play $s_{L_0}, s_{L_0+1}, \dots, s_{L_0+L_1}$, $L_1 \geq 1$ such that

$$\frac{f(s_{L_0}) + f(s_{L_0+1}) + \dots + f(s_{L_0+L_1-1})}{L_1} \geq V(s_{L_0}) - \varepsilon_2$$

and

$$V(s_{L_0+L_1}) \geq V(s_{L_0}) - \varepsilon_2.$$

Continuing inductively we end up with a play $(s_t)_{t \geq 0}$ and a sequence of positive integers L_0, L_1, \dots such that

$$S_{L_K-1}(s_{L_0+L_1+\dots+L_{K-1}}) \geq V(s_0) - \sum_{i=1}^K \varepsilon_i \geq V(s_0) - \varepsilon.$$

Therefore

$$\bar{V}(s_0) \geq \limsup_{t \rightarrow \infty} S_t(f(s)) \geq V(s_0) - \varepsilon.$$

As the last inequality holds for all $\varepsilon > 0$, $\bar{V} \geq V$.

7. AN APPLICATION TO ECONOMIC THEORY

In most applications of dynamic programming to economic theory (see, e.g., Stokey and Lucas with Prescott (1989)), the set of states is assumed to be a metric space endowed with the Borel σ -field, the set of actions is

a compact metric space, and the payoff function is continuous and bounded. In some models it is further assumed that the set of states is compact. Such a model will be called in the sequel a *compact-continuous model*. In Bhattacharya and Majumdar (1989b) it was proved, for the continuous model, that $V_\lambda(s) \rightarrow V(s)$, the lower long-run average value, and that there exists a constant c such that $V(s) = c$ for all $s \in S$, if the following two conditions are satisfied:

There exists $s_0 \in S$ and $M > 0$ such that

$$|V_\lambda(s) - V_\lambda(s_0)| \leq M(1 - \lambda) \quad (7.1)$$

for all $0 \leq \lambda < 1$ and for all $s \in S$.

The family of functions

$$\left\{ \frac{V_\lambda}{1 - \lambda} : 0 \leq \lambda < 1 \right\} \quad (7.2)$$

is equicontinuous on the set of states.

We now assume that the set of states is compact, remove the assumption (7.2), significantly relax the assumption (7.1), and get

PROPOSITION 7.1. *Suppose the following assumption is satisfied by a compact-continuous DP problem:*

There exists $s_0 \in S$ such that

$$M(s) = \sup_{0 \leq \lambda < 1} \frac{|V_\lambda(s) - V_\lambda(s_0)|}{1 - \lambda} < \infty \quad \text{for all } s \in S. \quad (7.3)$$

Then there exists a constant c such that

$$\bar{V}_M(s) = c \quad \text{for all } s \in S,$$

and

$$\lim_{\lambda \rightarrow 1} V_\lambda(s) = c \quad \text{for all } s \in S.$$

Proof. We need the following well-known lemma.

LEMMA 7.1. *Let $D \subseteq Y \subseteq X$, where X is a Banach space and Y is a closed subspace of X . Then D is relatively weakly compact in X iff D is relatively weakly compact in Y .*

We first show that the family $D = \{V_\lambda : 0 \leq \lambda < 1\}$ is relatively weakly

compact in $B(S, C)$, where C is the Borel σ -field in S . It can be easily verified that each V_λ is continuous on S . Therefore $D \subseteq C(S)$, the closed subspace of $B(S, C)$ consisting of all continuous functions. Therefore it suffices by Lemma 7.1 to prove that D is relatively weakly compact in $C(S)$. By Theorem 2.1 it suffices to show that for each sequence $\lambda_n \rightarrow 1$ there is a subsequence $(\lambda_{n_k})_{k=1}^\infty$ and a continuous function u on S such that $V_{\lambda_{n_k}} \rightarrow u$ weakly in $C(S)$.

By the theorem in Dunford and Schwartz (1988), it suffices to show that $V_{\lambda_{n_k}}(s) \rightarrow u(s)$ for all $s \in S$.

Let then $\lambda_n \rightarrow 1$. Because $(V_{\lambda_n}(s_0))_{n=1}^\infty$ is a bounded sequence of numbers, then there exists a real number c and a subsequence $(\lambda_{n_k})_{k=1}^\infty$ such that $V_{\lambda_{n_k}}(s_0) \rightarrow c$. By (7.3),

$$\lim_{K \rightarrow \infty} V_{\lambda_{n_k}}(s) = c \quad \text{for all } s \in S.$$

So, $V_{\lambda_{n_k}} \rightarrow u$ weakly in $C(S)$, where $u(s) = c$ for all $s \in S$.

Since D is relatively weakly compact in $B(S, C)$ then by Corollary E (in Section 3), the weak limit $\lim_{\lambda \rightarrow 1} V_\lambda$ exists. As $V_{\lambda_{n_k}} \rightarrow u$ weakly, we deduce that $\lim_{\lambda \rightarrow 1} V_\lambda = u$ weakly in $B(S, C)$. In particular,

$$\lim_{\lambda \rightarrow 1} V_\lambda(s) = c \quad \text{for all } s \in S.$$

By Corollary C we get

$$\bar{V}_M(s) = c \quad \text{for all } s \in S.$$

Proposition 7.1 is applicable to a large variety of economic models, e.g., an optimal growth model (Brock and Mirman, 1972), and an optimal research model (Burdett and Mortenson, 1980). Using the idea of the proof of Proposition 7.1, we can prove the following generalization.

PROPOSITION 7.2. *Suppose the following assumption is satisfied by a compact-continuous DP problem: There exist $s_0 \in S$ and $g \in C(S)$ such that*

$$\sup_{0 \leq \lambda < 1} \frac{|V_\lambda(s) - V_\lambda(s_0) - g(s)|}{1 - \lambda} < \infty \quad \text{for all } s \in S.$$

Then $V_\lambda \rightarrow \bar{V}_M$ weakly in $C(S)$.

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