Multipotential Games*
This version contains all proofs

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Abstract

We introduce and analyze q-potential games and q-congestion games, where q is a positive integer. A 1-potential (congestion) game is a potential (congestion) game. We show that a game is a q-potential game if and only if it is (up to an isomorphism) a q-congestion game. As a corollary, we derive the result that every game in strategic form is a q-congestion game for some q. It is further shown that every q-congestion game is isomorphic to a q-network game, where the network environment is defined by a directed graph with one origin and one destination. Finally we discuss our main agenda: The issue of representing q-congestion games with non-negative cost functions by congestion models with non-negative and monotonic facility cost functions. We provide some initial results in this regard.

1 Introduction

Models of congestion come with many real-life stories and in various mathematical forms. They seem to originate at transportation engineering [Wardrop, 1952], and they have been analyzed by several researchers from various additional fields, in particular computer science, communication networks, economics, and game theory.[1] Every congestion model gives rise to a game in strategic form (normal form).

Our starting point is the model of [Rosenthal, 1973]. A congestion form is defined by a finite set of players, each of which holding one unit of goods, a finite set of facilities, and per-unit cost functions associated with the facilities. Each player must use a subset of facilities in order to make its unit of goods valuable. The non-empty set of feasible subsets of facilities is player-specific. When a player chooses a subset of facilities her per-facility cost depends on the number of other players that decide to use the facility, and her total cost is the sum of costs of the facilities in this subset.

Each congestion form F defines a game in strategic form, Γ_F, which is called a congestion game. In Γ_F the strategy set of a player is her set of feasible subsets of facilities, and her cost function is as described above. The distinction between congestion forms and congestion games is important. The form, which is also a sort of game, contains more information than its associated game in strategic form. Two distinct forms may induce isomorphic games. This distinction resembles the distinction between an extensive-form game and its associated strategic form game. For example, a congestion game rarely reveals the structure of the facility cost functions. Indeed, one can think of various natural solution concepts for a congestion form that have no sense in its associated game. For example, the players can choose the facilities sequentially, which gives rise to a multistage game in which we can deal with subgame perfect equilibria and other solution concepts that are natural in multistage games. On the other hand, many interesting conceptual or computational concepts for congestion forms require as input only the game. For example, Nash equilibrium, strong equilibrium, correlated equilibrium, social surplus, and price of anarchy. Two congestion forms are said to be equivalent if they generate isomorphic games.[2]

Many applications of congestion forms/games come from networks. Hence, one may wish to consider a special type of congestion forms/games, in which facilities are edges in graphs and feasible subset of facilities are routes. It is natural to call such congestion forms network forms, and their associated congestion games network games. A natural question is how much we lose when we restrict attention to network forms. There are many modeling choices to make. We have decided to take the seemingly most restrictive definition: A network form is defined by a directed graph with only one origin and one destination. Nevertheless, we prove that dealing with networks does not restrict the generality. We prove that every congestion form is equivalent to a network form, or equivalently, that every congestion game is isomorphic to a network game. Hence, all other potential candidates for the term network games are isomorphic to network games in our sense, because they are in particular congestion games.

Much of the work on congestion games/forms has been inspired by the fact proved in [Rosenthal, 1973] that every such game has a potential function. The theory of general games

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[1] See [Altman and Wynter, 2004] for a commendable attempt to unify the research.

[2] Unfortunately, there is no explicit distinction in the CS/AI literature between forms and games. Both are referred to as congestion games.

that possess potential functions, i.e., of potential games was developed in [Monderer and Shapley, 1996], where the converse to Rosenthal's theorem was proved: Every potential game is isomorphic to a congestion game. However, there are two major differences between the proof that every potential game is isomorphic to a congestion game, and the proof that every congestion game is isomorphic to a network game. First, the proof that every congestion game is isomorphic to a network game is based on a transformation that transforms every congestion form to an equivalent network form. The number of facilities in the network form is twice the number of facilities in the congestion form. Hence we do not lose computational efficiency. In contrast the transformation given in [Monderer and Shapley, 1996] is not computationally efficient.[3]

The second difference concerns the issue of preserving economics properties. The transformation given in this paper that transforms congestion forms to equivalent network forms, preserves the properties of both non-negativity and monotonicity of facility-cost functions.As we show in this manuscript the transformation in [Monderer and Shapley, 1996] does not preserve any of these properties. As all results in CS/AI on network games were proved with various types of monotonicity and non-negativity assumptions, the topic of faithful representation of potential games by congestion games is important if one wants to generalize these results to general potential games. We present here some partial results in this regard.

In this paper we actually deal with the above mentioned issues in a more general context. We discuss a generalization of congestion forms (games), in which the facility cost functions are player-specific. This assumption on the facility cost functions is natural in many applications. Forms with player-specific cost functions are called PS-congestion forms, and their associated games in strategic form are called PS-congestion games.[4] We also define and discuss PS-congestion forms (games) of type q, or in short, q-congestion forms (games), where q is a positive integer. A PS-congestion form is of type q if the set of vectors of facility cost functions contains at most q distinct vectors, that is there exist at most q types of players. A 1-congestion form (game) is a congestion form (game).

Similarly we introduce a new type of games: q-potential games. A 1-potential game is simply a potential game. Roughly speaking, a game is a q-potential game if the set of players can be partitioned into q non-empty and mutually disjoint subsets N_j, $1 \le j \le q$, in such a way that if we fix the strategies of all players outside N_j, the remaining subgame is a potential game. We show that the proof of equivalence between potential games and congestion games can be extended. That is, every q-congestion game is a q-potential game, and every (finite) q-potential game is isomorphic to a q-congestion game. We notice that every game is an n-potential game, where n is the number of players. Therefore, we conclude that every game in strategic form is isomorphic to an n-congestion game. Hence, we get the somewhat surprising result that every game in strategic form is isomorphic to a PS-congestion game.[5]

We further prove that for every $q \ge 1$, every q-congestion form is equivalent to a q-network form. That is, every game in strategic form is isomorphic to a PS-network game.

2 q-Potential Games and q-Congestion games.

2.1 Isomorphic games

A *game in strategic form* is a tuple $\Gamma = (N, (X_i)_{i\in N}, (C_i)_{i\in N})$, where N is a finite set of players which, whenever convenient, we take to be $\{1, \cdots, n\}$; X_i is a set of strategies for i; and $C_i : X \to \mathbb{R}$ is the cost function of i, where $X = \times_{i\in N} X_i$. Γ is called a *finite game* if the sets of strategies are finite sets. We say that two games Γ^1 and Γ^2 are *isomorphic* if each of them is obtained from the other by changing the names of the players and the names of the strategies. That is, there exist bijection functions (i.e., functions which are both one-to-one and onto) $\tau : N^1 \to N^2$ and $\alpha_i : X_i^1 \to X_{\tau(i)}^2$, $i \in N^1$, such that:

for every $i \in N^1$ and for every $(x_j^1)_{j\in N^1} \in X^1$,

$$C_{\tau(i)}^2((\alpha_j(x_j^1))_{j\in N^1}) = C_i^1((x_j^1)_{j\in N^1}).$$

2.2 Potential games

Let $\Gamma = (N, (X_i)_{i\in N}, (C_i)_{i\in N})$ be a game in strategic form. Let X_{-i} denote the set of strategy profiles of all players but i. A function $P : X \to \mathbb{R}$ is a *potential function*[6] for i if for

[3]The transformation in the proof is linear in the parameters, and [Monderer and Shapley, 1996] gave a lower bound to the number of facilities required by this transformation to represent a given potential game by a congestion game. It is an open question to us whether a computationally efficient transformation exists.

[4]Simple and facility symmetric PS-congestion forms/games (see Definition 1) were already discussed, e.g., in [Milchtaich, 1996].

[5]In some contexts it is useful to consider congestion forms, which generalize PS-congestion forms. In such forms the facility cost functions are not only player-specific, but also depend on the identity of the users of the facility. We call such forms ID-congestion forms. Classical ID-congestion forms are congestion forms with weights. Relationships between congestion games, ID-congestion games, and the Shapley value were given for example in [Monderer and Shapley, 1996; Ui, 2000; Monderer, 2006]. Another type of generalization of congestion forms is local-effect games [Leyton-Brown and Tennenholtz, 2003], in which nodes in a graph represent actions, cost functions are associated with nodes, and every such cost function depends on the number of users of the neighboring nodes. These type of games were generalized in [Bhat and Leyton-Brown, 2004] to action-graph games, which were proved to have full expressive power.

[6]In [Monderer and Shapley, 1996], four types of potential functions were defined: exact potential, weighted potential, ordinal potential, and generalized ordinal potential. The term "potential" was used interchangeably with the term "exact potential". In recent literature, some other types of potentials have been considered. For example, best-response potential [Voorneveld, 2000], pseudo-potentials ([Dubey *et al.*, 2006]), ,various types of strong potentials [Holzman and Law-yone (Lev-tov), 1997], potential functions for mechanism design [Jehiel *et al.*, 2004], several types of potential functions that represent various acyclicity properties [Kukushkin, 2002; 2004], and generalized potential functions [Moriss and Ui, 2004]. In some other works, the term potential is used for ordinal or generalized ordinal potentials. In this paper we keep the terminology of [Monderer and Shapley, 1996].

every $x_i, y_i \in X_i$, and for every $x_{-i} \in X_{-i}$,

$$C_i(x_i, x_{-i}) - C_i(y_i, x_{-i}) = P(x_i, x_{-i}) - P(y_i, x_{-i}).$$

Following [Monderer and Shapley, 1996], Γ is a *potential game* if there exists a function P which is a potential for every player i.[7]

2.3 Congestion forms and congestion games

For the basic model of congestion forms and congestion games we follow [Rosenthal, 1973] and [Monderer and Shapley, 1996].

A *Congestion Form* is a tuple $F = (M, N, (\Sigma_i)_{i\in N}, (c_a)_{a\in M})$, where M is a finite set consisting of m elements, which are called *facilities*, N is a finite set consisting of n elements, which are called *players*; For every $i \in N$, $\Sigma_i \subseteq 2^M \setminus \{\emptyset\}$ is a non empty set of subsets of facilities, which is called the *feasible* set of i, and for every $a \in M$ $c_a : [0, \infty) \to \mathbb{R}$ is the per-unit facility cost function associated with $a \in M$; If k of the users choose a, each of them pays $c_a(k)$.

Every congestion form $F = (M, N, (\Sigma_i)_{i\in N}, (c_a)_{a\in M})$ defines a game in strategic form Γ_F, in which the set of players is N, Σ_i is the set of strategies of i, and for every $i \in N$ the cost function of player i is defined on $\Sigma = \times_{i\in N}\Sigma_i$ as follows:

$$C_i(A) = C_i(A_1, \cdots, A_n) = \sum_{a\in A_i} c_a(n_a(A)),$$

where $n_a(A) = |\{j \in N : a \in A_j\}|$.[8] A game Γ in strategic form is called a *congestion game* if $\Gamma = \Gamma_F$ for some congestion form F. Two congestion forms are *equivalent* if they generate isomorphic congestion games.

2.4 Player-specific facility cost functions

When the cost functions associated with the facilities are player-specific we get a *congestion form with player-specific facility cost functions* or, in short a *PS-congestion form*. Formally: A *PS-Congestion Form* is a tuple $F = (M, N, (\Sigma_i)_{i\in N}, ((c^i_a)_{a\in M})_{i\in N})$ such that all components except for the cost functions are defined as in a congestion form, and $c^i_a : [0, \infty) \to \mathbb{R}$ is i's facility cost function associated with $a \in M$; If k of the users choose a, agent i pays $c^i_a(k)$.

Definition 1 A PS-congestion form is *facility-symmetric* if $\Sigma_i = \Sigma_j$ for every $i, j \in N$. A PS-congestion form is *simple* if Σ_i contains only singletons for every player i.

[7] "Potential game" were defined in [Monderer and Shapley, 1996]. Potential games in the differentiable setup and discrete 2-person potential games were previously discussed in [Thépot, 1980; 1981], where they are called centralizable games. However, potential functions for various types of games have been used in the literature in several research fields much earlier. See e.g., [Wardrop, 1952; Beckmann *et al.*, 1956]. Additional references can be found in [Monderer and Shapley, 1996]. Non-atomic potential games were defined in [Sandholm, 2001].

[8] Hence, only the values of c_a on the set of integers $\{1, \cdots, n\}$ are relevant. However, it will be useful later, and it does not restrict the generality, to define c_a on the whole interval $[0, \infty)$.

Every PS-congestion form $F = (M, N, (\Sigma_i)_{i\in N}, ((c^i_a)_{a\in M})_{i\in N})$ uniquely defines a game in strategic form Γ_F, in which the set of players is N, Σ_i is the set of strategies of i, and for every $i \in N$ the cost function of player i is defined as follows:

$$c_i(A) = c_i(A_1, \cdots, A_n) = \sum_{a\in A_i} c^i_a(n_a(A)).$$

A game Γ in strategic form is called a *PS-congestion game* if $\Gamma = \Gamma_F$ for some PS-congestion form F. Thus, every congestion form is a PS-congestion form, and every congestion game is a PS-congestion game.

2.5 q-Congestion forms and q-congestion games

Roughly speaking, a PS-congestion form is of type q if the players can be partitioned into q types, where two players are of the same type if they share the same facility cost functions. Formally, Let $F = (M, N, (\Sigma_i)_{i\in N}, ((c^i_a)_{a\in M})_{i\in N})$ be a PS-congestion form. A finite set $K \subseteq (\mathbb{R}^{[0,\infty)})^M$ is a *cover* for F if for every player i there exists $c = (c_a)_{a\in M} \in K$ such that for every $a \in M$ $c^i_a = c_a$.

Let q be a positive integer. We say that F is a *q-congestion form* if it has a cover H with $|H| \leq q$. Obviously, if F is a $q-$congestion form, F is a $(q+1)$-congestion form. Because $K = \{(c^1_a)_{a\in M}, \cdots, (c^n_a)_{a\in M}\}$ is a cover for F, every $n-$person PS-congestion form is an n-congestion form. Let $1 \leq q(F) \leq n$ be the *index* of F, defined as the minimal cardinality of a cover for F. Obviously, F is a congestion form if and only if $q(F) = 1$. If F is a q-congestion form, Γ_F is called a *q-congestion game*.

2.6 q-Potential games

Let $\Gamma = (N, (X_i)_{i\in N}, (C_i)_{i\in N})$ be a game in strategic form. Let H be a set of real-valued functions defined on X. We say that H is a *cover* of Γ if for every $i \in N$ there exists $P \in H$, which is a potential function for i. Let q be a positive integer. We say that Γ is a *q-potential game* if it has a cover H with $|H| \leq q$. Obviously, if Γ is a $q-$potential game, Γ is a $(q+1)$-potential game. Because C_i itself is a potential for i, every $n-$person game is an n-potential game. Let $1 \leq q(\Gamma) \leq n$ be the *potential index* of Γ, defined as the minimal cardinality of a cover for Γ. Obviously, Γ is a potential game, if and only if Γ is a 1-potential game if and only if $q(\Gamma) = 1$.

Let Γ be a q-potential game . Let $H = \{P_1, \cdots, P_q\}$, $|H| = q$, be a cover of Γ, and let H_s be the set of all players i such that P_s is a potential function for i. A partition of the player set N to q nonempty and mutually disjoint subsets $\pi = (N_s)^q_{s=1}$, is *consistent with* H if P_s is a potential function for every $i \in N_s$, that is, $N_s \subseteq H_s$ for every $1 \leq s \leq q$. In a potential game with a potential function P, all players behave as if there exists one player whose goal is to minimize P over X. In a q-potential game with a cover $H = \{P_1, \cdots, P_q\}$ and an H-consistent partition $\pi = (N_s)^q_{s=1}$, the players behave as if there are q players, I_s, $1 \leq s \leq q$, playing a q-person game with the set of strategies $X_{[s]}$ for player I_s, where $X_{[s]} = \times_{i\in N_s} X_i$. For $x = (x_{[s]})_{1\leq s\leq q}$, the cost function of I_s is P_s. Note that every equilibrium x in the associated q-person game is also an equilibrium in the original game. Similarly, every correlated equilibrium corresponds to a correlated equilibrium.[9]

Unfortunately, as is shown in the next example, a cover with a minimal cardinality may have more than one consistent partition. In particular, the partition of the players' set to q subsets in a q-potential game with an index q is not uniquely determined by the game. Bellow is an example for a 3-player game with a potential index 2 in which both partitions $12, 3$ and $1, 23$ are consistent with the same cover H, where $|H| = 2$.

Example 1 We construct a game Γ. The strategy set of every player $1 \leq i \leq 3$ is $\{0, 1\}$. The cost functions are: $C_1(x) = C_1(x_1, x_2, x_3) = x_2 x_3$, $C_2(x) = x_1 x_3$, $C_3(x) = x_1 x_3 + x_1 x_2$. We first show that this is not a 1-potential game. Indeed, by [Monderer and Shapley, 1996] it suffices to show that there exists a closed path of strategy profiles in X, $\gamma = x(0), x(1), x(2), x(3), x(4)$ with $x(0) = x(4)$, such that $x(t+1)$ is obtained from $x(t)$ by changing the strategy of exactly one player, i_t, $0 \leq t \leq 3$, and such that $I(\gamma) \neq 0$, where

$$I(\gamma) = \sum_{t=0}^{3} [C_{i_t}(x(t+1)) - C_{i_t}(x(t))].$$

Indeed, for the path $\gamma = (0,1,0), (1,1,0), (1,1,1), (0,1,1), (0,1,0)$, $I(\gamma) = -1$. Observe that $P(x) = 0$ for every $x \in X$ is a potential function for both player 1 and player 2, and that C_2 is a potential function for both player 2 and player 3. Hence, $H = \{0, C_2\}$ is a cover for Γ, and both partitions $12, 3$ and $1, 23$ are consistent with H. ∎

2.7 Representation of q-potential games by q-congestion forms

It was proved in [Rosenthal, 1973] that every congestion game is a potential game. It was proved in [Monderer and Shapley, 1996] that every finite potential game is isomorphic to a congestion game. The two theorems are extended in this section.

Theorem 1

(1) *Every q-congestion game is a q-potential game.*

(2) *Every finite q-potential game is isomorphic to a q-congestion game.*

Proof of Theorem 1:

(1) Let $F = (M, N, (\Sigma_i)_{i \in N}, ((c_a^i)_{a \in M})_{i \in N})$ be a q-congestion form. Let $K = \{(c_a^{[s]})_{a \in M} : 1 \leq s \leq q\}$ be cover for F.

Let s be an integer, $1 \leq s \leq q$. Let $F^{[s]} = (M, N, (\Sigma_i)_{i \in N}, (c_a^{[s]})_{a \in M})$ be the congestion form obtained from F by replacing all cost functions in F with $(c_a^{[s]})_{a \in M}$.

Let P_s be the potential function associated by [Rosenthal, 1973] with the congestion form $F^{[s]}$. That is,

$$P_s(A) = P_s(A_1, \cdots, A_n) = \sum_{a \in \cup_{i=1}^n A_i} \sum_{k=1}^{n_A(a)} c_a^{[s]}(k).$$

Let $i \in N$, and let s satisfy $c_a^i = c_a^{[s]}$ for every $a \in M$. By [Rosenthal, 1973] P_s is a potential function for player i in the game $\Gamma_{F^{[s]}}$. Hence, $\{P_1, \cdots, P_q\}$ is a cover for Γ_F. That is, Γ_F is a q-potential game.

(2) We will basically use the proof of the case $q = 1$ given in [Monderer and Shapley, 1996], cleaning it a little bit.

Let $\Gamma = (N, (X_i)_{i \in N}, (C_i)_{i \in N})$ be a q-potential game in strategic form. Without lose of generality we can assume that the potential index of Γ equals q. Let $H = \{P_s : 1 \leq s \leq q\}$ be a cover for Γ, and let $\pi = (N_s)_{s=1}^q$ be a partition consistent with H. Then for every i there exists a unique integer s_i such that $i \in N_{s_i}$. Because P_{s_i} is a potential function for i, $C_i(x_i, x_{-i}) - P_{s_i}(x_i, x_{-i})$ does not depend on x_i. Therefore, there exists a function $f_i : X_{-i} \to \Re$ such that $C_i(x) = P_{s_i}(x) + f_i(x_{-i})$ for every $x \in X$.

We proceed to define a PS-congestion form. Let $M = 2^{X_1} \times \cdots \times 2^{X_n}$ be the set of facilities.[10] A generic element of M will be denoted by $T = (T_1, \cdots, T_n)$. Let

$$M_1 = \{T \in M : \ |T_i| = 1 \text{ for every } i \in N\},$$

and

$$M_2 = \{T \in M : \ \exists i \in N \text{ such that } T_i = X_i \text{ and } |T_j| = |X_j| - 1 \text{ for}$$

For every $i \in N$ and for every $x_i \in X_i$ let $A_{x_i} \subseteq M$ be defined as follows:

$$A_{x_i} = \{T \in M_1 \cup M_2 : \ x_i \in T_i\},$$

and let

$$\Sigma_i = \{A_{x_i} : \ x_i \in X_i\}.$$

Let $1 \leq s \leq q$. For every $T \in M$ we define $c_T^{[s]}$ on $\{1, \cdots, n\}$ (and extend arbitrarily to the interval $[0, \infty)$) as follows:

- For every $T \notin M_1 \cup M_2$, $c_T^{[s]}$ is identically zero.
- for $T \in M_1 \cup M_2$, $c_T^{[s]}(k) = 0$, for $1 < k < n$.
- For $T \in M_1$, $T = (\{x_1\}, \cdots, \{x_n\})$, $c_T^{[s]}(1) = 0$, and
$$c_T^{[s]}(n) = P_s(x_1, \cdots, x_n).$$
- For every $T \in M_2$, $c_T^{[s]}(n) = 0$. For such T there exists a unique $i \in N$, and a unique $x_{-i} \in X_{-i}$ such that $T_i = X_i$, and $T_j = X_j \setminus \{x_j\}$ for every $j \neq i$. If $i \notin N_s$, $c_T^{[s]}(1) = 0$. If $i \in N_s$, $c_T^{[s]}(1) = f_i(x_{-i})$.

[9]However, a mixed-strategy equilibrium in the associated q-person game corresponds only to a correlated equilibrium in the original game, and not necessarily to a mixed-strategy equilibrium.

[10]This set seems huge, however we are not actually using all facilities.

Let $c_T^i = c_T^{[s_i]}$. It is now clear that the PS-congestion form $F = (N, M, (\Sigma_i)_{i\in N}, ((c_T^i)_{T\in M})_{i\in N}$ is of type q, and Γ_F is isomorphic to Γ. ∎

We end with a somewhat surprising corollary:

Corollary 1 *Every game in strategic form is isomorphic to a PS-congestion game.*

Proof: As we noticed above, every game in strategic form is an n-potential game. By Theorem 1 every n-potential game is isomorphic to an n-congestion game. ∎

3 Network Forms and Network Games

Much of the literature about congestion forms has been motivated by transportation systems and by digital networks. In such models, facilities are edges in a graph, and feasible sets of facilities are routes. In this paper the terms network form and network game are defined in a specific way. All other graphical models fall under the category of congestion forms (games).

Consider a loop-free directed graph GR with a finite set of vertices $V = V_{GR}$, and a set of edges $E = E_{GR}$.[11] Every feasible subset of facilities for i represents a feasible route (a path with distinct vertices) in the graph. For every $o, d \in V$, $o \neq d$ we denote by $R(o, d)$ the set of all routes that connect o to d. A *PS-network form* is a PS-congestion form $F = (M, N, (\Sigma_i)_{i\in N}, ((c_a^i)_{a\in M})_{i\in N})$ for which there exists a directed graph, and two distinct vertices in this graph o and d, with $R(o, d) \neq \emptyset$ such that $M \subseteq E$ and $\Sigma_i \subseteq R(o, d)$ for every agent i. A game Γ in strategic form is called a *PS-network game* if $\Gamma = \Gamma_F$ for some PS-network form F. Naturally, a q*-network form* is a PS-network form of type q, and a q*-network game* is a PS-network game derived from a q-network form. A 1-network form is also called a *network form*, and a 1-network game is also called a *network game*.

Theorem 2 *Every q-congestion form is equivalent to a q-network form.*

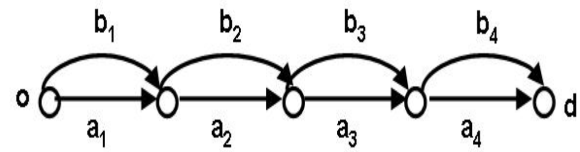
Proof: Let $F = (M, N, (\Sigma_i)_{i\in N}, ((c_a^i)_{a\in M})_{i\in N})$ be a q-congestion form. Assume $M = \{a_1, \ldots, a_m\}$, and let $K = \{b_1, \cdots, b_m\}$ be an arbitrary finite copy of M that does not intersect with M. We construct a graph GR as follows: The set of vertices is $V = \{1, \cdots, m+1\}$. For every $1 \leq j \leq m$ we connect j to $j+1$ with two edges, a_j, b_j. That is, j is the tail of both a_j and b_j, and $j+1$ is the head of both. We denote $o = 1$ and $d = m+1$. For every $A_i \in \Sigma_i$ we associate a route α_{A_i} as follows: $\alpha_{A_i} = z_1, z_2, \cdots, z_m$, where $z_j = a_j$ if $a_j \in A_i$, and $z_j = b_j$ if $a_j \notin A_i$. With the edges a_j we associate the cost function $(c_{a_j}^i)_{i\in N}$, and with the edges b_j we associate the cost functions which are constantly zero. It is obvious that we constructed a PS-network form of type q, and that the q-network game derived from this form is isomorphic to Γ_F. ∎

The following example together with its associated figure illustrates the proof of Theorem 2.

[11]One may use non-directed graphs. In this paper we basically follows the definition of [Holzman and Law-yone (Lev-tov), 2003].

Example 2 Consider the congestion form F,in which $N = \{1, 2\}$, $M = \{a_1, a_2, a_3, a_4\}$, $\Sigma_1 = \{\{a_1, a_2\}, \{a_3, a_4\}\}$, and $\Sigma_2 = \{\{a_1, a_3\}, \{a_2, a_4\}\}$. F is transformed to the network form shown in Figure 1 in which $\Sigma_1^* = \{a_1a_2b_3b_4, b_1b_2a_3a_4\}$, and $\Sigma_2^* = \{a_1b_2a_3b_4, b_1a_2b_3a_4\}$. The cost functions on the a_j-links, $j = 1, 2, 3, 4$ are the original cost functions, and the cost functions on the b_j-links are constantly zero.

Figure 1:



Corollary 2 *Every finite game in strategic form is isomorphic to a PS-network game.*[12]

Proof: The proof follows from combining Corollary 1 with Theorem 2. ∎

4 Faithful representations

In many real-life applications it is natural to assume that the facility cost functions of a PS-congestion form have a special structure. In particular, it is natural to assume that the facility cost functions are nonnegative and in addition are either non-decreasing or non-increasing, depending on the context. It is easy to check that the particular representation method described in the proof of Theorem 1 (or in the analogous proof in [Monderer and Shapley, 1996]) may represent a finite q-potential game with nonnegative costs by a q-congestion form in which some of the facility cost functions take negative values. Actually, the representation method depends on the choice of the potential functions. However, it can be seen that there exists a finite 1-potential game such that for every choice of a potential function the representation method yields a 1-congestion form with some facility cost functions that take negative values. This suggest three questions:

Question 1: *Can every finite q-potential game with non-negative cost functions be represented (up to an isomorphism) by a q-congestion form with non-negative facility cost functions*?

Question 2: *Can every finite q-potential game with non-negative cost functions be represented (up to an isomorphism) by a q-congestion form with non-negative and non-decreasing cost functions*?

[12]Other graphical representations of games have been analyzed in the literature of computer science and artificial intelligence. In some of these representations the focus is on dependencies among players' utility functions (see, e.g., [Kearns *et al.*, 2001; Koller and Milch, 2001; Mura, 2000; Vickrey and Koller, 2002]). Other types of representations focus on actions' dependencies– see [Leyton-Brown and Tennenholtz, 2003; Bhat and Leyton-Brown, 2004].

Question 3: *Can every finite q-potential game with non-negative costs be represented (up to an isomorphism) by a q-congestion form with non-negative and non-increasing cost functions*?

We show that the answer to Question 2 is negative by the next example:

Example 3 Consider the following parametric game with $z > 0$.

$\Gamma_z =$

	x_2	y_2
x_1	0, 0	0, z
y_1	z, 0	0, 0

Γ_z is a congestion game because it is a potential game with a potential function P_z, where

$P_z =$

	x_2	y_2
x_1	0	z
y_1	z	z

Let $N = \{1, 2\}$. Assume in negation that Γ_z is isomorphic to Γ_F, where $F = (M, N, (\Sigma_i)_{i=1}^2, (c_a)_{a\in M})$ is a congestion form in which the cost functions are non-negative and non-decreasing. In particular, for every facility a

$$0 \le c_a(1) \le c_a(2).$$

Because Γ_F is isomorphic to Γ_z, we may assume that the feasible sets in F are parameterized as follows: $\Sigma_1 = \{A_{x_1}, A_{y_1}\}$, and $\Sigma_2 = \{A_{x_2}, A_{y_2}\}$. Also, without loss of generality we can assume that $M = A_{x_1} \cup A_{y_1} \cup A_{x_2} \cup A_{y_2}$. Note that

$$\sum_{a\in A_{y_2}\setminus A_{y_1}} c_a(1) + \sum_{a\in A_{y_2}\cap A_{y_1}} c_a(2) = C_2(A_{y_1}, A_{y_2}) = 0.$$

Therefore, $c_a(1) = 0$ for every $a \in A_{y_2} \setminus A_{y_1}$, and $c_a(2) = 0$ for every $a \in A_{y_2} \cap A_{y_1}$. Since $c_a(1) \le c_a(2)$, we conclude that $c_a(1) = 0$ for every $a \in A_{y_2}$. It follows that

$$\sum_{a\in A_{y_2}\cap A_{x_1}} c_a(2) = C_2(A_{x_1}, A_{y_2}) = z > 0. \qquad < * >$$

On the other hand,

$$\sum_{a\in A_{x_1}\setminus A_{y_2}} c_a(1) + \sum_{a\in A_{x_1}\cap A_{y_2}} c_a(2) = C_1(A_{x_1}, A_{y_2}) = 0,$$

and therefore $\sum_{a\in A_{y_2}\cap A_{x_1}} c_a(2) = 0$, contradicting $< * >$.
∎

Note, however, that Γ_z in Example 3 can be represented by a simple congestion game with non-negative and non-increasing cost functions. Consider the congestion form F_z with two resources a, b. $c_a(1) = c_a(2) = 0$. $c_b(1) = z$, and $c_b(2) = 0$. It is easily verified that Γ_{F_z} is isomorphic to Γ_z. Hence, Question 1 and 3 can still have positive answers.

If we are less ambitious, and we allow representing q-potential games by r-congestion games with $r > q$, we show that the answer to the modified version of Question 1 and 2 is positive:

Lemma 1 *Every game in strategic form with non-negative cost functions is isomorphic to a PS-congestion game derived from a PS-congestion form with non-negative and non-decreasing facility cost functions.*

Proof: Let $\Gamma = (N, (X_i)_{i\in N}, (C_i)_{i\in N})$ be a game in strategic form with non-negative cost functions. We define a PS-congestion form as follows: Let the set of facilities be X. For every $i \in N$ and $x_i \in X_i$ let $A_{x_i} = \{x_i\} \times X_{-i} \subseteq X$. Let $\Sigma_i = \{A_{x_i} | \quad x_i \in X_i\}$. Finally, for every $x \in X$ let $c^i_x(k) = 0$ if $1 \le k < n$, and $c^i_x(n) = C_i(x)$, and extend c^i_x to the whole interval $[0, \infty)$ arbitrarily. Obviously the PS-congestion form just defined, F generates an PS-congestion game, which is isomorphic to Γ. Moreover the facility cost functions in F are non-negative and non-decreasing. ∎

We end this section with an example showing two congestion forms with the same combinatorial structure (same set of facilities and same feasible sets) and with positive cost functions that represent the same game. However, the facility cost functions in one of them are decreasing and in the other, increasing.

Example 4 The two congestion forms bellow represent the following potential game:

$\Gamma =$

	x_2	y_2
x_1	11, 11	7, 12
y_1	12, 7	8, 8

Consider the parametric congestion forms F, in which $M = \{a, b, c, d\}$, $\Sigma_1 = \{\{a, b\}, \{c, d\}\}$, $\Sigma_2 = \{\{a, c\}, \{d, d\}\}$. The cost functions are

$$c_z(1) = x_z, \text{ and } c_z(2) = y_z \text{ for every } z \in \{a, b, c, d\}.$$

Hence, Γ can be represented by this particular congestion form if there exists a solution to the following linear system with 8 equations and 8 variables:

$$\Gamma = \begin{array}{c|c|c|} & ac & bd \\ \hline ab & \begin{array}{l} y_a + x_b \\ \quad\quad y_a + x_c \end{array} & \begin{array}{l} y_b + x_a \\ \quad\quad y_b + x_d \end{array} \\ \hline cd & \begin{array}{l} y_c + x_d \\ \quad\quad y_c + x_a \end{array} & \begin{array}{l} y_d + x_c \\ \quad\quad y_d + x_b \end{array} \\ \hline \end{array}$$

It is easy to see that this system has solutions depending on two parameters s, t:

$$x_b = x_c = t,\ y_b = y_c = s,$$

$$y_a = 11 - t,\ y_d = 8 - t,\ x_a = 7 - s,\ x_d = 12 - s.$$

Obviously $t = 1, s = 6$ gives a congestion form with positive and increasing cost functions, while $t = 6, s = 1$ give a congestion form with positive and decreasing cost functions. ∎

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