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ASYMPTOTIC MEASURE-BASED VALUES OF NONATOMIC GAMES*†

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Let $v = \min(\mu_1, \mu_2, \ldots, \mu_n)$, where $\mu_1, \mu_2, \ldots, \mu_n$ are mutually singular nonatomic probability measures, i.e., v is the market game derived from an n-glove nonatomic market with transferable utility. We describe the set of all μ-asymptotic values of v, where μ ranges over all nonatomic probability measures for which μ_i is absolutely continuous with respect to μ and $d\mu_i/d\mu \in L_2(\mu)$ for all $1 \leqslant i \leqslant n$. This set is proved to be convex and relatively open.

0. Introduction. Values of market games have been extensively analyzed, in particular for "large" nonatomic games (e.g., Aumann and Shapley [2]). When the economy is sufficiently differentiable the market game derived from it has an asymptotic value (which is also the competitive distribution of the unique equilibrium price of the market (see [2])). However, it was proved in [2, Example 19.2] that the market game $v = \min(\mu_1, \mu_2, \mu_3)$ does not have an asymptotic value, where, for all $1 \leqslant i \leqslant 3$, μ_i is the normalized restriction of the Lebesque measure to the interval $[(i-1)/3, i/3]$.

In order to extend the class of games for which the asymptotic approach applies, the notion of measure-based asymptotic value has been introduced by Aumann and Kurz [1], and studied in Hart [3] (the axiomatic approach has been discussed in Monderer [5]). It takes into account, besides the coalitional worth function of the game, also the population measure μ of the underlying economy. This is done by considering only those partitions whose atoms have almost the same μ-measure. For example, it was proved in [3, Theorem 9.2] that the game $v = \min(\mu_1, \mu_2, \mu_3)$ has a μ-asymptotic value for every $\mu \in NA^1$ such that $\mu_i \in L_2(\mu)$ for every $1 \leqslant i \leqslant 3$. More generally, Hart proved that every market game derived from a market with population measure μ, has a μ-asymptotic value provided that the variance of the excess demand (for some competitive allocation) is finite (see [3, Main Theorem]).

In this paper we deal with one of the open problems introduced by Hart [3, Open problem B]: For a fixed market game v characterize the set of all μ-asymptotic values of v, where μ ranges over all nonatomic probability measure for which the μ-asymptotic value of v exists. We shall partially answer this question for an important class of market games—the minimum games. These are the games having the form $v = \min(\mu_1, \mu_2, \ldots, \mu_n)$, where $\mu_1, \mu_2, \ldots, \mu_n$ are mutually singular measures in NA^+, i.e., the exact games whose core is the convex hull of a finite number of mutually singular NA^1-measures. For every such game v we shall describe the set of all μ-asymptotic values of v for all $\mu \in NA^1$ for which the core of v is contained in $L_2(\mu)$. It turns out that this set is convex and relatively open.

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The paper is organized as follows: In §1 we state our main theorem and necessary preliminaries, and in §2 we prove the main theorem.

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1. μ-asymptotic values. We will use the terminology and notation of [2]. For a game v and a nonatomic probability measure μ (i.e., $\mu \in NA^1$), $\varphi_\mu v$ will denote the μ-asymptotic value of v as defined in [3]. Let R^n be the n-dimensional Euclidean space. For $x \in R^n$, x_i will denote its ith coordinate. We shall write $x \geqslant y$ if $x_i \geqslant y_i$ for all $1 \leqslant i \leqslant n$; $x > y$ if $x \geqslant y$ and $x \neq y$; $x \gg y$ if $x_i > y_i$ for all $1 \leqslant i \leqslant n$. Let R^n_+ be the nonnegative orthant of R^n, and let R^n_{++} be the positive orthant of R^n. A real-valued function f defined on a subset of R^n is *nondecreasing* if $f(x) \leqslant f(y)$ whenever $x \leqslant y$. A real-valued function f defined on R^n_+ which is concave, positively homogeneous of degree one, nondecreasing and with $f(0) = 0$, will be called a *market function*. MF_n will denote the set of all market functions defined on R^n_+.

Let f be a market function in MF_n. A vector $p \in R^n$ is a *super-gradient* of f at a point $x \in R^n_{++}$ if

$$f(y) - f(x) \leqslant p(y - x) \quad \text{for all } y \in R^n_+ .$$

The *gradient* of f at a point $x \in R^n_+$ is the set of all super-gradients of f at x. We shall denote it by $\Delta f(x)$. By Theorems 23.1 and 23.4 in [6] (see also [3, Chapter 6]), $\Delta f(x)$ is a nonempty compact convex set and

$$\Delta f(x) = \{ p \in R^n_+ : px = f(x) \text{ and } f(y) \leqslant py \text{ for every } y \in R^n_+ \}.$$

For every nonempty compact convex set K in R^n and for every $x \in R^n$ denote

$$P_K[x] = \{ p' \in K : p'x = \max\{ px : p \in K \} \},$$

and if $P_K[x]$ consists of only one point, denote this point by $P_K(x)$. $P_K(x)$ will be called the *maximizing function* of K. For any subset H of R^n let H^0 be the relative interior of H, and let $\overline{H}$ be the closure of H. In addition, let N_n be the standard normal probability distribution on R^n (when no confusion may result we shall simply denote it by N), and for every $n \times n$ positive definite matrix Σ let $N(\Sigma)$ be the normal probability distribution in R^n (possibly degenerated) with mean zero and covariance matrix Σ.

LEMMA 1 (Hart). *Lt $\mu_1, \mu_2, \ldots, \mu_n$ be nonzero measures in NA^+ which belong to $L_2(\mu)$, where $\mu \in NA^1$, and let g be a market function in MF_n. Then the μ-asymptotic value of $v = g \circ (\mu_1, \mu_2, \ldots, \mu_n)$ exists and $\varphi_\mu v = \sum_{i=1}^n \bar{p}_i \mu_i$, where $(\bar{p}_1, \bar{p}_2, \ldots, \bar{p}_n) = \bar{p} = \int_{R^n} P(z)\, dN(\Sigma)(z)$, where P is the maximizing function of $\Delta g(\lambda(I))$ (where $\lambda = (\mu_1, \mu_2, \ldots, \mu_n)$), and $\Sigma = (\sigma_{ij})$, where*

$$\sigma_{ij} = \int_I \left(\mu_i(I) 1_I - \frac{d\mu_i}{d\mu} \right) \left(\mu_j(I) 1_I - \frac{d\mu_j}{d\mu} \right) d\mu \quad \text{for all } 1 \leqslant i, j \leqslant n.$$

PROOF. The proof is given in the proof of [3, Theorem 9.2]. ■

We will need the following reformulation of Hart's theorem:

LEMMA 2. *Let $\mu_1, \mu_2, \ldots, \mu_n$ be mutually singular measures in NA^1 which belong to $L_2(\mu)$, where $\mu \in NA^1$. Let $g \in MF_n$ and denote $a_i = \int_I (d\mu_i/d\mu)^2\, d\mu$ for all $1 \leqslant i \leqslant n$.*

Then

$$\varphi_\mu v = \sum_{i=1}^{n} \beta_i \mu_i, \quad \textit{where}$$

$$(\beta_1, \beta_2, \dots, \beta_n) = \beta = \int_{R^n} P\left(\sqrt{a_1}\,x_1, \sqrt{a_2}\,x_2, \dots, \sqrt{a_n}\,x_n\right) dN_n(x),$$

where P *is the maximizing function of* $\Delta g(e)$ (*where* $e = (1, 1, \dots, 1)$).

PROOF. By Lemma 1 the μ-asymptotic value of v exists and $\varphi_\mu v = \sum_{i=1}^{n} \beta_i \mu_i$, where $\beta = \int_{R^n} P(z)\, dN(\Sigma_1)(z)$, for

$$\Sigma_1 = \begin{bmatrix} a_1 - 1 & -1 & \cdots & -1 \\ -1 & a_2 - 1 & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & a_n - 1 \end{bmatrix}.$$

Let $\bar{a} = \sum_{i=1}^{n}(1/a_i)$, then it can be easily verified that Σ_1 is invertible iff $\bar{a} \neq 1$. We divide the proof into two cases: $\bar{a} \neq 1$ and $\bar{a} = 1$.

The case $\bar{a} \neq 1$. Denote

$$\Sigma_2 = \begin{pmatrix} \Sigma_1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{and} \quad \overline{N}_2 = N(\Sigma_2).$$

Then

$$\beta = \int_{R^{n+1}} P(z_1, z_2, \dots, z_n)\, d\overline{N}_2(z_1, z_2, \dots, z_n, z_{n+1}).$$

For $1 \leqslant i \leqslant n$ let $u_i = z_i + z_{n+1}$, and let

$$u_{n+1} = c_1 z_1 + c_2 z_2 + \cdots + c_n z_n + z_{n+1},$$

where $(c_1, c_2, \dots, c_n)' = c$ is the unique solution of $\Sigma_1 c = -e'$, where $e = (1, 1, \dots, 1)$.

Now, it is obvious that

$$\int_{R^{n+1}} u_i u_j\, d\overline{N}_2 = 0 \quad \text{for all } 1 \leqslant i < j \leqslant n + 1;$$

$$\int_{R^{n+1}} u_j^2\, d\overline{N}_2 = a_j \quad \text{for all } 1 \leqslant j \leqslant n;$$

$$\int_{R^{n+1}} u_{n+1}^2\, d\overline{N}_2 = d > 0.$$

Let

$$\Sigma_3 = \begin{bmatrix} a_1 & & & & \\ & a_2 & & & \\ & & \ddots & & \\ & & & a_n & \\ & & & & d \end{bmatrix}.$$

then by change of variables formula,

$$\beta = \int_{R^{n+1}} P(u_1 - z_{n+1}, \ldots, u_n - z_{n+1})\, d\overline{N}_3(u_1, \ldots, u_{n+1}),$$

where $\overline{N}_3 = N(\Sigma_3)$.

Since for every real numbers $\alpha > 0$ and γ, $P(\alpha x + \gamma e) = P(x)$ for almost all x, we have:

$$\beta = \int_{R^{n+1}} P(u)\, d\overline{N}_3(u).$$

Hence

$$\beta = \int_{R^{n+1}} P(u)\, d\overline{N}_4(u),$$

where $\overline{N}_4 = N(\Sigma_4)$ and Σ_4 is the $n \times n$ matrix obtained from Σ_3 by eliminating the last row and the last column. Now by change of variables, $x_i = 1 \setminus \sqrt{a_i}\, u_i$ for all $1 \leqslant i \leqslant n$, we get the result:

$$\beta = \int_{R^n} P\left(\sqrt{a_1}\, x_1, \sqrt{a_2}\, x_2, \ldots, \sqrt{a_n}\, x_n\right) dN_n(x).$$

The case $\bar{a} = 1$. Let Q_1 be the $(n-1) \times (n-1)$ matrix

$$Q_1 = \begin{bmatrix} a_1 - 1 & -1 & \cdots & -1 \\ -1 & a_2 - 1 & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & a_{n-1} - 1 \end{bmatrix},$$

and let $M_1 = N(Q_1)$. Then

$$\beta = \int_{R^{n-1}} P\left(t_1, t_2, \ldots, t_{n-1}, -\frac{a_n}{a_1} t_1 - \frac{a_n}{a_2} t_2 - \cdots - \frac{a_n}{a_{n-1}} t_{n-1}\right) dM_1(t).$$

Let $M_2 = N(Q_2)$, where

$$Q_2 = \begin{pmatrix} Q_2 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then

$$(1.1)\quad \beta = \int_{R^{n-1}} P\left(t_1, t_2, \ldots, t_{n-1}, -\frac{a_n}{a_1} t_1 - \frac{a_n}{a_2} t_2 - \cdots - \frac{a_n}{a_{n-1}} t_{n-1}\right) dM_2(t).$$

By change of variables $u_i = t_i + t_n$ for $1 \leqslant i \leqslant n - 1$, and

$$u_n = -\frac{a_n}{a_1} t_1 - \frac{a_n}{a_2} t_2 - \cdots - \frac{a_n}{a_{n-1}} t_{n-1} + t_n,$$

we get from (1.1) that

$$\beta = \int_{R^n} P(u_1 - t_1, \ldots, u_n - t_n)\, dM_3(u),$$

where $M_3 = N(Q_3)$, and

$$Q_3 = \begin{bmatrix} a_1 & & & \\ & a_2 & & \\ & & \ddots & \\ & & & a_n \end{bmatrix},$$

and hence the result.

We need additional notations in order to state our main theorem. Set $P_n = \{\alpha \in R_+^n: \sum_{i=1}^n \alpha_i = 1\}$. For every $\mu_1, \mu_2, \dots, \mu_n \in NA^1$ and for every $\alpha \in R_+^n$ set $\mu_\alpha = \sum_{i=1}^n \alpha_i \mu_i$. Also, for $n \geqslant 3$ denote by K_n the set of all $\beta \in P_n^0$ such that $\sum_{j=1}^l \beta_{i_j} > 2^{-n+l}$ for all $1 \leqslant l \leqslant n-1$ and all $1 \leqslant i_1 < i_2 < \cdots < i_l \leqslant n$.

Main Theorem. *Let $v = \min(\mu_1, \mu_2, \dots, \mu_n)$, where $\mu_1, \mu_2, \dots, \mu_n$, $n \geqslant 3$ are mutually singular NA^1-measures. Then $A = B = C$, where*

$$A = \left\{\varphi_\mu v: \mu \in NA^1 \text{ and } \mu_i \in L_2(\mu)\ \forall 1 \leqslant i \leqslant n\right\};$$

$$B = \left\{\varphi_{\mu_\alpha} v: \alpha \in P_n^0\right\};$$

$$C = \left\{\sum_{i=1}^n \beta_i \mu_i: \beta = (\beta_1, \beta_2, \dots, \beta_n) \in K_n\right\}.$$

The proof of Main Theorem is given in the next section. An obvious (but relevant) remark is that, for $n = 2$, A $= \{\frac{1}{2}\mu_1 + \frac{1}{2}\mu_2\}$.

2. The proof.

Proof of $A = B$. Obviously $B \subseteq A$. As for the inclusion $A \subseteq B$, consider a probability measure $\mu \in NA^1$ such that $\mu_i \in L_2(\mu)$ for all $1 \leqslant i \leqslant n$. Denote $\bar{a} = \sum_{i=1}^n (1/\alpha_i)$, where, for $1 \leqslant i \leqslant n$, $a_i = \int_I (d\mu_i/d\mu)^2\, d\mu$. Denote

$$\alpha = \left(\frac{1}{\bar{a}a_1}, \frac{1}{\bar{a}a_2}, \dots, \frac{1}{\bar{a}a_n}\right).$$

It is easily verified that for all $1 \leqslant i \leqslant n$

$$\bar{a}\int_I \left(\frac{d\mu_i}{d\mu}\right)^2 d\mu = \int_I \left(\frac{d\mu_i}{d\mu_\alpha}\right)^2 d\mu_\alpha$$

and hence, by Lemma 2, $\varphi_{\mu_\alpha} v = \varphi_\mu v$. ∎

Proof of $B = C$. Denote $\varphi_\alpha = \varphi_{\mu_\alpha}$. Let $X_1, X_2, \dots, X_n$ be i.i.d. normal variables with mean 0 and standard deviation 1. By Lemma 2, for all $\alpha \in P_n^0$

$$\varphi_\alpha v = \sum_{i=1}^n \beta_i(\alpha)\mu_i, \tag{2.1}$$

where for every $1 \leqslant i \leqslant n$

$$\beta_i(\alpha) = \operatorname{Prob}\left(\frac{1}{\sqrt{\alpha_i}} X_i > \max_{i \neq j} \frac{1}{\sqrt{\alpha_j}} X_j \right). \tag{2.2}$$

Since X_i is a symmetric random variable for all $1 \leqslant i \leqslant n$, and since $\operatorname{Prob}(X_i = X_j) = 0$ for all $i \neq j$, we have

$$\beta_i(\alpha) = \operatorname{Prob}\left(\frac{1}{\sqrt{\alpha_i}} X_i \leqslant \min_{j=1}^{n} \frac{1}{\sqrt{\alpha_j}} X_j \right). \tag{2.3}$$

Denote $\beta(\alpha) = (\beta_1(\alpha), \beta_2(\alpha), \ldots, \beta_n(\alpha))$ and let M_n be the range of the function β. That is $M_n = \{\beta(\alpha)\colon \alpha \in P_n^0\}$. By (2.1), in order to prove that $B = C$ it suffices to prove that $M_n = K_n$. Obviously $M_n \subseteq P_n$. We now show that

$$M_n \subseteq K_n. \tag{2.4}$$

By symmetry arguments it suffices to prove that for every $\alpha \subsetneq P_n^0$ and for every $1 \leqslant l \leqslant n-1$

$$\sum_{i=1}^{l} \beta_i(\alpha) > 2^{-n+l}. \tag{2.5}$$

Let then $\alpha \in P_n^0$. Denote $Y_i = X_i / \sqrt{\alpha_i}$. Then (2.5) is equivalent to

$$\operatorname{Prob}\left(\min_{1 \leqslant i \leqslant l} Y_i < \min_{l < j \leqslant n} Y_j \right) > 2^{-n+l}. \tag{2.6}$$

For $l = n-1$ we have:

$$\operatorname{Prob}\left(\min_{1 \leqslant i \leqslant n-1} Y_i < Y_n \right) \geqslant \operatorname{Prob}(Y_1 < Y_n) + \operatorname{Prob}(Y_2 < Y_n < Y_1) \geqslant 2^{-1} + \epsilon > 2^{-1}.$$

Therefore (2.6) holds (recall that $n \geqslant 3$). Assume now that $1 \leqslant l \leqslant n-2$. Then

$$\operatorname{Prob}\left(\min_{1 \leqslant i \leqslant l} Y_i < \min_{l < j \leqslant n} Y_j \right) \geqslant \operatorname{Prob}\left(Y_1 < \min_{l < j \leqslant n} Y_j \right).$$

Thus it suffices to show that

$$\operatorname{Prob}\left(Y_1 < \min_{l < j \leqslant n} Y_j \right) > 2^{-n+l}. \tag{2.7}$$

For $1 \leqslant l \leqslant n-2$ define

$$f_l(x) = \frac{1}{2} \prod_{i=l}^{n} x_i + \frac{1}{2} \prod_{i=l}^{n} (1 - x_i), \qquad x \in \left[0, \tfrac{1}{2}\right]^{n-l+1}. \tag{2.8}$$

For every $x \neq \frac{1}{2}e$ (where $e = (1, 1, \ldots, 1)$), $\partial f / \partial x_i < 0$ for every $l \leqslant i \leqslant n$ and the minimal value of f is obtained at $x = \frac{1}{2}e$. Therefore f is decreasing and $f(x) > 2^{-n+l-1}$ for all $x \neq \frac{1}{2}e$.

For $2 \leqslant i \leqslant n$ set $Z_i(y)$
and, for all y, $\operatorname{Prob}(-|y| <$
for the conditional probabili

$\operatorname{Prob}\left(Y_1 < \min_{i > l} Y_i \,|\, |Y_1| \right)$

which proves (2.7).
We now show that

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Theorem 12.7.9]) to show th

CLAIM. *Let*

$F(a) = \operatorname{Prob}\left(X\right.$

Then F is decreasing on R_+^{n-}
PROOF OF CLAIM. Let b
least one j. Denote $Y_1 = X_1$

(2.10) $\operatorname{Prob}\left(Y\right.$

Indeed, denote $\bar{Z}_j(y) = \text{Pro}$
Y_j), $j \geqslant 2$). Then for every
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with probability one:

$\operatorname{Prob}(Y_1 < \epsilon_j Y_j \; \forall j$

which implies (2.10). ∎
We are now able to show
$1 \leqslant i \leqslant n$ satisfying:

Without loss of generality v

For $2 \leqslant i \leqslant n$ set $Z_i(y) = \text{Prob}(|y| < Y_i)$. Then with probability 1, $Z_i(Y_i) < \frac{1}{2}$, and, for all y, $\text{Prob}(-|y| < Y_i) = 1 - Z_i(y)$ and thus the following inequality holds for the conditional probability:

$$\text{Prob}\Big(Y_1 < \min_{i > l} Y_i \,|\, |Y_1|\Big) = \frac{1}{2} \prod_{i=l+1}^{n} Z_i(Y_i) + \frac{1}{2} \prod_{i=l+1}^{n} (1 - Z_i(Y_i)) > 2^{-n+l},$$

which proves (2.7).

We now show that

$$(2.9) \qquad M_n \text{ is open in } K_n.$$

As β is a continuous function it suffices by The Invariance of Domain Theorem (see [4, Theorem 12.7.9]) to show that β is one to one. We first prove the following claim:

CLAIM. *Let*

$$F(a) = \text{Prob}\Big(X_1 < \min_{j > 1} a_j X_j\Big), \qquad a = (a_2, \ldots, a_n) \in R_+^{n-1}.$$

Then F is decreasing on R_+^{n-1}.

PROOF OF CLAIM. Let $b < a$. Then $b_j = \epsilon_j a_j$, where $0 < \epsilon_j \leqslant 1$ and $\epsilon_j < 1$ for at least one j. Denote $Y_1 = X_1$ and, for $j \geqslant 2$, $Y_j = a_j X_j$. We have to show that

$$(2.10) \qquad \text{Prob}\Big(Y_1 < \min_{j \geqslant 2} \epsilon_j Y_j\Big) > \text{Prob}\Big(Y_1 < \min_{j \geqslant 2} Y_j\Big).$$

Indeed, denote $\overline{Z}_j(y) = \text{Prob}(|y| < \epsilon_j Y_j)$, $j \geqslant 2$ (and recall that $Z_j(y) = \text{Prob}(|y| < Y_j)$, $j \geqslant 2$). Then for every $y \neq 0$, $0 \leqslant \overline{Z}_j(y) \leqslant Z_j(y) < \frac{1}{2}$, and for at least one value of j, $\overline{Z}_j(y) < Z_j(y)$. As the function f_2 defined in (2.8) is decreasing we have that with probability one:

$$\begin{aligned}
\text{Prob}\big(Y_1 < \epsilon_j Y_j \; \forall j \geqslant 2 \,|\, |Y_1|\big) &= \frac{1}{2} \prod_{j=2}^{n} \overline{Z}_j(Y_1) + \frac{1}{2} \prod_{j=2}^{n} \big(1 - \overline{Z}_j(Y_1)\big) \\
&> \frac{1}{2} \prod_{j=2}^{n} Z_j(Y_1) + \frac{1}{2} \prod_{j=2}^{n} \big(1 - Z_j(Y_1)\big) \\
&= \text{Prob}\big(Y_1 < Y_j \; \forall j \geqslant 2 \,|\, |Y_1|\big),
\end{aligned}$$

which implies (2.10). ∎

We are now able to show that β is one to one. Indeed, let $\alpha \neq \overline{\alpha} \in P_n^0$. There exists $1 \leqslant i \leqslant n$ satisfying:

$$\frac{\alpha_i}{\overline{\alpha}_i} = \max\left\{ \frac{\alpha_j}{\overline{\alpha}_j} : 1 \leqslant j \leqslant n \right\}.$$

Without loss of generality we may assume $i = 1$. Therefore

$$\frac{\alpha_j}{\alpha_1} \geqslant \frac{\overline{\alpha}_j}{\overline{\alpha}_1} \quad \forall j \geqslant 2,$$

and since $\alpha \neq \bar{\alpha}$ we have

$$\left(\frac{\alpha_2}{\alpha_1}, \dots, \frac{\alpha_n}{\alpha_1}\right) < \left(\frac{\bar{\alpha}_2}{\bar{\alpha}_1}, \dots, \frac{\bar{\alpha}_n}{\bar{\alpha}_1}\right).$$

Thus by the above claim, $\beta_1(\alpha) > \beta_1(\bar{\alpha})$, which implies $\beta(\alpha) \neq \beta(\bar{\alpha})$. This completes the proof that M_n is open in K_n.

We now prove that

$$K_n \setminus M_n \text{ is open in } K_n. \tag{2.11}$$

By (2.2) one can easily deduce the following boundary property of β:

Let (α^m) be a sequence of points in P_n^0 which converges to $\alpha \in P_n \setminus P_n^0$, and suppose $\beta(\alpha^m) \to u$. Then $u \in \bar{K}_n \setminus K_n$, where $\bar{K}_n$ denotes the closure of K_n.

We proceed to prove (2.11). Let $u \in K_n \setminus M_n$, and suppose there is no open ball S in K_n such that $u \in S \subseteq K_n \setminus M_n$. Then there exists a sequence (u^m) in M_n such that $u^m \to u$. For every $m \geqslant 1$ let $\beta(\alpha^m) = u^m$, where $\alpha^m \in P_n^0$. As P^n is compact, we may assume without loss of generality that $\alpha^m \to \alpha \in P_n$. If $\alpha \in P_n^0$, then by the continuity of β, $\beta(\alpha) = u$, which contradicts the assumption that $u \in K_n \setminus M_n$. If $\alpha \in P_n \setminus P_n^0$, then by the boundary property of β, $u \in \bar{K}_n \setminus M_n$, which contradicts the assumption that $u \in K_n$. Therefore $K_n \setminus M_n$ is open in K_n.

Since K_n is connected topological space (as it is convex) and since $M_n \neq \varnothing$, then (2.9) and (2.11) imply $M_n = K_n$. ∎

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JOHN

New upper bounds are gi
employ subgradient informati
explicit moment information
previous bounds with differen

1. **Introduction.** Evaluat
requirement in utility theory
ming (see, for example, Dem
the expectation of some func
We assume that this functi
function and the underlying
bounds on this expectation a
ent and conjugate function.
not easily computable and
extremal values are not avail

The most basic bound on
which requires knowing only
following Edmundson [9], ga
This bound again requires fi
rectangular domain of sto
Hochman [2] extended and
information of the expected
its mean. Gassmann and Zi
independence of n-dimens
extension of the Edmundso
the joint expectations of the

The general process of o
described in Birge and Wets
and Wets [5]. Explicit solut
Nedeva [10] and Cipra [6
expectation of convex fun
moment information.

Our results differ from
information but instead inf

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IAOR 1973 subject classification.
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