# Asymptotically Optimal Multi-Object Auctions *

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**Abstract**

A seller who wishes to sell a set of (distinct or similar) items can attempt to do it by a variety of auction mechanisms, such as combinatorial auctions, sequential auctions or parallel auctions. We provide a proof for the intuitive result that if the potential buyers are risk averse, then for every mechanism, the expected revenue of the seller in equilibrium is bounded above by the expected maximal surplus of the potential buyers, provided that the expected utility of every agent in this equilibrium is non negative. In our main result, we prove that if the number of potential buyers is sufficiently large and there are no externalities, then in the valuation-symmetric independent-private-values model, in which the valuation functions over allocations of items are independent across players, this upper bound is almost achieved by the Vickrey-Clarke [VC] mechanism. Hence, in an environment with many potential buyers, e.g., the Internet, a monopolist who uses the VC mechanism can extract almost all social surplus from the agents

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who consider participation. This result is generalized to the case in which there is a random number of agents, and the utility functions of the agents (and not only their valuation functions) are their private information.

# 1 Introduction

Auctions have been extensively studied in economic theory for many years (see e.g, [13, 15, 17, 33, 8] for surveys). A central issue in auction theory is the study of optimal, revenue maximizing, auctions. However, the characterization of optimal auctions has turned out to be extremely challenging. In a highly influential work [21], Myerson characterized optimal auctions in a model for a single good where agents are risk-neutral and have independent private values. However, this leaves open many cases that are of interest in real resource allocation contexts; In particular:

1. Most work in economics assumes that agents are risk-averse. However, there is very little work on auctions with risk-averse agents, and almost nothing is known about the properties of optimal auctions with risk-averse agents.

2. Recent auction theory deals with multi-object auctions, where agents can express their preferences over bundle of goods by using valuation functions, which are not necessarily additive. Almost nothing is known about optimal auctions in this case, even if the buyers are risk-neutral.

In this paper we tackle the problem of optimal auctions for multi-object auctions where the buyers might be risk-averse agents, addressing the above two major challenges. The assumption that we adopt is that the number of participants is large, and therefore we are interested in asymptotic optimality. We prove that if the number of potential buyers is sufficiently large and there are no externalities, then in the valuation-symmetric independent-private-values model, in which the valuation functions over allocations of items are independent across players, the famous Vickrey-Clarke [VC] mechanism [31, 1] is close to optimal. This is obtained by first showing an upper bound on the revenue that may be obtained by any auction mechanism, and then showing that the VC auction obtains a revenue that approaches this upper bound.

In the next section we present our assumptions and results. Sections 3–7 will provide the reader with full formal details and discussion of all concepts, definitions, and theorems required for establishing the above-mentioned contributions. Proofs appear in an appendix.

# 2 The Main Results

In a typical auction setting, an organizer designs a mechanism in order to sell $k$ objects to a set of potential buyers. The outcome of each such mechanism is an allocation of the objects to the agents and to the organizer herself, and a transfer of money from the agents to the organizer (we do not exclude negative transfers).

A potential buyer is an agent who considers participation in the market organized by the organizer. That is, he collects information about the value of the objects and about the other agents, and he learns about the rules of trade. However, he may still decide not to participate.

It is assumed that every agent knows his monetary value for each possible outcome, and that the transfers of the other agents do not enter his own preferences. However, the goods may have externalities; An agent may have preferences that depend on the allocation of goods to the other agents too. Hence, each agent $i$ has a valuation function $w_i$ defined over the set of allocations. Each agent is also assumed to have a von-Newmann-Morgenstern utility function $u_i$ for money, which is normalized to be zero at zero. The utility of agent $i$ if the allocation $\pi$ is chosen and he transfers $t$ monetary units to the organizer is $u_i(w_i(\pi) - t)$.

The valuation functions of the agents are their private information. These valuation functions are drawn from a compact set of possible valuation functions. If the (vector valued) random variables that determine the agents' valuation functions are independent, then the model is known as the independent-private-values model. If, in addition, these random variables are identically distributed, we refer to the model as a valuation-symmetric independent-private-values model[1]. The behavior of the agents depends on the auction mechanism, their utility and valuation functions, and their information about the value of the objects and about the types of the other agents.

We initially consider the following additional assumptions:

- **A1** No externalities: The value of an allocation to an agent depends only on the goods allocated to this agent. Hence the valuation function of $i$ is a function $v_i : 2^G \rightarrow R$, where $G$ is the set of goods and $2^G$ is the set of subsets of $G$. We always use the normalization $v_i(\emptyset) = 0$.
- **A2** Free disposal: The valuation function of every agent is non decreasing[2].
- **A3** valuation-symmetric independent-private-values model.
- **A4** Agents are risk averse (recall that a risk-neutral agent is a simple, extreme case of a risk-averse agent).

In addition, we make use of the following terminology. For a fixed number of agents, for each allocation of the goods to the agents, we define the surplus variable at this allocation to be the sum of the valuation functions of the agents, where each agent's valuation function is computed at the set of goods allocated to him. The maximum, over all allocations, of the surplus variable is a random variable which depends on the valuation functions of the agents. We refer to it as the *maximal surplus variable*. The maximal value of this random variable is denoted by $S^*$ and is referred to as the *ultimate surplus*[3]. Note that if **A1** holds , and the number of agents exceeds the number of goods, the ultimate surplus does not depend on the number of agents.[4]

[1] We use the notion "valuation-symmetric" because all agents have the same distribution over valuation functions, but the agents may have different attitude to risk, i.e., different utility functions for money.

[2] Note that **A1-A2** imply that valuations are non negative.

[3] The maximal value of a random variable is not necessarily well-defined. Therefore we define the ultimate surplus $S^*$ as the essential sup of $\tilde{S}$, where $\tilde{S}$ is the maximal surplus variable. That is, $S^*$ is defined by the following two conditions: $Prob(\tilde{S} \leq S^* = 1)$, and for every $\varepsilon > 0$, $Prob(\tilde{S} \leq S^* - \varepsilon) < 1$.

[4] For example, when $k = 1$, if all random valuations $(\tilde{v}_i)_{i=1}^n$ are supported at some interval $[a, b]$, then the $n$-th random maximal surplus is $max\{\tilde{v}_1, \tilde{v}_2, \ldots, \tilde{v}_n\}$, and the ultimate surplus equals $b$.

The organizer is assumed to be a risk-neutral economic agent. Hence, the objective of the organizer is to select an optimal auction mechanism, that is an auction mechanism that will maximize her expected revenue. However, the form of an optimal auction in the setup described by **A1-A4** is not known. The optimality problem was solved only for very special environments. In all of these environments, **A4** is not satisfied. That is, the agents are assumed to be **risk-neutral**[5]:

In his pioneering work [21], Myerson characterized optimal auctions in a model for a single good ($k = 1$) that satisfies **A1-A2,** independent-private-values and **ARN**, where

- **ARN** Agents are risk-neutral.

In particular, he proved that if in addition the model is valuation symmetric (i.e., **A3** is satisfied), an optimal auction mechanism is a second-price auction with an appropriate reservation price[6].

It can be deduced from Krishna and Perry [9] that in a multi-object model, which satisfies assumptions **A1-A3** and **ARN** the VC mechanism is optimal among the efficient mechanisms[7].

Our work is concerned with asymptotic optimality. We prove:

- **R1** If **A1-A2** and **A4** hold, then for a fixed number of participants, in every auction mechanism the organizer's expected revenue in equilibrium is bounded above by the expected maximal surplus. Consequently, the expected revenue is bounded above by the ultimate surplus.

  Notice that this result deals with the general case where we have arbitrary risk-averse agents, rather than only with the extreme case of linear utility functions. Indeed, in some classical contexts agents' bids are expected to increase (relative to the case of risk-neutral agents) due to risk-aversion.

[5]There are only few papers that deal with auctions with risk-averse agents (see, e.g., [25, 10, 28]). The issue of optimality in such auctions is discussed in [11].

[6]This result can be easily extended to the case where we have multiple units of a good, in which every agent wishes to purchase only one unit (see e.g. [32]).

[7]The VC mechanism is sometime referred to as the Vickrey-Clarke-Groves (VCG) mechanism. The Vickrey mechanism is the Clarke mechanism for one good, and the Clarke mechanism is the Groves mechanism [3] with a particular vector of transfer functions.

- **R2** Suppose there are infinitely many agents , and **A1-A4** hold. Then when the VC mechanism is applied to an infinite sequence of auctions with an increasing number of agents, the organizer's revenue, as well as the maximal surplus, converge almost surely to the ultimate surplus, when the number of participants converges to infinity. In particular, the upper bound established in **R1** is almost achieved by the VC Mechanism if the number of agents is sufficiently large.

Our method of proof of **R2** shows that:

- **R3** The rate of convergence in **R2** depends on the number of goods $k$, and on the distribution of types, but not on the utility functions.

Our results generalize [19]. In [19], we proved **R1** for a single good model (satisfying **A1-A4**), and we proved a weaker version of **R2** for a single good model. In this weaker version (which was independently proved by Neeman [22]), we established that the ratio between the expected revenue and the expected maximal surplus converges to 1, while in the current paper we deal with almost surely convergence in a multi-object model. In a later independent work, Jackson and Kremer [4] provide a setting from which R2 can be deduced for the case of multiple units of the same good. Our result however deals with arbitrary sets of goods.

Our theorem implies that in an environment in which **A1-A4** are satisfied, and in addition

- **A5** There are "many" players,

a monopolist can (with high probability) extract almost all social surplus.

As the VC mechanism is efficient, **R2** implies that it is also an asymptotically optimal efficient mechanism[8].

[8]Asymptotic efficiency properties of auctions are discussed e.g., by Rustichini, Satterthwaite and Williams [26], in the framework of a model of double auctions, and by Swinkels[30, 29] in the framework of multi-unit auctions (i.e., $k$ identical objects). These works prove asymptotic efficiency in independent-private-values models with risk neutral agents. The main novelty in Swinkels's works is the removal of the symmetry assumption.

Note again that all optimality results (including **R2**) assume the independent-private-values model[9]. On the other hand, the inequality **R1** does not assume it, and it holds in the most general information structure.

It may not be reasonable to discuss auctions with many participants in which the number of potential buyers and their utility functions are not commonly known. Indeed, in Section 6, we show how our results **R1** and **R2** in Sections 3 and 5, respectively, can be generalized to a model with a random number of potential participants, in which the type of each agent $i$ has two components: a valuation function $v_i$ and a utility function $u_i$.[10]

In Section 7 we deal with the issue of externalities. There are two steps in the proof of the asymptotic optimality of the VC mechanism. In the first step we establish the upper bound **R1** for every auction mechanism. It turns out, that this proof can be extended to a very general case. In Section 7 we prove **R1** without assuming **A1-A3**. That is, we do not require independence, symmetry, no-externalities or free disposal. All these assumptions are replaced by a single assumption:

- **E** The expected utility of each agent in equilibrium is non-negative.

Thus we prove that if the agents are risk-averse (i.e. **A4** holds), then **R1** holds in any equilibrium profile in which **E** is satisfied. We provide two examples that show that these assumptions (**A4** and **E**) are also necessary; that is, the theorem may not hold if one of them is not satisfied.

It is also possible to prove **R2-R3** in a model with externalities, if we assume anonymous externalities and nonnegative valuation functions. Since we believe that the real interest with externalities is with negative externalities, we do not discuss this issue in this paper[11].

[9] See e.g., [16, 5, 12, 2, 23, 24] for discussions of models in which types are correlated. Neeman [22] discusses the asymptotic ratio between the expected revenue and the expected maximal surplus in such models.

[10] A model with a random number of participants is discussed, e.g., in [14, 30]. We do not know on any model in auction theory that deals with random utility functions.

[11] An independent-private-values model with one object, in which each agent is risk-neutral and has a negative externality on the other agents was discussed by Jehiel, Moldovanu and Stacchetti in [6]. Their paper deals with auction mechanisms in which every agent can give only a single numerical bid. Under these assumption it characterizes

# 3 An Upper Bound on the Seller's Expected Revenue in Equilibrium

In this section we assume **A1-A2** and **A4**. Consider a seller denoted by 0, who wishes to sell a set $G$ of $k$ goods to a group of potential buyers, $N = \{1, 2, \cdots, n\}$, where a potential buyer is an agent who considers participation in the market organized by the seller. That is, he learns about the rules of trade and collects information about other agents. However, he may still decide not to participate.

Let $N^0 = N \cup \{0\}$. An *allocation* of goods is a function $\pi : G \to N^0$. That is, if the allocation $\pi$ is chosen, $i \in N$ receives the good $a \in G$ if and only if $\pi(a) = i$. Thus, if $\pi(a) = 0$ it means that no agent gets this item. The set of all allocations is denoted by $\Pi$, and for a subset $N'$ of $N$ we denote by $\Pi_{N'}$ the set of all allocation $\pi : G \to N' \cup \{0\}$. For each allocation $\pi \in \Pi$ and for each $i \in N^0$ we denote by $\pi_i$ the set of goods that are allocated to $i$ by $\pi$. That is,

$$\pi_i = \{a \in A : \quad \pi(a) = i\}.$$

Every agent $i \in N$ has a valuation function $v_i : 2^G \to R_+$ normalized by $v_i(\emptyset) = 0$. In addition, $i$ has a strictly increasing concave von-Neumann Morgenstern utility function for money, $u_i : R \to R$, which is normalized by $u_i(0) = 0$.

The utility of $i$ if the center chooses the allocation $\pi$ and $i$ pays $t$ is $u_i(v_i(\pi_i) - t)$.

The valuation functions of the agents are private information. Let $R^\Pi$ be the set of all possible valuation functions of an agent. We assume that for every agent $i$ there exists a compact subset, $V_i$ of $R^\Pi$ of feasible valuation functions. Each agent $i$'s valuation function $v_i \in V_i$ is determined by a random (vector valued) variable, $\tilde{v}_i$. The random valuation functions $(\tilde{v}_i)_{i \in N}$ are distributed on $V = V_1 \times V_2 \times \ldots \times V_n$ according to the probability measure

an optimal auction. Krishna and Perry [9] discuss a general independent-private-values model with risk neutral agents, in which there are many objects and externalities are possible. However, they incorporate into their model the assumption that every agent has an *exogenously given* and type-dependent outside option function. Under these conditions they proved that an appropriate Groves mechanism is optimal among the efficient mechanisms.

$q$. That is for each measurable[12] subset $B$ of $V$, $Prob((\tilde{v}_1, \tilde{v}_2, \cdots, \tilde{v}_n) \in B) = q(B)$. Without loss of generality we assume that each $\tilde{v}_i$ is actually defined on $V$, that is $\tilde{v}_i : V \to V_i$ is defined by $\tilde{v}_i(v_1, v_2, \ldots, v_n) = v_i$.

Let $\tilde{S}$ be the random variable defined on $V$ by

$$\tilde{S}(v) = \max_{\pi \in \Pi} \sum_{i=1}^{n} v_i(\pi_i). \tag{3.1}$$

That, is for each vector of valuation functions $v = (v_1, v_2, \ldots, v_n)$, $\tilde{S}$ is the maximal-surplus variable. The expected value of $\tilde{S}$ is denoted by $E(\tilde{S})$, that is

$$E(\tilde{S}) = \int_V \tilde{S}(v) dq(v),$$

and the essential sup of $\tilde{S}$ is denoted by $S^*$. That is $S^*$ is defined by the following conditions:

$$Prob(\tilde{S} \leq S^*) = 1,$$

and

$$Prob(\tilde{S} \leq S^* - \varepsilon) < 1, \quad \varepsilon > 0.$$

We refer to $S^*$ as the ultimate surplus.

An *auction mechanism* is defined by a set of messages $(M_i)_{i \in N}$, one set for each agent, and by two functions $(g, t)$.

The *outcome function* $g$ determines a random allocation, that is $g : M \to \Delta(\Pi)$, where $M = M_1 \times M_2 \times \cdots \times M_n$ and $\Delta(\Pi)$ is the set of all probability distributions $p$ on $\Pi$, that is $p = (p(\pi))_{\pi \in \Pi}$, where $\sum_{\pi \in \Pi} p(\pi) = 1$ and $p(\pi) \geq 0$ for every $\pi \in \Pi$.

The *transfer function* $t$ determines the transfers of the agents to the organizer as a function of the messages and of the random choice of allocations. That is, $t = (t_1, t_2, \ldots, t_n)$, where $t_i : M \times \Pi \to R$.

Thus, if the agents send the vector of messages $m = (m_1, m_2 \cdots, m_n)$ then the seller conducts a lottery over allocations with the distribution function

[12] See the remark at the end of this section.

$g(m)$. The probability that the allocation $\pi$ is chosen is $g_\pi(m)$. If the allocation $\pi$ is chosen, then agent $i$ transfers the amount $t_i(m, \pi)$ to the center. We assume that each $M_i$ contains a distinguished message denoted by $\varphi$, which means "no participation". Note that $\varphi$ is not a real message; it is just a notation for non-participation. Naturally we require that a non participating agent does not receive any item and pays nothing. That is, if $m_i = \varphi$, $\pi_i = \emptyset$ for all $\pi \in \Pi$ for which $g_\pi(m) > 0$, and $t_i(m, \pi) = 0$ for every $\pi \in \Pi$.

Every auction mechanism $(M, g, t)$ defines a Bayesian game. In this game a strategy of agent $i$ is a function $b_i : V_i \to M_i$. Let us denote the set of possible strategies of agent $i$ by $\Sigma_i$, and let $\Sigma = \Sigma_1 \times \Sigma_2 \times \cdots \times \Sigma_n$ be the set of strategy profiles. If the agents use the vector of strategies $b = (b_1, b_2, \cdots, b_n) \in \Sigma$, then the expected utility of $i$ is denoted by $L_i(b)$, that is

$$L_i(b) = \int_V \sum_{\pi \in \Pi} u_i[v_i(\pi_i) - t_i(b(v), \pi)] g_\pi(b(v)) dq(v), \tag{3.2}$$

where $b(v) = (b_1(v_1), b_2(v_2), \ldots, b_n(v_n))$. A vector of strategies $b^e \in \Sigma$ is in equilibrium if for every $i \in N$, $\max_{b_i \in \Sigma_i} L_i(b_i, b^e_{-i})$ is attained at $b^e_i$.

Note that our setup is general. It includes sequential auctions and parallel auctions, in which the interaction between the seller and the buyers cannot be summarized by a description of one message which is sent by every agent to the seller. With each such auction we associate a fictitious auction mechanism, in which the message spaces of the agents are the sets of strategies in the Bayesian game associated with the auction. Though, formally the agents are not required to submit their full strategies, it is clear that their equilibrium behavior in the true auction can be analyzed within the framework of the associated fictitious auction mechanism.

**Remark**

In order to guarantee that the right hand side of (3.2) is well-defined, one has to impose measurability properties on the utility functions, outcome function and transfer functions, as well as on the strategies. However, since we don't have existence results in this paper, but rather we prove that certain properties hold in equilibrium, whenever an equilibrium exists, then all these assumptions can be captured by the assumption, that an equilibrium exists. Alternatively, we can assume a discrete model in which the valuation functions are restricted to take values in some finite set, and therefore all integrals make sense.

The main theorem of this section will make use of the following notation:

For a given auction mechanism and a given equilibrium $b^e$ (of the corresponding Bayesian game), we denote by $\tilde{R}$ the random revenue variable of the organizer. That is,

$$\tilde{R}(v) = \sum_{i=1}^{n} \left( \sum_{\pi \in \Pi} t_i(b^e(v), \pi) g_\pi(b^e(v)) \right),$$

and we denote the expected value of $\tilde{R}$ by $E(\tilde{R})$. We can establish the following theorem whose proof is given in section 8.

**Theorem 1** *Assume* **A1**, **A2**, *and* **A4**. *Then the expected revenue of the seller in equilibrium is bounded above by the expected maximal surplus. That is,*

$$E(\tilde{R}) \leq E(\tilde{S}).$$

*Consequently, the expected revenue in equilibrium is bounded above by the ultimate surplus, that is*

$$E(\tilde{R}) \leq S^*.$$

# 4 The VC mechanisms

Consider the environment described in Section 3, defined by the set of goods, the set of agents, the utility functions of the agents and the distribution of valuation-functions. In particular we assume **A1-A2**.

In a VC Mechanism the set of real messages of $i$ is its set of valuation-functions, that is $M_i = V_i \cup \{\varphi\}$. For every $m \in M$ we denote by $N(m)$ the set of all active agents, that is the set of all $i \in N$ for which $m_i \neq \varphi$. The outcome function $g$ is deterministic, that is $g : M \rightarrow \Pi$, and for $m = (m_1, m_2, \ldots, m_n)$ with $N(m) \neq \emptyset$, $g(m)$ is an allocation that maximizes $\sum_{i \in N(m)} v_i(\pi_i)$ over $\pi \in \Pi_{N(m)}$. If $N(m) = \emptyset$, then all goods go to the organizer, that is $g(m)(a) = 0$ for every $a \in G$.

We proceed to define the transfer functions in a VC mechanism. For every $i \in N$ we define $t_i(m, \pi)$ only for $\pi = g(m)$, and we denote $t_i(m) =$

$t_i(m, g(m))$. The transfer of $i$ to the seller is defined as follows. If $N(m) = \{i\}$, then $t_i(m) = 0$. Otherwise, let $N_{-i}(m) = N(m) \setminus \{i\}$ and define:

$$t_i(m) = \max_{\pi \in \Pi_{N_{-i}(m)}} \sum_{j \in N_{-i}(m)} v_j(\pi_j) - \sum_{j \in N_{-i}(m)} v_j(g(m)_j).$$

That is, $i$ pays the loss of surplus of the other agents that it causes.

For every $i \in N$ let $b_i^e$ be the truth revealing strategy of $i$. That is, $b_i^e(v_i) = v_i$ for every $v_i \in V_i$. It is well-known (see e.g. [1]) that $b^e = (b_1^e, b_2^e, \cdots, b_n^e)$ is an equilibrium vector of strategies independently of the probability of types $q$ and of the utility functions $(u_i)_{i \in N}$. This is so, because $b_i^e$ is a weakly dominating strategy for each $i$. When we refer in this paper to the expected revenue of the seller in a VC mechanism, we implicitly assume that all participants reveal their true valuation functions. Hence,

$$\tilde{R} = \sum_{i=1}^{n} t_i(\tilde{v}, g(\tilde{v})),$$

where $\tilde{v} = (\tilde{v}_1, \cdots, \tilde{v}_n)$. We will need the following representations for the revenue: For $v \in V$,

$$\tilde{R}(v) = \sum_{i=1}^{n} \left( \max_{\pi \in \Pi_{N \setminus \{i\}}} \sum_{j \neq i} v_j(\pi_j) - \sum_{j \neq i} v_j(g(v)) \right).$$

Note that,

$$\sum_{i=1}^{n} \left( \sum_{j \neq i} v_j(g(v)_j) \right) = (n-1)\tilde{S}(v),$$

Hence

$$\tilde{R}(v) = \tilde{S}(v) - \sum_{i=1}^{n} \left( \tilde{S}(v) - \max_{\pi \in \Pi_{N \setminus \{i\}}} \sum_{j \neq i} v_j(\pi_j) \right). \tag{4.1}$$

Note that because of the possibility of ties, there may be many VC mechanisms, where each one is defined by the choices it makes for vector of types that yield more than one allocation that maximizes the sum of valuations. The utility of a participating agent may depend on the particular chosen VC mechanism. However, by (4.1), the organizer's revenue depends only on $v$

(and not on $g(v)$). Hence, the revenue of the seller does not depend on the particular choice of the tie breaking rule[13]. From (4.1) we deduce:

**Lemma 1** *In a VC mechanism,*

$$\tilde{R}(v) \leq \tilde{S}(v) \leq S^*, \quad v \in V.$$

# 5 Asymptotic Optimality of VC Mechanisms

In this section we assume **A1-A3**. We deal with asymptotic properties of the seller's revenue in the VC mechanism applied to a sequence of auctions with an increasing number of agents. It is convenient to have an infinite set of players $N_\infty = \{1, 2, \cdots, \}$. The $n^{th}$ auction is conducted for the set of agents $N_n = \{1 \ldots, n\}$. Because we assume A3, we can assume without loss of generality that $V_i = V_j$ for all agents $i, j$. We denote by $V_*$ the set of feasible valuation functions for an agent, that is $V_* = V_i$ for every $i$. Let $(\tilde{v}_i)_{i=1}^{\infty}$ be the sequence of independent identically distributed random valuation functions of the agents in $N_\infty$. We assume that each $\tilde{v}_i$ is distributed on $V_*$ according to the probability measure $\lambda$. As $V_*$ is compact, and in particular it is bounded, there exists $c \in R_+^{2^G}$ such that for every $v \in V_*$

$$v(A) \leq c(A) \quad \text{for every } A \in 2^G.$$

Without loss of generality, assume that the constants $c(A)$, $A \in 2^G$ are tight. That is, for every $\varepsilon > 0$, and for every agent $i$, $Prob(\tilde{v}_i(A) \leq c(A) - \varepsilon) < 1$. We further assume that at least for one $A \subseteq G$, $c(A) > 0$, otherwise, with probability 1, for every agent $i$, $v_i(A) = 0$ for every $A \subseteq G$.

Let $V^\infty = V_*^{N_\infty}$ be the infinite Cartesian product of the agents' type sets, and let $q$ denotes the product probability on $V^\infty$. Let $\tilde{S}_n$ denote the maximal surplus random variable, when the set of agents is $N_n$. It is convenient to consider $\tilde{S}_n$ as a random variable defined on $V^\infty$, which depends only on the first $n$ coordinates. That is, for $v = (v_1, v_2, \ldots) \in V^\infty$,

$$\tilde{S}_n(v) = \max_{\pi \in \Pi_n} \sum_{i=1}^{n} v_i(\pi_i),$$

[13]This holds for every announcements of the agents and not only for their behavior in equilibrium.

where $\Pi_n$ is the set of all allocations $\pi : G \rightarrow N_n \cup \{0\}$. Let $S_n^*$ be the ultimate surplus in the $n^{th}$ auction, that is,

$$Prob(\tilde{S}_n \leq S_n^*) = 1 \quad \text{and } Prob(\tilde{S}_n \leq S_n^* - \varepsilon) < 1 \text{ for every } \varepsilon > 0.$$

Let

$$\Theta^* = \max \sum_{i=1}^{l} c(G_i),$$

where the max ranges over all ordered partitions $(G_1, \ldots, G_l)$ of $G$ (that is, $G_s \neq \emptyset$ for every $s$, $G_s \cap G_j = \emptyset$ for every $s \neq j$, and $\cup_{s=1}^{l} G_s = G$). Because we assume $c(A) > 0$ for some $A \subseteq G$, $\Theta^* > 0$. Because of our assumption on the tightness of the constants $c(A)$, $A \in 2^G$, the following lemma holds:

**Lemma 2** *In the valuation-symmetric private-values-model,* $S_n^* \leq \Theta^*$ *for every* $n \geq 1$*. In addition, for every* $n \geq k$*,*

$$S_n^* = \Theta^*.$$

We will denote the revenue obtained in the $n$-th auction by $\tilde{R}_n$.

The proof of the following theorem is given in Section 8:

**Theorem 2** *Consider the VC mechanism applied to a sequence of auctions, where the* $n^{th}$ *auction is conducted for the set of agents* $N_n$*,* $n \geq 2$*. Assume* **A1-A4***. Then,*

$$\lim_{n\to\infty} \tilde{R}_n = \Theta^* \quad a.s. \tag{5.1}$$

*Consequently,*

$$\lim_{n\to\infty} \frac{\tilde{R}_n}{\tilde{S}_n} = 1 \quad a.s., \tag{5.2}$$

*and*

$$\lim_{n\to\infty} \frac{E(\tilde{R}_n)}{E(\tilde{S}_n)} = 1. \tag{5.3}$$

# 6 Extensions: random utility functions, and random number of potential participants

In Section 6.1 we show that an appropriate versions of theorems 1 and 2 hold in an extended model of the one given in Section 3. In this new model the utility functions of the agents are their private information. In section 6.2 we extend theorems 1 and 2 to a model with a random number of potential participants. Combining these two subsections show that theorem 1 and 2 are generalized to an extended model in which we have both, random utility functions and a random number of potential participants.

### 6.1 Random utility functions

In all previous sections it was assumed that the utility functions of the agents (in contrast to their valuation functions) are commonly known. This assumption does not make sense when the number of participants is large. To deal with this we assume that every agent $i$ is characterized by a pair of random variables $(\tilde{v}_i, \tilde{u}_i)$, which is distributed on $T_i = V_i \times U_i$, where $U_i$ is a set of utility functions[14], that is every $u_i \in U_i$ satisfies $u_i(0) = 0$ and $u_i$ is strictly increasing. This additional feature does not change the conclusions of Theorem 1 and 2. However, it is necessary to modify the notations: Let $T = T_1 \times \cdots \times T_n$, and assume that $((\tilde{v}_i, \tilde{u}_i))_{i=1}^n$ is distributed on $T$ according to the probability measure $q$. Every auction mechanism defines a Bayesian game, in which a strategy of $i$ is a function $b_i : T_i \to M_i$. For each profiles of strategy $b$ and each $u_i \in U_i$ we denote the expected utility of $i$ given that its utility function is $u_i$ by $L_i(b|u_i)$. Thus, $b^e$ is in equilibrium, if for every agent $i$, for every strategy $b_i$ of $i$,

$$L(b^e|u_i) \geq L((b_i, b^e_{-i})|u_i) \quad u_i\text{-a.s.} \tag{6.1}$$

Note that the definition (6.1) is neither the ex post definition in (3.2), nor the equivalent interim definition, which would require that (6.1) holds given the full type $(v_i, u_i)$ of $i$. However, it is obviously equivalent to these two common definitions. The proof of the following theorem is given in Section 8:

[14]Of course, $U_i$ should be associated with a $\sigma$-field of measurable sets.

**Theorem 3** *Assume* **A1**, **A2**, *and that every agent $i$ is risk averse with probability 1. Then the expected revenue of the seller in equilibrium is bounded above by the expected maximal surplus. That is,*

$$E(\tilde{R}) \leq E(\tilde{S}).$$

*Consequently, the expected revenue in equilibrium is bounded above by the ultimate surplus, that is*

$$E(\tilde{R}) \leq S^*.$$

We proceed to generalize Theorem 2 to the case of random utility functions. We replace assumption **A3** with

- **A3'** Symmetric independent-private-values model.

That is, the types are identically distributed and independent.

As the surplus variable does not depend on the utility functions of the agents, the proof of the following theorem is analogous to the proof of Theorem 2, and therefore it is omitted.

**Theorem 4** *Consider the VC mechanism applied to a sequence of auctions, where the $n^{th}$ auction is conducted for the set of agents $N_n$, $n \geq 2$. Assume* **A1-A2**, **A3'**, *and that every agent is risk averse with probability 1. Then,*

$$\lim_{n\to\infty} \tilde{R}_n = \Theta^* \quad a.s.$$

*Consequently,*

$$\lim_{n\to\infty} \frac{\tilde{R}_n}{\tilde{S}_n} = 1 \quad a.s.,$$

*and*

$$\lim_{n\to\infty} \frac{E(\tilde{R}_n)}{E(\tilde{S}_n)} = 1.$$

### 6.2 Random number of potential buyers

Let $A$ be the set of agents in the world. If the seller organizes an auction, only some of these agents consider participation. Many of them are not aware of the existence of this auction, and some of those who know about the auction do not go to the auction place (that may be an home page), without making any strategic considerations. In our previous results we take the group of agents as the group of agents who consider participation. The number of agents in this group was implicitly assumed to be commonly known. However, this common assumption in the theory of auctions is not reasonable in many setups, and in particular in Internet auctions. Neither the organizer, nor the potential buyers can possibly know the number of potential buyers. However, the number of agents (that is, the number of all Internet users) is commonly known (or, at least , commonly estimated). In order to capture this feature of many auctions, we assume that there exists a sequence $(\tilde{z}_i)_{i \in A}$ of $\{0,1\}$-random variables, such that agent $i$ considers participation if and only if $\tilde{z}_i = 1$. We do not assume that $\tilde{z}_i$ and $\tilde{v}_i$ are independent. Adding this feature to the model does not change our main results or their proofs, if we define the surplus as the surplus of the agents in $A$ that consider participation. Thus, the definition of the maximal-surplus variable in (3.1) is changed as follows: $\tilde{S} : V \times \{0,1\}^A \to R$, is defined by

$$\tilde{S}(v, z) = \max_{\pi \in \Pi_{A(v)}} \sum_{i \in A(v)} v_i(\pi),$$

where

$$A(v) = \{i \in A : \quad \tilde{z}_i = 1\}.$$

To see this we define a setup, like in Section 3, in which every agent $i \in A$ considers participation, but in the modified model the new random valuation function of agent $i$ with the random valuation function $\tilde{v}_i$ is $\tilde{v}_i \tilde{z}_i$, We then apply our theorems to this new setup. Note that in our optimality theorem we have to assume that the sequence of random variables $(\tilde{v}_i, \tilde{z}_i)_{i=1}^{\infty}$ is identically distributed and independent. However, for every agent $i$, the value of $\tilde{z}_i$ may depend on its valuation function.

# 7 Externalities: An Upper Bound on the Seller's Expected Revenue in Equilibrium

In this section we prove that the upper bound on the organizer's revenue established in Theorem 1, under assumption **A1-A2** and **A4** remains valid, when **A1-A2** are replaced by assumption **E**. That is, we show that the expected revenue in equilibrium is bounded above by the expected maximal surplus, if the expected utility in this equilibrium is non negative for every participating agent. As in Section 3, we consider a seller denoted by 0, who wishes to sell a set $G$ of $k$ goods to a group of potential buyers, $N = \{1, 2, \cdots, n\}$, and we set $N^0 = N \cup \{0\}$. An allocation of goods is a function $\pi : G \to N^0$, and the set of all allocations is denoted by $\Pi$. We deviate from the notations established in Section 3 when defining valuation functions. A valuation function of $i$ depends on the full distribution of goods amongst the agents, and not only on the set of goods allocated to him. In such a case, it is not reasonable to assume non negativity of valuations or the free disposal assumption[15]. Thus, a valuation function of $i$ is a function $w_i : \Pi \to R$. In addition, $i$ has a strictly increasing von-Neumann Morgenstern utility function for money, $u_i : R \to R$, which is normalized by $u_i(0) = 0$. The utility of $i$ if the center chooses the allocation $\pi$ and $i$ pays $t$ is $u_i(v_i(\pi) - t)$.

The valuation functions of the agents are private information. Let $R^\Pi$ be the set of all possible valuation functions of an agent. We assume that for every agent $i$ there exists a compact subset, $W_i$ of $R^\Pi$ of feasible valuation functions. Each agent $i$'s valuation function $w_i \in W_i$ is determined by a random (vector valued) variable, $\tilde{w}_i$. The random valuation functions $(\tilde{w}_i)_{i \in N}$ are distributed on $W = W_1 \times W_2 \times \ldots \times W_n$ according to the probability measure $q$. That is for each measurable[16] subset $B$ of $W$, $Prob((\tilde{w}_1, \tilde{w}_2, \cdots, \tilde{w}_n) \in B) = q(B)$. Without loss of generality we assume that each $\tilde{w}_i$ is actually defined on $W$, that is $\tilde{w}_i : W \to W_i$ is defined by $\tilde{w}_i(w_1, w_2, \ldots, w_n) = w_i$.

Let $\tilde{S}$ be the random variable defined on $W$ by

$$\tilde{S}(v) = \max_{\pi \in \Pi} \sum_{i=1}^{n} w_i(\pi).$$

That, is for each vector of valuation functions $w = (w_1, w_2, \ldots, w_n)$, $\tilde{S}$ is the maximal-surplus variable. The expected value of $\tilde{S}$ is denoted by $E(\tilde{S})$, and the essential sup of $\tilde{S}$ is denoted by $\tilde{S}^*$.

[15] An agent may prefer less goods if these goods are allocated to particular agents, because this may increase competition amongst the other agents.

[16] See the remark at Section 3.

Every auction mechanism $(M, g, t)$, as defined in Section 3 defines a Bayesian game. In this game a strategy of agent $i$ is a function $b_i : V_i \to M_i$. Let us denote the set of possible strategies of agent $i$ by $\Sigma_i$, and let $\Sigma = \Sigma_1 \times \Sigma_2 \times \cdots \times \Sigma_n$ be the set of strategy profiles. If the agents use the vector of strategies $b = (b_1, b_2, \cdots, b_n) \in \Sigma$, then the expected utility of $i$ is denoted by $L_i(b)$, that is

$$L_i(b) = \int_V \sum_{\pi \in \Pi} u_i[w_i(\pi) - t_i(b(w), \pi)] g_\pi(b(w)) dq(w),$$

where $b(w) = (b_1(w_1), b_2(w_2), \ldots, b_n(w_n))$. A vector of strategies $b^e \in \Sigma$ is in equilibrium if for every $i \in N$, $\max_{b_i \in \Sigma_i} L_i(b_i, b^e_{-i})$ is attained at $b^e_i$.

For a given auction mechanism and a given equilibrium $b^e$ in this mechanism, we denote by $\tilde{R}$ the random revenue variable of the organizer. That is,

$$\tilde{R}(w) = \sum_{i=1}^{n} \left( \sum_{\pi \in \Pi} t_i(b^e(w), \pi) g_\pi(b^e(v) \right),$$

and we denote the expected value of $\tilde{R}$ by $E(\tilde{R})$. We can establish the following:

**Theorem 5** *Assume all agents are risk averse (**A4**). Consider an auction mechanism and an equilibrium profile $b^e$ in the Bayesian game determined by this mechanism, in which $U_i(b^e) \geq 0$ for every $i \in N$, where $U_i(b^e)$ is the expected utility of agent $i$ in $b^e$. Then the expected revenue of the seller in this equilibrium is bounded above by the expected maximal surplus. That is,*

$$E(\tilde{R}) \leq E(\tilde{S}).$$

*Consequently, the expected revenue in equilibrium is bounded above by the ultimate surplus, that is*

$$E(\tilde{R}) \leq S^*.$$

The proof of Theorem 5 is given in Section 8. Note that in Theorem 1 we prove that $U_i(b^e) \geq 0$, while in Theorem 5 we have to assume it.

The proof of Theorem 5 uses the risk-aversion of the agents and the non-negativity of the agents' expected utilities in equilibrium. We proceed to show that these conditions are necessary for the conclusion of the theorem.

In Example 1 we present an example to an auction mechanism and to a distribution of types that yield an equilibrium, in which the expected utilities of the agents are negative. In this example the expected revenue of the seller exceeds the upper bound $E(\tilde{S})$.

**Example 1**

There is only one good ($k = 1$) and two risk neutral agents ($n = 2$). Hence $\Pi$ contains only 3 allocations denoted by $1, 2, 0$. That is, the allocation in which $i$ gets the object is denoted by $i$. The distribution over types is concentrated at a single vector of valuation functions $(w_1, w_2)$ (that is $q(\{(w_1, w_2)\}) = 1$), which are defined as follows: For $i = 1, 2$, $w_i(0) = 0$, $w_i(i) = 1$, and $w_i(3-i) = -3$.[17]. Consider the following auction mechanism: The message spaces are $M_i = \{w_i, \varphi\}$, $i = 1, 2$. If both players participate, that is they reveal their valuation functions, the center chooses $\pi = 0$, and each player pays 1. If only one player participates, it gets the object and pays 0. if both agents do not participate, the center chooses $\pi = 0$. The (degenerate) Bayesian game associated with this mechanism has a unique equilibrium, in which each agent declares its true valuation function. Actually, choosing $w_i$ strongly dominates $\varphi$ for $i = 1, 2$. In equilibrium, the center chooses the allocation $\pi = 0$ and each agent $i$ pays 1. Hence the expected revenue of the seller in this equilibrium is $t_1 + t_2 = 2$. However the expected surplus is $E(\tilde{S}) = \max_{j=0}^{3}(w_1(j) + w_2(j)) = 0$. ∎

In Example 2, which is taken from [19], there is a single good and the auction mechanism is a third-price auction. In such an auction the winning bid is the highest bid, and the price paid for the good is the third highest bid (see e.g., [33, 7, 18]). In this example the agents are risk seeking, their expected utility in equilibrium is non-negative, but the seller's expected revenue in equilibrium exceeds $E(\tilde{S})$.

It is well-known that a risk-neutral seller can sell lottery tickets with negative expected gain to a risk-seeking agent and obtain as a result very high gains. As pointed out in [20], this example shows that a third-price auction can serve as an implicit lottery mechanism for risk seeking agents.

**Example 2**

[17] The situation described in this example is referred to by [6] as the *negative externalities case.*

We will discuss the equilibrium set of a third-price auction for a single good with three participants, $N = \{1,2,3\}$. We use the independent-private-values model, and we assume that there are no externalities, and that the valuation of each agent is distributed in the interval [0,1] according to the uniform distribution, $F(x) = x$ for every $x \in [0,1]$. Every agent $i \in N$ uses the convex utility function $u_\alpha : R \to R$, $\alpha > 1$, where $u_\alpha(x) = x$ when $x \leq 0$, and $u_\alpha(x) = \alpha x$ for $x > 0$. By a slight modification in the proof of Theorem AT in [18] [18] it can be shown that a continuous function $s_\alpha$ defined on $[0,1]$ constitutes a symmetric equilibrium strategy in the auction game if and only if $s_\alpha(0) = 0$, $s_\alpha$ is increasing, and

$$\int_{t=0}^{v} u_\alpha(v - s_\alpha(t))dt = 0$$

for every $v \in [0,1]$. Solving this integral equation yields the unique solution:

$$s_\alpha(v) = (1 + \sqrt{\alpha})v, \quad v \in [0,1].$$

It is easily verified that the expected utility of each agent in this equilibrium is positive.

The expected revenue of the seller in this equilibrium is

$$E_\alpha(\tilde{R}) = (1 + \sqrt{\alpha}) \cdot E(v_{[3]}),$$

where $E(v_{[3]})$ is the expected value of the third-order statistics (i.e., the expected value of the type of the agent with the third-highest evaluation). As $lim_{\alpha \to \infty} E_\alpha(\tilde{R}) = \infty$, for sufficiently large $\alpha$,

$$E_\alpha(\tilde{R}) > E(\max(\hat{v}_1, \hat{v}_2, \hat{v}_3)).$$

Actually, in our case the expected first-order statistics is $\frac{3}{4}$, while the third-order statistics is $\frac{1}{4}$. Hence, if $\alpha > 4$ the expected revenue of the seller exceeds the expected maximal surplus. ▮

[18] This theorem is proved under the assumption that the utility functions are twice continuously differentiable.

# 8 Proofs

**Proof of Theorem 1**

Consider a fixed auction mechanism and a fixed equilibrium profile $b^e$. Let $i \in N$. Because $i$ is risk averse, $u_i$ is concave. Therefore, for every $v \in V$

$$\sum_{\pi\in\Pi} u_i(v_i(\pi_i) - t_i(b^e(v),\pi))g_\pi(b^e(v)) \leq u_i\left(\sum_{\pi\in\Pi}(v_i(\pi_i) - t_i(b^e(v),\pi))g_\pi(b^e(v))\right).$$

Because $i$ can deviate to the strategy of always not participating, its expected utility $L_i(b^e)$ is non negative. Therefore (3.2) yields:

$$\int_V u_i\left(\sum_{\pi\in\Pi}(v_i(\pi_i) - t_i(b^e(v),\pi)g_\pi(b^e(v))\right) dq(v) \geq 0. \tag{8.1}$$

By applying Jensen inequality (i.e., $u_i(E(\cdot)) \geq E(u_i(\cdot))$) to (8.1),

$$u_i\left(\int_V (\sum_{\pi\in\Pi}(v_i(\pi_i) - t_i(b^e(v),\pi))g_\pi(b^e(v)))dq(v)\right) \geq 0.$$

As $u_i$ is increasing and $u_i(0) = 0$,

$$\int_V (\sum_{\pi\in\Pi}(v_i(\pi_i) - t_i(b^e(v),\pi))g_\pi(b^e(v)))dq(v) \geq 0. \tag{8.2}$$

Let $\tilde{R}^i$ be the expected payment of $i$, and let $E(\tilde{R}^i)$ denotes the expected value of $\tilde{R}^i$. That is

$$E(\tilde{R}^i) = \int_V \sum_{\pi\in\Pi} t_i(b^e(v),\pi)g_\pi(b^e(v))dq(v).$$

By (8.2),

$$E(\tilde{R}^i) \leq \int_V (\sum_{\pi\in\Pi} v_i(\pi_i)g_\pi(b^e(v)))dq(v).$$

As $E(\tilde{R}) = \sum_{i=1}^n E(\tilde{R}^i)$,

$$E(\tilde{R}) \leq \int_V (\sum_{\pi\in\Pi}\left(\sum_{i=1}^n v_i(\pi_i)\right) g_\pi(b^e(v)))dq(v).$$

Because a convex combination of a set of numbers is less or equals the maximal number in this set,

$$E(\tilde{R}) \leq \int_V (\max_{\pi\in\Pi} \sum_{i=1}^{n} v_i(\pi_i))dq(v).$$

Hence

$$E(\tilde{R}) \leq E(\tilde{S}).$$

∎

**proof of Theorem 2**

We will prove (5.1). Hence, Lemma 1 implies $\lim_{n\to\infty} \tilde{S}_n = \Theta^*$ a.s., which implies (5.2), because $\Theta^* > 0$. By the Lebesque convergence theorem, (5.1) yields $\lim_{n\to\infty} E(\tilde{R}_n) = \lim_{n\to\infty} E(\tilde{S}_n) = \Theta^*$, and therefore (5.3) holds. We proceed to prove (5.1). By Lemma 2 there exists an ordered partition $(G_1, G_2, \ldots, G_l)$ of the set of goods such that

$$\Theta^* = \sum_{s=1}^{l} c(G_s).$$

For a finite set of agents $B$, let $\tilde{S}_B$ be the random variable defined on $V^\infty$, that describes the surplus of the agents in $B$. That is,

$$\tilde{S}_B(v) = \max_{\pi\in\Pi_b} \sum_{i\in B} v_i(\pi_i), \quad v \in V^\infty.$$

When $B = N \setminus \{i\}$ we denote $\tilde{S}_B$ by $\tilde{S}_{n,-i}$. By (4.1),

$$\tilde{R}_n = \tilde{S}_n - \sum_{i=1}^{n}(\tilde{S}_n - \tilde{S}_{n,-i}).$$

We will show that $\tilde{S}_n \to \Theta^*$ a.s., and that $\sum_{i=1}^{n}(\tilde{S}_n - \tilde{S}_{n,-i}) \to 0$ a.s.

It can be easily deduced from Theorem 1 (Page 251) in [27], that a sequence of random variables $(X_n)_{n=1}^{\infty}$ converges to a random variable $X$ if for every $\epsilon > 0$,

$$\sum_{n=1}^{\infty} Prob(|X - X_n| \geq \epsilon) < \infty.$$

Let then $0 < \epsilon < \Theta^*$. Obviously,

$$Prob(|\Theta^* - \tilde{S}_n| \geq \epsilon) = Prob(\tilde{S}_n \leq \Theta^* - \epsilon).$$

Note that if $\tilde{S}_n(v) \leq \Theta^* - \epsilon$, then for every ordered set of $l$ agents in $N_n$, $F = \{i_1, i_2, \ldots, i_l\}$, $i_1 < \cdots < i_l \leq n$, $\sum_{j=1}^{l} v_{i_j}(G_j) \leq \Theta^* - \epsilon$. Let $n \geq k$, and let $r_n$ be a positive integer satisfying $r_n k \leq n < (r_n + 1)k$. Then

$$Prob(\tilde{S}_n \leq \Theta^* - \epsilon) \leq Prob(\sum_{i=1}^{l} \tilde{v}_{jk+i}(G_i) \leq \Theta^* - \epsilon \quad \text{for every } 0 \leq j \leq r_n - 1).$$

Because of our independence assumption,

$$Prob(\tilde{S}_n \leq \Theta^* - \epsilon) \leq (Prob(\sum_{i=1}^{l} \tilde{v}_i(G_i) \leq \Theta^* - \epsilon))^{r_n}.$$

Therefore,

$$Prob(\tilde{S}_n \leq \Theta^* - \epsilon) \leq (Prob(\sum_{i=1}^{l} \tilde{v}_i(G_i) \leq \Theta^* - \epsilon))^{\frac{n}{k}-1}. \tag{8.3}$$

Let

$$\alpha(\varepsilon) = Prob(\sum_{i=1}^{l} \tilde{v}_i(G_i) \leq \Theta^* - \epsilon). \tag{8.4}$$

We now show that

$$\alpha(\varepsilon) < 1. \tag{8.5}$$

which combined with (8.4) proves that

$$\sum_{n=k}^{\infty} Prob(\tilde{S}_n \leq \Theta^* - \varepsilon) < \sum_{n=k}^{\infty} \alpha(\varepsilon)^{\frac{n}{k}-1} < \infty,$$

and hence that

$$\sum_{n=1}^{\infty} Prob(\tilde{S}_n \leq \Theta^* - \varepsilon) < \infty.$$

To prove (8.5), recall that $\Theta^* = c(G_1) + \cdots + c(G_l)$. Hence

$$\alpha(\varepsilon) = Prob(\sum_{i=1}^{l} \tilde{v}_i(G_i) \leq \Theta^* - \epsilon) \leq Prob(\exists 1 \leq i \leq l, \tilde{v}_i(G_i) \leq c(G_i) - \frac{\epsilon}{l}).$$

Hence,

$$\alpha(\varepsilon) \leq 1 - \prod_{i=1}^{l}(1 - Prob(\tilde{v}_i(G_i) \leq c(G_i) - \frac{\epsilon}{l}) < 1,$$

by our assumption on the tightness of $c(A)$, $A \in 2^G$.

We proceed to show that $\tilde{M}_n = \sum_{i=1}^{n}(\tilde{S}_n - \tilde{S}_{n,-i})$ converges to 0 almost surely. It suffices to show that for every $\epsilon > 0$

$$\sum_{n=1}^{\infty} Prob(\tilde{M}_n \geq \epsilon) < \infty. \tag{8.6}$$

For every $n \geq k$ and for every $1 \leq j \leq k$, let $\tilde{\delta}_j^n$ be the random $\{0,1\}$-variable defined on $V^\infty$ by : $\tilde{\delta}_j^n(v) = 1$ if and only if $g(v)_j \neq \emptyset$ in the auction conducted for $N_n$. Let $\tilde{F}_n = \{j \in N_n : \tilde{\delta}_j^n = 1\}$. Obviously, $|\tilde{F}_n| \leq k$, and

$$Prob(\tilde{M}_n \geq \varepsilon) = \sum_{F \in 2^{N_n}, |F| \leq k} Prob(\tilde{M}_n \geq \epsilon, \tilde{F}_n = F).$$

We will show that for every $n \geq k$, for every $\varepsilon > 0$, and for every $F \in 2^{N_n}$ with $|F| \leq k$,

$$Prob(\tilde{M}_n \geq \varepsilon, \tilde{F}_n = F) \leq \alpha(\frac{\varepsilon}{k})^{\frac{n}{k}-2}, \tag{8.7}$$

where $\alpha(.)$ is defined in (8.4). As the number of $F \subseteq N_n$ with $|F| \leq k$ is bounded above by $n^k$, (8.7) implies

$$Prob(\tilde{M}_n \geq \varepsilon) \leq \ n^k \alpha(\frac{\varepsilon}{k})^{\frac{n}{k}-2}.$$

As $\alpha(\frac{\varepsilon}{k}) < 1$,

$$\sum_{n=k}^{\infty} n^k \alpha(\frac{\varepsilon}{k})^{\frac{n}{k}-2} < \infty,$$

which proves (8.6).

Indeed, let $F = \{i_1, \ldots, i_s\} \subseteq N_n$, and let $D = N_n \setminus F$. Note that

$$Prob(\tilde{M}_n \geq \varepsilon, \tilde{F}_n = F) = Prob(\sum_{i=1}^{n} \tilde{S}_{n,-i} \leq n\tilde{S}_n - \epsilon, \tilde{F}_n = F).$$

If $i \in D$, $\tilde{S}_{n,-i} = \tilde{S}_n$ on $F$, hence

$$Prob(\sum_{i=1}^{n} \tilde{S}_{n,-i} \leq nS_n - \epsilon, \tilde{F}_n = F) = Prob(\sum_{j=1}^{s} \tilde{S}_{n,-i_j} \leq s\tilde{S}_n - \epsilon, \tilde{F}_n = F).$$

As $\tilde{S}_n \leq \Theta^*$,

$$Prob(\tilde{M}_n \geq \epsilon, \tilde{F}_n = F) \leq Prob(\sum_{j=1}^{s} S_{n,-i_j} \leq s\Theta^* - \epsilon, \tilde{F}_n = F).$$

As $\tilde{S}_{n,D} \leq \tilde{S}_{n,-i_j}$ for every $1 \leq j \leq s$,

$$Prob(\tilde{M}_n \geq \epsilon, \tilde{F}_n = F) \leq Prob(\tilde{S}_{n,D} \leq \Theta^* - \frac{\epsilon}{s}, \tilde{F}_n = F),$$

implying that

$$Prob(\tilde{M}_n \geq \epsilon, \tilde{F}_n = F) \leq Prob(\tilde{S}_{n,D} \leq \Theta^* - \frac{\epsilon}{k}, \tilde{F}_n = F).$$

Therefore, by partitioning $D$ to $r_n - 1$ subsets with cardinality of $k$ each,

$$Prob(\tilde{M}_n \geq \epsilon, \tilde{F}_n = F) \leq \alpha(\frac{\varepsilon}{k})^{\frac{n}{k}-2},$$

which proves (8.7). ▮

**Proof of Theorem 3**

Consider a fixed auction mechanism and a fixed equilibrium profile $b^e$. Let $i \in N$. For every $v \in V$, every $u_{-i} \in U_{-i}$, and every concave function $u_i \in U_i$,

$$\sum_{\pi \in \Pi} u_i \; (v_i(\pi_i) - t_i(b^e(v,u),\pi) \;) g_\pi(b^e(v,u)) \leq u_i \left( \sum_{\pi \in \Pi} (v_i(\pi_i) - t_i(b^e(v,u),\pi)) g_\pi(b^e(v,u)) \right),$$

where $u = (u_i, u_{-i})$. Because $i$ can deviate to the strategy of always not participating, its expected utility $L_i(b^e|u_i)$ is almost surely non negative. Therefore (6.1) yields that $u_i$-almost-surely,

$$\int_{V \times U} u_i \left( \sum_{\pi \in \Pi} (v_i(\pi_i) - t_i(b^e(v,u),\pi) g_\pi(b^e(v,u)) \right) dq((v,u)|u_i) \geq 0. \quad (8.8)$$

By applying Jensen inequality (i.e., $u_i(E(\cdot|u_i)) \geq E(u_i(\cdot)|u_i)$ to (8.8),

$$u_i\left(\int_{V\times U}(\sum_{\pi\in\Pi}(v_i(\pi_i) - t_i(b^e(v,u),\pi))g_\pi(b^e(v,u)))dq((v,u)|u_i)\right) \geq 0, \quad u_i\text{-a.s.}$$

As $u_i$ is increasing and $u_i(0) = 0$,

$$\int_{V\times U}(\sum_{\pi\in\Pi}(v_i(\pi_i) - t_i(b^e(v,u),\pi))g_\pi(b^e(v,u)))dq((v,u)|u_i) \geq 0, \quad u_i\text{-a.s.} \tag{8.9}$$

Therefore, the expected value of the left-hand side of (8.9) is non negative. that is,

$$\int_{V\times U}(\sum_{\pi\in\Pi}(v_i(\pi_i) - t_i(b^e(v,u),\pi))g_\pi(b^e(v,u)))dq((v,u)) \geq 0. \tag{8.10}$$

Let $\tilde{R}^i$ be the expected payment of $i$, and let $E(\tilde{R}^i)$ denotes the expected value of $\tilde{R}^i$. That is

$$E(\tilde{R}^i) = \int_{V\times U}\sum_{\pi\in\Pi}t_i(b^e(v,u),\pi)g_\pi(b^e(v,u))dq(v,u).$$

By (8.10),

$$E(\tilde{R}^i) \leq \int_{V\times U}(\sum_{\pi\in\Pi}v_i(\pi_i)g_\pi(b^e(v,u)))dq(v,u).$$

As $E(\tilde{R}) = \sum_{i=1}^n E(\tilde{R}^i)$,

$$E(\tilde{R}) \leq \int_{V\times U}(\sum_{\pi\in\Pi}\left(\sum_{i=1}^n v_i(\pi_i)\right)g_\pi(b^e(v,u)))dq(v,u).$$

Because a convex combination of a set of numbers is less or equals the maximal number in this set,

$$E(\tilde{R}) \leq \int_{V\times U}(\max_{\pi\in\Pi}\sum_{i=1}^n v_i(\pi_i))dq(v,u).$$

Hence

$$E(\tilde{R}) \leq E(\tilde{S}).$$

∎

**Proof of Theorem 5**

Consider a fixed auction mechanism and a fixed equilibrium profile $b^e$, for which $U_i(b^e) \geq 0$ for every $i \in N$. The proof can be produced word by word from the proof of Theorem 1, by replacing the terms $\tilde{v}_i$, $v_i(\pi_i)$, $v$, $V_i$ and $V$ in the proof of Theorem 1 with the terms $\tilde{w}_i$, $w_i(\pi)$, $w$,$W_i$, and $W$ respectively. ∎

Note again that in Theorem 1 we prove that $U_i(b^e) \geq 0$, while in Theorem 5 we have to assume it.

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