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k-Price Auctions*

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In this paper we analyze the equilibria structure of k-price auctions, $k \geq 3$, under the independent-private-value assumption. We discuss agents with an arbitrary attitude toward risk. That is, agents may be risk averse or risk seeking, or they may have an alternating attitude toward risk. We provide a characterization of a continuous symmetric equilibrium, prove that there exists at most one such equilibrium, and show that every such equilibrium is differentiable and increasing. We also show some additional general properties of the equilibrium strategies in these auctions. *Journal of Economic Literature* Classification Numbers: C72, C73, D83. © 2000 Academic Press

1. INTRODUCTION

The analysis of k-price auctions for $k \geq 3$ has been considered to be an intellectual exercise which does not have economic application.[1] Monderer and Tennenholtz (1998) suggest that such auctions may play an important role in the new economics evolving in the Internet.[2] In that paper we focus on (on-line) Internet auctions, which are widely used as a selling mechanism for relatively cheap items like TV sets or computer products.[3] We end the paper on Internet auctions with a recommendation to auction organizers to use k-price auctions, $k \geq 3$. Our main motivation for analyz-

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* Some of the results in this paper were previously contained in an earlier draft of the manuscript "Internet Auctions: Are They Gamblers' Attraction?" (Monderer and Tennenholtz, 1998). We thank Arkadi Nemirovski for helpful discussions.

[1] See Wolfstetter (1996).

[2] The Internet suggests new challenges to economic theory and artificial intelligence research, as it exhibits new forms of interaction which are not captured by existing models (see, e.g., Varian, 1995; Boutilier *et al.*, 1997; Monderer and Tennenholtz, 1999).

[3] See http://auction.eecs.umich.edu/other-auctions.html for a listing of some of the related Internet auctions and Internet auction houses.

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ing k-price auctions is the paper Internet Auctions; we give in Section 7 a brief description of the results obtained there.

In the current paper we discuss the issue of existence and uniqueness of equilibrium strategies in k-price auctions, $k \geq 3$.[4] We discuss agents with an arbitrary attitude toward risk. That is, agents may be risk averse or risk seeking, or they may have an alternating attitude toward risk. We use the independent-private-value assumption, which seems to be the right one in auctions with many anonymous participants. We prove that there exists at most one continuous and symmetric equilibrium, and we provide a characterization for such an equilibrium.

Our results on existence and uniqueness are augmented with additional results on the structure of equilibrium strategies. We show that for any distribution function (that satisfies some technical conditions) on the types in a third-price auction we have that (a) agents with any attitude toward risk overbid in equilibrium, and (b) the bids are decreasing with the number of participants when the agents are risk averse. This extends the results obtained by Kagel and Levin (1993), who proved (a) and (b) assuming the uniform distribution function and that the agents are either risk neutral or have constant absolute risk aversion.

2. k-PRICE AUCTIONS

In this section we present the basic definitions needed for the analysis of k-price auctions of a single item in the independent-private-value model. We assume that the seller is risk-neutral with zero valuation of the object, and that he sets a zero reservation price. All of the results in this paper are naturally extended to positive reservation prices. In a k-price auction, $k \geq 1$, the winner (i.e., the agent who gets the object) is the one with the highest bid. In a tie, the winner is determined by a lottery with equal probability for each participant with the maximum bid. The winner pays the k-order statistics of the sequence of bids.[5] There are n potential

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[4] Existence and/or uniqueness of equilibrium for first- or second-price auctions with risk-averse or risk-neutral agents were discussed, e.g., in Riley and Samuelson (1981) and in Maskin and Riley (1984), and more recently, e.g., in Maskin and Riley (1996), Lebrun (1996, 1999), Bajari (1996), Lizzeri and Persico (1997), Athey (1997), and Reny (1998). Equilibrium in k-price auctions, $k \geq 3$, was discussed in Kagel and Levin (1993), Wolfstetter (1996), and implicitly in Athey (1997) and Reny (1998), who discussed general existence theorems of equilibrium in games with incomplete information (see the discussion on noncontinuous equilibrium strategies in Section 8).

[5] As is common in auction theory, we define the k-order statistics of a sequence of bids as the kth highest bid. According to the standard definition in statistics, the k-order statistics should be the kth lowest bid.

buyers of the object, whom we refer to as *agents*, denoted by $1, 2, \ldots, n$, $n \geq 2$. The set of agents is denoted by N. The valuation v_i of agent i is drawn from the interval $[0, 1]$ according to a random variable $\tilde{v}_i$, which is distributed according to a distribution function F; F satisfies the following properties:

D1 F is twice continuously differentiable in $[0, 1]$.

D2 $F'(x) > 0$ for every $x > 0$.

D3 $\lim_{x \to 0}(F(x)/F'(x)) = 0$.

Note that in most of the auction literature D1, D2, and $F'(0) > 0$ are assumed. In such a case assumption D3 is satisfied. However, we make the weaker assumption D3 because it implies the following desired property: If F satisfies D1–D3, so does F^n for every integer $n \geq 1$, where $F^n(x) = (F(x))^n$. By a repeated application of L'Hôpital's rule it can be verified that if F satisfies D1–D2 and there exists an integer $n \geq 1$ for which the nth derivative $F^{(n)}$ of F exists in $[0, 1]$ and $F^{(n)}(0) \neq 0$, then F satisfies D3. Note that D3 implies that the function which is $F(x)/F'(x)$ for $0 < x \leq 1$ and 0 at $x = 0$ is a continuous function on $[0, 1]$. We denote this function by $F(x)/F'(x)$. Hence, $F(0)/F'(0) = 0$.[6] We will denote by P_F the probability measure induced by F on $[0, 1]$, and for any subset of agents $M \subseteq N$, we denote by P_F^M the product probability measure induced by F on $[0, 1]^M$.

We assume that all agents have the same von Neumann–Morgenstern utility function $u(x)$, $-\infty < x < \infty$, which satisfies the following assumptions:

U1 $u(0) = 0$.

U2 u is twice continuously differentiable.

U3 $u'(x) > 0$ for every $x \in R$.

The agents are risk averse if $u'' \leq 0$. The set of all such functions u is denoted by RA. The agents are strictly risk averse, or $u \in SRA$, if $u'' < 0$. Similarly, the agents are risk seeking, or $u \in RS$, if $u'' \geq 0$, and the agents are strictly risk seeking, or $u \in SRS$, if $u'' > 0$. Finally, the agents are risk neutral, or $u \in RN$, if $u'' = 0$ (that is, $u(x) = ax$ for all x, for some $a > 0$). For every $k \geq 1$, $n \geq \max\{2, k\}$, a utility function u which satisfies U1–U3, and a distribution function F which satisfies D1–D3, we denote by $A(k, n, u, F)$ the k-price auction with these parameters. When some of the parameters are clear, we may omit them.

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[6] Actually, all of the results in this paper can be proved under weaker conditions than D1–D3. These conditions allow both F' and F'' to get the value ∞ at 0.

A *strategy* (of any agent) in the auction $A(K, n, u, F)$ is a *continuous* function[7] b: $[0,1] \to [0, \infty)$. For a strategy b, we denote $b_i = b$ for every $i \in N$. The sequence of strategies $(b_i)_{i \in N}$ is denoted by $\hat{b}$, and for every agent i the sequence $(b_j)_{j \neq i}$ is denoted by $\hat{b}_{-i}$. Similarly, for every $v = (v_i)_{i \in N} \in [0,1]^N$ we denote by $v_{-i} = (v_j)_{j \in N \setminus \{i\}}$ the sequence of all other players' types.[8] For every $n \geq 2$ and any sequence of n random variables $Z = (Z_1, Z_2, \dots, Z_n)$ we denote the k-order statistic of this sequence by $Z_{[k]}$, $1 \leq k \leq n$. That is, if $Z_{i_1} \geq Z_{i_2} \geq \cdots \geq Z_{i_n}$, $Z_{[k]} = Z_{i_k}$. Let b be a strategy. For every possible type of agent i, v_i, and for every possible bid x_i, denote by $E_i^k(x_i, b | v_i)$ the expected utility of agent i in the k-price auction $A(K, n, u, F)$ given that his type is v_i, he bids x_i, and all other agents use the strategy b.[9] That is,

$$E_i^k(x_i, b|v_i) = \int_{[0,1]^{N \setminus \{i\}}} \frac{1}{\tilde{s}\left(\hat{b}_{-i}(v_{-i}), x_i\right)} u\left(v_i - \hat{b}_{-i[k-1]}(v_{-i})\right) \times 1_{\{\hat{b}_{-i[1]} \leq x_i\}}((v_{-i}))\, dP_F^{N \setminus \{i\}}(v_{-i}), \tag{2.1}$$

where for $k = 1$, $\hat{b}_{-i[k-1]}(v_{-i}) = x_i$ for all v_{-i}, 1_A denotes the characteristic function of a set A, and $\tilde{s}$ is the integer-valued random variable counting the number of maximum bids, that is,

$$\tilde{s}(z_1, \dots, z_n) = \#\left\{j \in N: z_j = \max_{l \in N} z_l\right\}. \tag{2.2}$$

If b does not have any atom (that is, $P_F(b = x) = 0$ for every bid x), then $\tilde{s}(\hat{b}_{-i}(v_{-i}), x_i) = 1$ almost surely on the set of all v_{-i} for which $\hat{b}_{-i[1]} \leq x_i$. Therefore,

$$E_i^k(x_i, b|v_i) = \int_{[0,1]^{N \setminus \{i\}}} u\left(v_i - \hat{b}_{-i[k-1]}(v_{-i})\right) 1_{\{\hat{b}_{-i[1]} \leq x_i\}}((v_{-i}))\, dP_F^{N \setminus \{i\}}(v_{-i}). \tag{2.3}$$

A continuous function b is an equilibrium[10] strategy in $A(k, n, u, F)$ if for every agent i, and every $v_i \in [0,1]$, $\max_{x_i \in [0, \infty)} E_i^k(x_i, b|v_i)$ is attained at $x_i = b(v_i)$.

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[7] We discuss noncontinuous equilibrium strategies in Section 8.

[8] We apologize for the convenient inconsistency in our notations: b denotes a single strategy, while v denotes a vector of types.

[9] Technically, the conditional expectation function is not uniquely defined at a single type v_i. As it is commonly done, we pick the particular natural version of the conditional expectation given in (2.1) (see the discussion of noncontinuous strategies in Section 8).

[10] Note that we deal with symmetric equilibrium only.

3. UNIQUENESS AND CHARACTERIZATION

THEOREM A. *Let* $k \geq 3$. *Let* u *be a utility function with* $k-1$ *continuous derivatives. A strategy* b *is an equilibrium strategy in the* k*-price auction* α *if and only if the following two conditions hold*:

E1 b *is increasing in the interval* $[0,1]$.

E2 $\int_{t=0}^{x} u(x-b(t))F(t)^{n-k}(F(x)-F(t))^{k-3}F'(t)\,dt = 0$, *for every* $0 \leq x \leq 1$.

Moreover, E2 *has at most one solution in the set of all strategies, and such a solution* (*and in particular every equilibrium strategy*) *satisfies*

(1) $b(0)=0$.

(2) $b(x) > x$ *for every* $0 < x \leq 1$.

(3) $b'(x)$ *exists for every* $0 < x \leq 1$.

The proof of Theorem A for $k \geq 3$ is similar to the proof for $k=3$, except that the main ideas can easily be seen in the case $k=3$, where they are less hidden by the variety of indices. We therefore prove the case $k=3$ only. For further references we state Theorem A separately for $k=3$:

THEOREM AT ($k=3$). *A strategy* b *is an equilibrium strategy in the third-price auction* $A(3,n,u,F)$ *if and only if the following two conditions hold*:

ET1 b *is increasing in the interval* $[0,1]$.

ET2 $\int_{t=0}^{x} u(x-b(t))F(t)^{n-3}F'(t)\,dt = 0$ *for every* $0 \leq x \leq 1$.

Moreover, ET2 *has at most one solution in the set of all strategies, and such a solution* (*and in particular every equilibrium strategy*) *satisfies*

(1) $b(0)=0$.

(2) $b(x) > x$ *for every* $0 < x \leq 1$.

(3) $b'(x)$ *exists for every* $0 < x \leq 1$.

Before we prove Theorem AT we provide the following simple observation, which will be used in the proof of Theorem AT and in the proofs of other results in the paper.

LEMMA B. b *is a solution to* ET2 *when there are* n *agents and the distribution function is* F *if and only if* b *is a solution to* ET2 *when* $n=3$ *and the distribution function is* $G = F^{n-2}$.

Proof of Theorem AT. By Lemma B, it suffices to prove this theorem in the case $n=3$ only. The proof follows from the following seven claims, A1–A7. The proofs of claims A1, A2, and A3 require tools different from those mainly used in this section, and therefore these proofs will be given only after the proof of the other claims.

Claim A1. Every equilibrium strategy is nondecreasing.

A point $v \in [0,1]$ is called a *single* point for b if $b(w) \neq b(v)$ for every $w \neq v$ in $[0,1]$.

Claim A2. Let b be an equilibrium strategy and let v be a single point for b; then

$$\int_{t=0}^{v} u(v - b(t)) F'(t)\, dt = 0.$$

Consequently, if b is increasing in the interval $[0, v_1]$ for some $0 < v_1 \leq 1$, then

$$\int_{t=0}^{v} u(v - b(t)) F'(t)\, dt = 0, \qquad \text{for every } 0 \leq v \leq v_1. \tag{3.1}$$

Claim A3. Every equilibrium strategy is increasing.

Claim A4. Every equilibrium strategy b satisfies ET2 for $n = 3$. That is, $\mathrm{ET2}_{n=3}$ $\int_{t=0}^{x} u(x - b(t)) F'(t)\, dt = 0$ for every $0 \leq x \leq 1$.

Proof. The assertion follows from claims A2 and A3. ■

Claim A5. Every solution b of $\mathrm{ET2}_{n=3}$ in the set of strategies, and in particular every equilibrium strategy, satisfies

(1) $b(0) = 0$.

(2) $b(x) > x$ for every $0 < x \leq 1$.

(3) $b'(x)$ exists for every $0 < x \leq 1$.

Proof. Since b is continuous, we can differentiate both sides of $\mathrm{ET2}_{n=3}$ to get

$$u(x - b(x)) F'(x) + \int_{0}^{x} u'(x - b(t)) F'(t)\, dt = 0, \qquad x \in [0,1].$$

Therefore,

$$b(x) = x - u^{-1}\left(-\frac{1}{F'(x)} \int_{0}^{x} u'(x - b(t)) F'(t)\, dt \right), \qquad x \in (0,1]. \tag{3.2}$$

For every $x > 0$,

$$\frac{1}{F'(x)}\int_0^x u'(x-b(t))F'(t)\,dt \le M\frac{F(x)}{F'(x)},$$

where M is an upper bound of u' in the range of the function $(z,t) \to z - b(t)$, $t, z \in [0,1]$. Therefore by D3 we have

$$\lim_{x\to 0}\frac{1}{F'(x)}\int_0^x u'(x-b(t))F'(t)\,dt = 0.$$

Since b is continuous, the last inequality implies that $b(0) = 0$. Since $u' > 0$ and $F'(x) > 0$ for $x > 0$ $(1/F'(x))\int_0^x u'(x-b(t))F'(t)\,dt > 0$. Therefore $b(x) > x$ for $x > 0$, because $u^{-1}(-z) \le 0$ for $z > 0$. $b'(x)$ exists for $x > 0$ since $F''(x)$ exists, $F'(x) > 0$, and u^{-1} is differentiable. ■

Claim A6. There exists at most one solution to $\mathrm{ET2}_{n=3}$ in the set of strategies. Consequently, there exists at most one equilibrium strategy.

Proof. Assume b_1 and b_2 are strategies satisfying $\mathrm{ET2}_{n=3}$. Then by Claim A5, $b_i(0) = 0$ for $i = 1, 2$, and by (3.2),

$$b_i(x) = x - u^{-1}\left(-\frac{1}{F'(x)}\int_0^x u'(x-b_i(t))F'(t)\,dt\right), \qquad x \in [0,1].$$

Since the b_i's are bounded and nonnegative, there exists some constant m such that for every $t \le x$ in $[0,1]$, $m \le x - b(t) \le 1$. Since $u'(x) > 0$ for every $-\infty < x < \infty$, u^{-1} is a differentiable function. Since both u^{-1} and u' are continuously differentiable, two applications of the Mean Value Theorem yield the existence of a positive constant M such that

$$|b_1(x) - b_2(x)| \le M\frac{1}{F'(x)}\int_0^x |b_1(t) - b_2(t)|F'(t)\,dt, \qquad x \in [0,1]. \tag{3.3}$$

Now, replace every t in (3.3) with an s, replace every x with a t, plug in the resulting inequality in the right-hand side of (3.3), and get

$$|b_1(x) - b_2(x)| \le M^2\frac{1}{F'(x)}\int_0^x |b_1(t) - b_2(t)|F'(t)(x-t)\,dt.$$

Repeating this procedure again, we get

$$|B_1(x) - b_2(x)| \le M^3\frac{1}{F'(x)}\int_0^x |b_1(t) - b_2(t)|F'(t)\frac{(x-t)^2}{2}\,dt.$$

By repeating the above procedure $n - 1$ times we get

$$|b_1(x) - b_2(x)| \le M^n \frac{1}{F'(x)} \int_0^x |b_1(t) - b_2(t)| F'(t) \frac{(x-t)^{n-1}}{(n-1)!}\, dt.$$

Therefore,

$$|b_1(x) - b_2(x)| \le \frac{M^n}{(n-1)!} \frac{1}{F'(x)} \int_0^x |b_1(t) - b_2(t)| F'(t)\, dt, \qquad x \in [0,1]. \tag{3.4}$$

The right-hand side of (3.4) converges to 0 when $n \to \infty$ for every x. Thus $b_1(x) = b_2(x)$ for all x. ■

Claim A7. Every increasing strategy b which satisfies $\mathrm{ET2}_{n=3}$ is an equilibrium strategy.

Proof. Let $0 < v < 1$. Define for $0 \le z \le 1$,

$$L(z) = \int_{v_2=0}^{z} \left(\int_{v_3=0}^{v_2} u(v - b(v_3)) F'(v_3)\, dv_3 \right) F'(v_2)\, dv_2.$$

To show that b is an equilibrium strategy it suffices to prove that $\max L(z)$, $0 \le z \le 1$, is attained at $z = v$. We show that L is increasing on the interval $[0, v]$ and decreasing on $[v, 1]$. Note that $L'(z) = F'(z)g(z)$, where $g(z) = \int_{v_3=0}^{z} u(v - b(v_3))F'(v_3)\, dv_3$. Note that $g'(z) = u(v - b(z))F'(z) > 0$ for $0 < z < b^{-1}(v)$ and $g'(z) \le 0$ for $b^{-1}(v) < z < 1$. As $g(0) = g(v) = 0$ and $b^{-1}(v) < v$ (by Claim A5), $g(z) > 0$ for $0 < z < v$ and $g(z) < 0$ for $z > v$. Therefore $L'(z) > 0$ for $0 < z < v$ and $L'(z) < 0$ for $z > v$. ■

Proof of Claim A1. Assume agents 2 and 3 use the equilibrium strategy b. Let $v_1 \in [0,1]$. For every $0 \le x$ denote by $E(v_1, x)$ the expected utility of agent 1 when he bids x, given that his valuation is v_1. That is,

$$E(v_1, x) = I_1(v_1, x) + J_1(v_1, x) + \tfrac{1}{2} I_2(v_1, x) + \tfrac{1}{3} I_3(v_1, x), \tag{3.5}$$

where

$$I_1(v_1, x) = 2 \int_{b(v_2) < x} \left(\int_{b(v_3) < b(v_2)} u(v_1 - b(v_3)) F'(v_3)\, dv_3 \right) F'(v_2)\, dv_2, \tag{3.6}$$

$$J_1(v_1, x) = \int_{b(v_2)<x}\left(\int_{b(v_3)=b(v_2)} u(v_1 - b(v_3))F'(v_3)\,dv_3\right)F'(v_2)\,dv_2, \tag{3.7}$$

$$I_2(v_1, x) = 2\int_{b(v_2)=x}\left(\int_{b(v_3)<x} u(v_1 - b(v_3))F'(v_3)\,dv_3\right)F'(v_2)\,dv_2, \tag{3.8}$$

and

$$I_3(v_1, x) = \int_{b(v_2)=x}\left(\int_{b(v_3)=x} u(v_1 - x)F'(v_3)\,dv_3\right)F'(v_2)\,dv_2. \tag{3.9}$$

Note that if x is a single point for b,

$$E(v_1, x) = I_1(v_1, x) + J_1(v_1, x).$$

Assume in negation that there exists $v_1 < \bar{v}_1$ with $x_1 = b(v_1) > b(\bar{v}_1) = \bar{x}_1$. Note that there are at most countably many nonnegative numbers z for which $P_F(b = z) > 0$, where P_F is the probability measure induced by F on $[0, 1]$. Because b is continuous and not constant in the interval $[v_1, \bar{v}_1]$, we can assume without loss of generality that $P_F(b = x_1) = P_F(b = \bar{x}_1) = 0$. Therefore $E(v, x_1) = I_1(v, x_1) + J_1(v, x_1)$ and $E(v, \bar{x}_1) = I_1(v, \bar{x}_1) + J_1(v, \bar{x}_1)$ for every $v \in [0, 1]$. As b is an equilibrium strategy,

$$I_1(v_1, x_1) + J_1(v_1, x_1) \geq I_1(v_1, \bar{x}_1) + J_1(v_1, \bar{x}_1), \tag{3.10}$$

and

$$I_1(\bar{v}_1, \bar{x}_1) + J_1(\bar{v}_1, \bar{x}_1) \geq I_1(\bar{v}_1, x_1) + J_1(\bar{v}_1, x_1). \tag{3.11}$$

Adding the inequalities (3.10) and (3.11) yields $A + B \geq 0$, where

$$A = \int_{\bar{x}_1 \leq b(v_2) < x_1}\left(\int_{b(v_3)<b(v_2)} 2\big(u(v_1 - b(v_3)) - u(\bar{v}_1 - b(v_3))\big) \times F'(v_3)\,dv_3\right)F'(v_2)\,dv_2,$$

and

$$B = \int_{\bar{x}_1 \leq b(v_2) < x_1}\left(\int_{b(v_3)=b(v_2)} \big(u(v_1 - b(v_3)) - u(\bar{v}_1 - b(v_3))\big)F'(v_3)\,dv_3\right) \times F'(v_2)\,dv_2.$$

As u is increasing, $u(v_1 - b(v_3)) - u(v_1 - b(v_3)) < 0$ for all v_3, implying

$$P_{F\times F}(A) = P_{F\times F}\big(\{(v_2, v_3)\colon \bar{x}_1 \le b(v_2) < x_1, b(v_3) < b(v_2)\}\big) = 0. \tag{3.12}$$

Let $z = (x_1 + \bar{x}_1)/2$. Given that b is continuous, there exists $v_1 < v_0 < \bar{v}_1$ with $b(v_0) = z$ Therefore one can find $\delta_1, \delta_2, \varepsilon > 0$ such that $v_1 < v_0 - \delta_1 < v_0 + \delta_1 < \bar{v}_1 - \delta_2$ such that for every $v_0 - \delta_1 < v_2 < v_0 + \delta_1$ and for every $\bar{v}_1 - \delta_2 < v_3 < \bar{v}_1$, $b(v_3) < z - \varepsilon < b(v_2) < z + \epsilon$. Hence,

$$P(A) \ge \big(F(v_0 + \delta_1) - F(v_0 - \delta_1)\big)\big(F(\bar{v}_1) - F(\bar{v}_1 - \delta_2)\big) > 0,$$

contradicting (3.12). ■

Proof of Claim A2. Let b be an equilibrium strategy. Since b is continuous and nondecreasing and $P_F(b = y) > 0$ only for countably many values y, it can easily be verified that the set of single points for b is dense in itself in the following sense:

S1 If $0 < v \le 1$ is a single point, then there exists an increasing sequence $(v_n)_{n=1}^{\infty}$ of single points with $v_n < v$ for all n and $\lim_{n\to\infty} v_n = v$.

S2 If $0 \le v < 1$ is a single point, then there exists a decreasing sequence $(v_n)_{n=1}^{\infty}$ of single points with $v_n > v$ for all n and $\lim_{n\to\infty} v_n = v$.

Let v be a single point. By S1 and S2 if we prove the assertion for $0 < v < 1$, then it holds for $v = 0$ and $v = 1$ too. Assume therefore without lose of generality that $0 < v < 1$.

For every $z \in [0,1]$ define

$$L(2) = \int_0^z g(v_2)F'(v_2)\,dv_2 = \int_0^z \big(g_1(v_2) + g_2(v_2)\big)F'(v_2)\,dv_2,$$

where

$$g_1(v_2) = 2\int_{b(v_3)<b(v_2)} u\big(v - b(v_3)\big)F'(v_3)\,dv_3,$$

and

$$g_2(v_2) = 2\int_{b(v_3)=b(v_2)} u\big(v - b(v_3)\big)F'(v_3)\,dv_3.$$

Obviously $L'(z) = g(z)F'(z)$ at every continuity point of g. Let $\{x_n\colon n \ge 1\} = \{x\colon P_F(b = x) > 0\}$, and for every $n \ge 1$ let $[a_n, b_n] = \{v \in [0,1]\colon b(v) = x_n\}$. Then g is continuous at every $v \in [0,1]$ except for v in the countable set $\{a_n, b_n\}_{n=1}^{\infty}$. In particular, g is continuous at every single point. Note also that for every single point z, $g_1(z) = 2\int_0^z u(v - b(v_3))F'(v_3)\,dv_3$, and $g_2(z) = 0$. As for every single point z, $L(z) = E(v, b(z))$, $L(v) \geq L(z)$ for every single point z. By S1 there exists an increasing sequence of single points $(v_n)_{n=1}^{\infty}$ with $\lim_{n\to\infty} v_n = v$. Therefore

$$L'(v) = \lim_{n\to\infty} \frac{L(v) - L(v_n)}{v - v_n} \geq 0.$$

Similarly by S2 we get $L'(v) \leq 0$. Hence $L'(v) = 0$. Hence

$$0 = L'(v) = g(v)F'(v) = g_1(v)F'(v).$$

As $F'(v) > 0$, $g_1(v) = 0$. Therefore

$$\int_0^v u(v - b(v_3))F'(v_3)\,dv_3 = 0. \tag{3.13}$$

As for the second assertion of the claim, if b is increasing in $[0, v_1]$, then every $0 \leq v < v_1$ is a single point and therefore it satisfies (3.13). As the left-hand side of (3.13) is continuous in v, (3.13) is satisfied for $v = v_1$ as well. ■

Proof of Claim A3. Assume in negation that there exists $v_1 < \bar{v}_1$ with $b(v) = x$ for every $v \in [v_1, \bar{v}_1]$. Without loss of generality we can assume that $[v_1, \bar{v}_1]$ is the maximum interval at which b is constantly x, that is, $b(v) < x$ for $v < v_1$ and $b(v) > x$ for $v > \bar{v}_1$. Assume that each of the agents 2 and 3 uses the strategy b and consider agent 1, who computes his expected utility according to (3.5). We show that $x \geq \bar{v}_1$. Assume to the contrary that $x < \bar{v}_1$. Then

$$I_2(\bar{v}_1, x) > 0 \quad \text{and} \quad I_3(\bar{v}_1, x) > 0. \tag{3.14}$$

As b has only a countable number of atoms, we can find a decreasing sequence $(\varepsilon_n)_{n=1}^{\infty}$ with $\varepsilon_n > 0$, $\lim_{n\to\infty} \varepsilon_n = 0$, such that $P_F(b = x + \varepsilon_n) = 0$ and $x + \varepsilon_n < \bar{v}_1$. As b is an equilibrium strategy, for every $n > 1$,

$$E(\bar{v}_1, x) \geq E(\bar{v}_1, x + \varepsilon_n).$$

Therefore

$$\begin{aligned} I_1(\bar{v}_1, x) + J_1(\bar{v}_1, x) + \tfrac{1}{2}I_2(\bar{v}_1, x) + \tfrac{1}{3}I_3(\bar{v}_1, x)& \\ \geq I_1(\bar{v}_1, x + \varepsilon_n) + J_1(\bar{v}_1, x + \varepsilon_n).& \end{aligned} \tag{3.15}$$

Note that for every $n \geq 1$,

$$I_1(\bar{v}_1, x + \varepsilon_n) - I_1(\bar{v}_1, x) = 2\int_{x \leq b(v_2) < x + \varepsilon_n} H(v_2)\, dv_2,$$

where

$$H(v_2) = \int_{b(v_3) < b(v_2)} u(\bar{v}_1 - b(v_3))F'(v_3)\, dv_3.$$

As H is bounded and does not depend on n,

$$\lim_{n \to \infty} \left(I_1(\bar{v}_1, x + \varepsilon_n) - I_1(\bar{v}_1, x) \right) = I_2(\bar{v}_1, x).$$

Similarly

$$\lim_{n \to \infty} \left(J_1(v_1, x + \varepsilon_n) - J_1(\bar{v}_1, x) \right) = I_3(\bar{v}_1, x).$$

Therefore by (3.15)

$$\tfrac{1}{2} I_2(\bar{v}_1, x) + \tfrac{2}{3} I_3(\bar{v}_1, x) \leq 0,$$

contradicting (3.14). Hence $x \geq \bar{v}_1$.

Since $b(v_1) = x \geq \bar{v}_1 > 0$, $v_1 > 0$ because otherwise the expected utility of agent 1 at $v_1 = 0$ is negative (which is impossible in equilibrium, because he can deviate to the zero bid, which yields a zero expected utility). As b has only a countable number of atoms, there exists a decreasing sequence $(\varepsilon_n)_{n=2}^{\infty}$ with $\varepsilon_n > 0$, $\lim_{n \to \infty} \varepsilon_n = 0$, such that $P_F(b = x - \varepsilon_n) = 0$ and $b(0) < x - \varepsilon_n$. Let $(v_n)_{n=2}^{\infty}$ be an increasing sequence with $b(v_n) = x - \varepsilon_n$. Then, $\lim_{n \to \infty} v_n = v_1$. As b is an equilibrium strategy,

$$E(v_1, x) \geq E(v_1, x - \varepsilon_n), \qquad n \geq 2.$$

Therefore, by taking the limit of the right-hand side of the above inequality when $n \to \infty$,

$$\tfrac{1}{2}(v_1, x) + \tfrac{1}{3}(v_1, x) \geq 0.$$

Similarly the inequalities

$$E(v_n, x - \varepsilon_n) \geq E(v_n, x), \qquad n \geq 2$$

imply

$$\tfrac{1}{2} I_2(v_1, x) + \tfrac{1}{3} I_3(v_1, x) \leq 0.$$

Therefore

$$\tfrac{1}{2}I_2(v_1, x) + \tfrac{1}{3}I_3(v_1, x) = 0. \tag{3.16}$$

By Claim A2,

$$\int_0^{v_n} u(v_n - b(v_3))F'(v_3)\,dv_3 = 0 \qquad \text{for } n > 2.$$

Therefore (from continuity considerations)

$$\int_0^{v_1} u(v_1 - b(v_3))F'(v_3)\,dv_3 = 0.$$

Hence

$$I_2(v_1, x) = 2\int_{v_2=v_1}^{\bar{v}_1}\left(\int_{v_3=0}^{v_1} u(v_1 - b(v_3))F'(v_3)\,dv_3\right)F'(v_2)\,dv_2 = 0.$$

Therefore by (3.15) $I_3(v_1, x) = 0$, implying that $u(v_1 - x) = 0$. Hence $x = v_1 < \bar{v}_1$, in contradiction to the inequality $x \geq \bar{v}_1$ we proved. ■

4. THIRD-PRICE AUCTIONS: EXISTENCE IN SPECIAL CASES

Before we deal (in Section 5) with analyzing the issue of the existence of equilibrium, we provide some examples. In this section we relate the following condition to the problem of existence of equilibrium in some special cases:

(MD) $x \to F(x)/F'(x)$ is increasing in $[0, 1]$.

4.1. *Risk Neutrality*

When the agents are risk neutral, equation ET2 can be solved easily and explicitly, yielding the following result:

PROPOSITION C. *Let $u \in RN$. Then, for every $n \geq 3$, the unique solution of ET2 is given by*

$$b_n^{RN}(x) = x + \frac{1}{n-2}\,\frac{F(x)}{F'(x)} \qquad \textit{for all } x \in [0, 1]. \tag{4.1}$$

Consequently, if (MD) *is satisfied, there exists an equilibrium and b_n^{RN} is the equilibrium strategy.*[11]

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[11] Kagel and Levin (1993) proved that b_n^{RN} is an equilibrium strategy when F is the uniform distribution.

Note that by Theorem AT, for $n = 3$, (MD) is also a necessary condition for the existence of equilibrium.

4.2. *Constant Absolute Risk Attitude*

Roughly speaking, an agent has a constant risk attitude if his attitude toward risk does not depend on his level of wealth. Namely, an agent with a utility function u has constant absolute risk seeking (resp. risk aversion) if u''/u' is a positive (resp. negative) constant function. For a detailed discussion of this concept (for risk aversion) see, e.g., Kreps (1988). For $\lambda > 0$ let

$$u_\lambda(x) = 1 - e^{-\lambda x}. \tag{4.2}$$

Technically speaking, it turns out that an agent has a *constant absolute risk aversion* if he uses a utility function of the form au_λ, $a > 0$.[12] Similarly let

$$v_\lambda(x) = e^{\lambda x} - 1. \tag{4.3}$$

An agent has a *constant absolute risk seeking* if he uses a utility function of the form av_λ, $a > 0$. A direct calculation reveals

PROPOSITION D. *Let $\lambda > 0$ and let $u = u_\lambda$. Then, for every $n \geq 3$, the unique solution of* ET2 *is given by*

$$b_n^{CARA}(x) = x + \frac{1}{\lambda}\ln\left(1 + \frac{\lambda}{n-2}\frac{F(x)}{F'(x)}\right) \qquad \text{for all } x \in [0,1].$$

Consequently, if (MD) *is satisfied there exists an equilibrium and b_n^{CARA} is the equilibrium strategy.*[13]

PROPOSITION E. *Let $\lambda > 0$ and assume that $1 - \lambda(F(x)/F'(x)) > 0$, $x > 0$. Let $u = v_\lambda$. Then, for every $n \geq 3$, the unique solution of* ET2 *is given by*

$$b_n^{CARS}(x) = x - \frac{1}{\lambda}\ln\left(1 - \frac{\lambda}{n-2}\frac{F(x)}{F'(x)}\right) \qquad \text{for all } x \in [0,1].$$

Consequently, if (MD) *is satisfied, then there exists an equilibrium and b_n^{CARS} is the equilibrium strategy.*

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[12] Some of the auction theory that deals with risk-averse agents focuses only on agents with constant absolute risk aversion (see, e.g., Matthews, 1983).

[13] Kagel and Levin (1993) proved that b_n^{CARA} is an equilibrium strategy when F is the uniform distribution.

5. EXISTENCE OF A SOLUTION TO E2

Our characterization of equilibrium (Theorem A) cannot be considered a direct existence theorem. In this section we make a first step toward an existence theorem. By Theorem A, there exists at most one solution to E2 In this section we provide sufficient conditions for the existence of a solution when the utility functions are in $RA \cup RS$. The conditions we give are not necessary conditions. They are also not the tighter conditions under which our method of proof works. They are, however, easily testable conditions. As was done in previous proofs, we will first reduce the general existence problem to the problem of existence in third-price auctions with three participants. In this particular case (see (3.2)) b_0 is a solution of E2 if and only if b_0 is a fixed point of the operator $T = T_{u,F}$ defined on the set of strategies b as follows:

$$T_{(u,F)}b(x) = Tb(x) = x - u^{-1}\left(-\frac{1}{F'(x)} \int_0^x u'(x - b(t))F'(t)\,dt \right). \tag{5.1}$$

For the proof of our theorems we need a few notations concerning such operators. Let $C = C([0,1])$ be the Banach space of continuous functions on $[0,1]$ with the max norm. That is, for $b \in C$, $\|b\| = \max_{x \in [0,1]}|b(x)|$. All topological notations in C refer to this norm. Note that C is a complete metric space, and therefore each of its closed subsets is a complete metric space. Let D be a closed subset of C and let S: $D \to D$. We denote by S^n: $D \to D$ the nth iteration of S. That is, $S^1 = S$, and for $n \geq 2$, $S^n(b) = S(S^{n-1}(b))$ for all $b \in D$. We need the following well-known extension of the contraction mapping theorem:

THE EXTENDED CONTRACTION MAPPING THEOREM. *Let D be a closed subset of C and let S: $D \to D$. Assume there exists $n \geq 1$ and $0 \leq a < 1$ such that for every b_1, b_2 in D,*

$$\|S^n(b_2) - S^n(b_1)\| \leq a\|b_2 - b_1\|.$$

Then S has a unique fixed point $b_0 \in D$. Moreover, for every $b \in D$,

$$\lim_{m\to\infty} \|S^{mn}b - b_0\| = 0.$$

We denote by C_I the closed subset of C consisting of all $b \in C$ with $b(x) \geq x$ for every $x \in [0,1]$. By our characterization theorem every equilibrium strategy b must belong to C_I. Therefore we will be interested in the fixed point of T in C_I. However, we first need to know that Tb is well defined for $b \in C_I$. To prove that Tb is well defined, we have to show that $(-(1/F'(x))\int_0^x u'(x - b(t))F'(t)\,dt)$ belongs to the domain of definition of u^{-1} for all $x \in [0,1]$. Note that u^{-1} is defined on an interval containing $(M_{-\infty}(u), 0]$, where

$$M_{-\infty}(u) = \lim_{x \to -\infty} u(x).$$

Thus, if $M_{-\infty}(u) = -\infty$, Tb is well defined. This in particular shows that Tb is well defined when $u \in RA$. For $u \in RS$ we will impose a condition that guarantees that Tb is well defined for all $b \in C_I$, e.g., $M_{-\infty}(u) = -\infty$. Since both u and F are increasing, $Tb(x) \geq x$ for every $x \in [0,1]$. Therefore we have established that $T(C_I) \subseteq C_I$ Hence T: $C_I \to C_I$. We proceed to discuss a few properties of the operator T.

Claim F1. Let F satisfy D1–D3. Let $u \in RS$ with $M_{-\infty}(u) = -\infty$ or $u \in RA$. Let D be a closed subset of C_I such that $T(D) \subseteq D$, where T is given in (5.1). If there exists $M > 0$ such that for every $b_1, b_2 \in D$,

$$|Tb_1(x) - Tb_2(x)| \leq M \frac{1}{F'(x)} \int_0^x |b_1(t) - b_2(t)| F'(t)\,dt, \qquad x \in [0,1], \tag{5.2}$$

then there exists $n \geq 1$ such that T^n is a contraction mapping.

Proof. The proof is similar to the proof of Claim A6. By (5.2),

$$|T^2 b_1(x) - T^2 b_2(x)| \leq M \frac{1}{F'(x)} \int_0^x |Tb_1(t) - Tb_2(t)| F'(t)\,dt.$$

By substituting (5.2) in the right-hand side we get

$$|T^2 b_1(x) - T^2 b_2(x)| \leq M^2 \frac{1}{F'(x)} \int_0^x |b_1(t) - b_2(t)|(x - t) F'(t)\,dt.$$

Repeating this procedure n times yields

$$\begin{aligned} &|T^n b_1(x) - T^n b_2(x)| \\ &\quad \leq M^n \frac{1}{F'(x)} \int_0^x |b_1(t) - b_2(t)| \frac{(x-t)^{n-1}}{(n-1)!} F'(t)\,dt \\ &\quad \leq \frac{M^n}{(n-1!)} \frac{F(x)}{F'(x)} \int_0^x |b_1(t) - b_2(t)|\,dt. \end{aligned}$$

Since $F(x)/F'(x)$ is continuous on $[0,1]$, it is bounded by a positive constant L. Therefore

$$\|T^n b_1 - T^n b_2\| \leq \frac{M^n}{(n-1)!} L \|b_1 - b_2\|.$$

Since $\lim_{n\to\infty}(M^n/(n-1)!) = 0$, the assertion follows. ■

Claim F2. Let F satisfy D1–D3. Let $u \in RS$, with $M_{-\infty}(u) = -\infty$ or $u \in RA$. Let T be the operator defined in (5.1).

(1) If $u \in RA$, then for every $b_1 \leq b_2$ in C_I, $Tb_1 \leq Tb_2$.

(2) If $u \in RS$ and u satisfies $M_{-\infty} = -\infty$, then for every $b_1 \leq b_2$ in C_I, $Tb_1 \geq Tb_2$.

Proof. The assertions are true because u' is nonincreasing for $u \in RA$, and u' is nondecreasing for $u \in RS$. ■

THEOREM F. *Let $k \geq 3$. Consider the k-price auction $A(k,n,u,F)$ with a utility function u that has $k-1$ continuous derivatives. If $u \in RS$ and u satisfies $M_{-\infty}(u) = -\infty$ or $u \in RA$ and all jth-order derivatives*, $2 \leq j \leq k-1$, *are bounded on* $(-\infty, 1]$, *then Eq.* E2 *has a unique solution*, $b(k,n,u,F)$, *in the set of all strategies. Consequently*, $b(k,n,u,F)$ *is the equilibrium strategy if and only if it is increasing in* $[0,1]$.

As in the proof of Theorem A, we prove Theorem F for the case $k = 3$ only (Theorem FT). However, in this case we give a stronger version of Theorem F. We replace the assumption that for $u \in RS$ $M_{-\infty} = -\infty$ with a weaker assumption that still guarantees that T (defined in (5.1)) is well defined for $u \in RS$. Under this assumption, the distribution function F and the utility function u satisfy the following property:

DU $(1/F'(x))\int_0^x u'(x-t)F'(t)\,dt < -M_{-\infty}(u)$ for every $x \in [0,1]$.

If $M_{-\infty}(u) = -\infty$, u satisfies DU for every F. However, DU is not necessarily satisfied in the risk-seeking case.[14] Note that if F and u satisfy DU, so do F^n and u for every $n \geq 1$. If $u \in RS$, u satisfies DU; then because u' is nondecreasing and because of assumption DU,

$$\begin{aligned}\frac{1}{F'(x)}\int_0^x u'(x-b(t))F'(t)\,dt &\leq \frac{1}{F'(x)}\int_0^x u'(x-t)F'(t)\,dt \\ &< -M_{-\infty}(u)\end{aligned}$$

for all $x \in [0,1]$. Hence Tb is well defined.

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[14] For example, when F is the uniform distribution, u satisfies DU if and only if $u(1) < -M_{-\infty}(u)$. Thus, if for $\lambda > 0$, $u_\lambda(x) = e^{\lambda x} - 1$, satisfies U1–U3, but it satisfies DU for the uniform distribution only if $e^\lambda < 2$.

THEOREM FT. *Consider the third-price auction* $A(3, n, u, F)$. *Assume, in addition, that* $u \in RS$ *and that* F *and* u *satisfy DU or* $u \in RA$ *and the second derivative of* u *is bounded on* $(-\infty, 1]$. *Then Eq.* ET2 *has a unique solution,* $b(3, n, u, F)$, *in the set of all strategies. Consequently,* $b(3, n, u, F)$ *is the equilibrium strategy if and only if it is increasing in* $[0, 1]$.

Proof of Theorem FT. As in the proof of Theorem A it suffices to consider the case $n = 3$. By (3.2) (which by Claim A5 is satisfied for $x = 0$ also), $b \in C_I$ satisfies $\mathrm{ET2}_{n=3}$ if and only if b is a fixed point of $T = T_{(u,F)}$ (defined in (5.1)). By the Extended Contraction Mapping Theorem, it suffices to show that there exists a closed subset $D \subseteq C_I$ such that $T(D) \subseteq D$ and such that for some $n \geq 1$, T^n is a contracting mapping on D. By Claim F1 it suffices to show the existence of $M > 0$ for which (5.2) is satisfied. We separate the proof for risk-seeking and risk-averse agents. Let then $u \in RS$ satisfy DU. Assume that $T(D) \subseteq D$. If all functions in D are uniformly bounded, that is, there exists $M_0 > 0$ such that $\|b\| \leq M_0$ for every $b \in D$, then (5.2) is satisfied by some $M > 0$. Denote $I(x) = x$. By Claim F2, $Tb \leq TI$ for all $b \in C_I$. Hence $D = C_{[I, TI]}$ is uniformly bounded and T maps D into D. Suppose $u \in RA$, and the second derivative of u is bounded on $(-\infty, 1]$. Then the first derivative of u^{-1} is bounded above by $1/u'(0)$ on $(-\infty, 0]$ and the absolute value of the second derivative of u is bounded above on $(-\infty, 1]$. Hence, in this case there exists $M > 0$ such that (5.2) is satisfied for every $b_1, b_2 \in C_I$. ■

Note that $u \in RA \cup RS$ satisfies the conditions stated in Theorem FT if there exists $-\infty < c < \infty$ such that u is an affine function in the interval $(-\infty, c]$.

6. THIRD-PRICE AUCTIONS WITH VARYING NUMBERS OF PARTICIPANTS

It can be easily derived from Propositions C, D, and E that for $u \in RN \cup CARA \cup CARS$ the following properties are satisfied in a third-price auction:

(1) If an equilibrium exists for some n, then it exists for any larger number of participants $m > n$.

(2) The equilibrium bid is a decreasing function of the number of participants.[15]

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[15] (1) and (2) have already been proved by Kagel and Levin (1993) for $u \in RN \cup CARA$ and the uniform distribution. For this distribution an equilibrium exists for every number of agents and it is given in the risk-neutral case by $b_n(x) = ((n-1)/(n-2))x$.

We now prove that (2) holds for $u \in RA$ when the second derivative of u is bounded on $(-\infty, 1]$. We do not know whether (1) holds for $u \in RA \cup RS$, or whether (2) holds for $u \in RS$. We actually prove a stronger theorem by proving the monotonicity property for solutions of ET2 that are not necessarily increasing.

THEOREM G. *Let $u \in RA$ have a bounded second derivative in $(-\infty, 1]$, and let F be a distribution function satisfying* D1–D3. *For every $n > 3$ let b_n be a solution of* ET2 *in the third-price auction $A(3, n, u, F)$. Then for every $n \geq 3$,*

$$b_n(x) > b_{n+1}(x), \qquad \textit{for every } 0 < x \leq 1. \tag{6.1}$$

Proof. Note that by Theorem AT, b_n is the unique solution of ET2. We prove (6.1) for $n = 3$. The proof of the general case, $n > 3$, is very similar and therefore will be omitted. Thus we prove that $b_3(x) > b_4(x)$ for all $0 < x \leq 1$. Let

$$h(x) = \int_{t=0}^{x} u(x - b_3(t))F(t)F'(t)\,dt. \tag{6.2}$$

Note that h is a continuously differentiable function on $[0, 1]$ satisfying

$$\lim_{x \to 0} F\frac{h'(x)}{(x)F'(x)} = 0. \tag{6.3}$$

And moreover,

Claim G1. $h'(x) < 0$ for every $0 < x \leq 1$ and $h'(0) = 0$.

Proof of Claim G1. By (6.2), for every $0 \leq x \leq 1$,

$$h'(x) = u(x - b_3(x))F(x)F'(x) + \int_{t=0}^{x} u'(x - b_3(t))F(t)F'(t)\,dt. \tag{6.4}$$

Since $F(t) < F(x)$ for every $0 < t < x$, we have

$$h'(x) < u(x - b_3(x))F(x)F'(x) + F(x)\int_{t=0}^{x} u'(x - b_3(t))F'(t)\,dt, \qquad x > 0.$$

Therefore

$$h'(x) < F(x)\frac{d}{dx}\left(\int_{t=0}^{x} u(x-b_3(t))F'(t)\,dt\right).$$

As b_3 is the unique solution of $\mathrm{ET2}_{n=3}$,

$$\int_{t=0}^{x} u(x-b_3(t))F'(t)\,dt = 0, \qquad 0 \le x \le 1.$$

Hence $h'(x) < 0$ for every $0 < x \le 1$, and by (6.4) $h'(0) = 0$. ■

Consider the equation with the unknown function b:

EH $\int_{t=0}^{x} u(x-b(t))F(t)F'(t)\,dt = h(x)$ for every $0 \le x \le 1$.

Proceeding as in the proof of Claim A6 and Theorem FT, we can prove that there exists a unique continuous function b that satisfies (EH). Obviously this unique solution is b_3, and b_3 is also the unique fixed point of the operator T_h defined as follows:

$$T_h b(x) = x - u^{-1}\left(\frac{h'(x)}{F(x)F'(x)} - \frac{1}{F(x)F'(x)}\int_0^x u'(x-b(t))F(t)F'(t)\,dt\right).$$

Let

$$Tb(x) = x - u^{-1}\left(-\frac{1}{F(x)F'(x)}\int_0^x u'(x-b(t))F(t)F'(t)\,dt\right).$$

By $\mathrm{ET}_{n=4}$, b_4 is a fixed point of T. By Theorem A, b_4 is the unique fixed point of T. By Claim G1, for every $0 < x \le 1$,

$$T_h b(x) > Tb(x) \tag{6.5}$$

for every continuous function b. Applying (6.5) to $b = b_3$ yields

$$b_3(x) > Tb_3(x), \qquad 0 < x \le 1. \tag{6.6}$$

Note that $T = T_{(u, F^2)}$, defined in (5.1). Therefore, by Claim F2 and (6.6), for every $n \ge 1$,

$$b_3(x) > T^n b_3(x) \ge T^{n+1} b_3(x), \qquad 0 < x \le 1. \tag{6.7}$$

By Claim F1, there exists $m \geq 1$ for which T^m is a contraction mapping and therefore, by the extended contraction mapping theorem,

$$\lim_{n \to \infty} T^{mn} b_3(x) = b_4(x)$$

for every $x \in [0, 1]$. Hence, by (6.7),

$$b_3(x) > b_4(x), \qquad \text{for all } 0 < x \leq 1.$$

■

Remark. By Claim F2, Theorem G holds for $u \in RS$, if it can be shown that (6.6) holds with T^2 instead of T, that is, if for every $0 < x \leq 1$, $b_3(x) > T^2 b_3(x)$. We do not know if this inequality holds.

7. *k*-PRICE AUCTIONS AND INTERNET AUCTIONS

In the paper “Internet Auctions: Are They Gamblers’ Attraction?” (Monderer and Tennenholtz, 1998) we recommend to auction organizers that they conduct k-price auctions, $k \geq 3$. In this section we briefly discuss the logic beyond this recommendation. The paper uses the independent-private-value model, which seems to be the appropriate one on the Internet, which involves relatively cheap items (without an obvious common value) and many anonymous participants.

The paper has two parts. In the first part it is shown that if agents are risk averse, an auction organizer should prefer a first-price auction to a second-price auction. For the case where this organizer is a monopolist this result was previously proved by Riley and Samuelson (1981) and Maskin and Riley (1984). We show that this result continues to hold in an ogopolistic setup if the buyers have constant absolute risk aversion. More precisely, we deal with a two-stage game (the auction selection game). At stage 1 every organizer can choose either a first- or a second-price auction. At the second stage each customer chooses an auction place and a bid as a function of his type. The customers make their choices simultaneously. We prove that there exists a unique continuous subgame perfect equilibrium in this game (if agents have constant absolute risk aversion). In this equilibrium all organizers choose to conduct a second-price auction. Recall that first-price auctions are equivalent to Dutch auctions and that in the independent-private-value model, second-price auctions are equivalent to English auctions. Therefore this result is not consistent with what we see on the Internet, where most of the auctions are English and only a few of them are Dutch. There are many possible explanations for this phenomenon. One such explanation is that participants in such auctions are not risk averse. It seems to us that indeed, Internet auctions attract people who like lotteries (i.e., risk-seeking agents). We have not had the tools to test such a hypothesis, but to some extent, this special characteristic may be explained by the lack of commitment power of Internet auction organizers. We support this hypothesis by showing that when agents are risk seeking, an auction organizer should prefer a second-price auction to a first-price auction in both monopolistic and competitive setups. In the second part of the paper we deal with auction theory with risk-seeking agents. We show that in such a case, an auction organizer should prefer a k-price auction to a second-price auction for every $k \geq 3$. We could not rank the revenues obtained in such auctions for different values of $k \geq 3$.

8. REMARKS

8.1. *Limitations*

8.1.1. *Symmetry*

As is common in the auctions literature, we assume that all agents have the same utility function. We could not handle the nonsymmetric case. Even in our symmetric model we assume, rather than prove, a symmetric equilibrium. We conjecture, however, that it can be proved in our setting that every equilibrium profile is symmetric.[16]

8.1.2. *Random Number of Participants*

It is not reasonable to model many auctions, such as Internet auctions, under the assumption that the number of participants is fixed and commonly known. However, since we assume the independent-private value model of information, all, of our results can easily be extended to the case where the number of participants is randomly determined (see McAfee and McMillan, 1987).

8.2. *Nonexistence of Equilibrium*

One can detect two types of nonexistence. In one of them a solution to the equilibrium equation E2 (or ET2) exists, but this unique solution is not increasing. For the other type, a solution does not exist.

EXAMPLE H. Consider a third-price auction with three risk-neutral agents, in which the distribution function is F. By Proposition C, the

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[16] Griesmer *et al.* (1967), Myerson (1981), and Maskin and Riley (1996) allow nonsymmetric information.

unique solution of $\mathrm{ET2}_{n=3}$ is

$$b(x) = x + \frac{F(x)}{F'(x)} \qquad \text{for all } x \in [0,1].$$

Thus, if b is not increasing (e.g., when $F(x) = (x/(4-3x))^{1/4}$), then an equilibrium does not exist. We conjecture, however, that allowing the utilization of mixed strategies will resolve this nonexistence phenomenon.

EXAMPLE I. Consider a third-price auction with three agents and with the utility function $u(x) = e^x - 1$. In this case $\mathrm{ET2}_{n=3}$ does not have a solution. However, the function $b(v) = v - \ln(1-v)$ is an increasing function which solves $\mathrm{ET2}_{n=3}$ for every $0 \le v < 1$. This function cannot be extended to the whole interval because $\lim_{v \to 1} b(v) = \infty$. So here the problem is that agents' bids are too high. We conjecture that such a problem can be resolved by imposing natural capacity constraints on the players' resources.

We do not know of any example of $u \in RA$ for which equation E2 does not have a solution.

8.2.1. *Noncontinuous Strategies*

In this paper we deal only with equilibrium with continuous strategies. When one is interested in equilibrium with merely measurable strategies, it is reasonable to slightly change the definition of equilibrium as follows:

(AE) In equilibrium every player i is maximizing his conditional expectation given his type v_i, for *almost every* type v_i.

Note that an equilibrium strategy satisfies (AE) if and only if it is an ex ante equilibrium. So, in our paper (as well as in most of the literature of auction theory) we require optimality everywhere with respect to given versions of the conditional probability distributions[17] (see, e.g., Billingsley, 1979), and therefore we have to be sure that our results do not depend on the particular version of the conditional expectation given in (2.1), while in Definition AE, we require optimality only almost surely, and therefore it does not matter which version of the conditional expectation is chosen. If we restrict attention to continuous strategies, then in our setup Definition AE and the standard definition of equilibrium yield the same equilibria set. However, if we allow discontinuous strategies, Definition AE may yield a bigger equilibria set. Consider Example I: by assigning an arbitrary value to $b(1)$ we get an equilibrium in the sense of (AE), which is not an

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[17] One for each player.

equilibrium under the standard definition.[18] Reny (1998) gives a general ex ante equilibrium existence theorem in the general measurable setup, which when applied to k-price auctions with *risk-neutral* agents, yields that every such auction has an equilibrium with respect to definition (AE). This does not contradict what we have just said, because Reny also uses a different tie-breaking rule, according to which ties are broken according to the *true* types of the players. Hence, if we consider any of the standard tie-breaking rules (i.e., a rule which depends only on the bids), then the question of existence of (AE) equilibrium in the risk-neutral case is still open. Needless to say, the existence problem is open for agents with other attitudes toward risk.

Athey (1997) provided several sufficient conditions for the existence of equilibrium in games with incomplete information. She uses the standard definition of equilibrium. Her theorems require the Single Crossing Condition, which is based on the definition of Milgrom and Shannon (1994) of "single crossing property of incremental returns." It is easily verified that k-price auctions, $k \geq 3$, satisfy the Single Crossing Condition. Therefore, Theorem (2.1) of Athey shows that under our conditions on F and u, if we restrict our attention to a finite set of bids, every k-price auction has a Nash equilibrium with nondecreasing functions. However, when we deal with the model of our paper, that is, with a continuum of actions, Athey's theorems cannot be applied to k-price auctions because they do not satisfy the basic property (3.1) required in her paper. This property means that the payoff of an agent depends on the other players' bids only through the determination of the winner, not through payoffs directly.

REFERENCES

Athey, S. (1997). "Single Crossing Properties and the Existence of Pure Strategy Nash Equilibria in Games of Incomplete Information," Working Paper 97-11, Massachusetts Institute of Technology.

Bajari, P. (1996). "Properties of the First Price Sealed Bid Auctions with Asymmetric Bidders," mimeo, University of Minnesota.

Billingsley, P. (1979). *Probability and Measure*. Wiley, New York.

Boutilier, C., Shoham, Y., and Wellman, M. P. (1997). "Economic Principles of Multi-Agent Systems," special issue, *Artificial Intelligence* **94**.

Griesmer, J. H., Levitan, R. E., and Shubik, M. (1967). "Towards a Study of Bidding Processes. Part Four: Games with Unknown Costs," *Naval Res. Logistics Quart.* **14**, 415–433.

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[18] It is not clear whether there exists an equilibrium in the auction of Example H, where the problematic issue is monotonicity.

Kagel, J. H., and Levin, D. (1993). "Independent Private Value Auctions: Bidder Behavior in First-, Second-, and Third-Price Auctions with Varying Numbers of Bidders," *Econ. J.* **103**, 868–879.

Kreps, D. M. (1988). *Notes on the Theory of Choice*. Boulder, CO: Westview Press.

Lebrun, B. (1996). "Existence of an Equilibrium in First Price Auctions," *Econ. Theory* **7**, 421–443.

Lebrun, B. (1999). "First Price Auctions in the Asymmetric *N* Bidder Case," *Internat. Econ. Rev.* **40**, 125–142.

Lizzeri, A., and Persico, N. (1997). "Uniqueness and Existence of Equilibrium in Auctions with a Reserve Price," mimeo, UCLA.

Maskin, E., and Riley, J. (1984). "Optimal Auctions with Risk-Averse Buyers," *Econometrica* **52**, No. 6, 1473–1518.

Maskin, E., and Riley J. (1996). "Asymmetric Auctions," working paper, Harvard University.

Matthews, S. A. (1983). "Selling to Risk-Averse Buyers with Unobservable Tastes," *J. Econ. Theory* **30**, 370–400.

McAfee, R. P., and McMillan, J. (1987). "Auctions with a Stochastic Number of Bidders," *J. Econ. Theory* **43**, 1–19.

Milgrom, P., and Shannon, C. (1994). "Monotone Comparative Statics," *Econometrica* **62**, 157–180.

Monderer, D., and Tennenholtz, M. (1998). "Internet Auctions: Are They Gamblers' Attraction?" working paper, Technion.

Monderer, D., and Tennenholtz, M. (1999). "Distributed Games," *Games Econ. Behav.* **28**, 55–72.

Myerson, R. (1981). "Optimal Auction Design," *Math. Oper. Res.* **6**, 58–73.

Reny, P. J. (1998). "On the Existence of Pure Mixed Strategy Nash Equilibria in Discontinuous Games," working paper, Pittsburgh University.

Riley, J. G., and Samuelson, W. F. (1981). "Optimal Auction," *Am. Econ. Rev.* **71**, 381–392.

Varian, H. R. (1995). "Economic Mechanism Design for Computerized Agents," working paper, University of California, Berkeley.

Wolfstetter, E. (1996). "Auctions: An Introduction," *J. Econ. Surv.* **10**, No. 4, 367–420.