WEIGHTED VALUES AND THE CORE

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1. Introduction. Let G be the set of all TU games in coalitional form on the finite set N of players. Let $w^N \in Int(\Delta^N)$–the relative interior of the unit simplex Δ^N in R^N. The *weighted Shapley value* $\phi(w^N) : G \longrightarrow R^N$ is defined in Shapley [9] as the unique linear operator satisfying $\phi_i^{u_S}(w^N) = w_i^N/w^N(S)$ for all $i \in S$ and $\phi_i^{u_S}(w^N) = 0$ for all $i \notin S$, where $u_S(T) = 1$ if $T \supseteq S$ and $u_S(T) = 0$ if $T \not\supseteq S$.

Generalized weighted Shapley values (weighted values for short) extend weighted Shapley values to weight systems that include hierarchies of zero weights. Generalized values were defined already in Shapley [10] and were axiomatized recently by Kalai–Samet [4]. They are defined as follows:

A *weight system* on N is a vector $w = (w^S)_{S \subseteq N}$ in $\prod_{S \subseteq N} \Delta^S$ satisfying:

$$w_i^S = w_i^T/w^T(S)$$

whenever $i \in S \subseteq T$ and $w^T(S) > 0$.

The set of all weight systems on N will be denoted by $\mathcal{W}$. For each $w \in \mathcal{W}$ the *weighted value* $\varphi(w) : G \longrightarrow R^N$ is the unique linear operator satisfying $\varphi_i^{u_S}(w) = w_i^S$ whenever $i \in S$ and $\varphi_i^{u_S}(w) = 0$ whenever $i \notin S$. It is easily verified (e.g., see [4]) that the weighted Shapley value $\phi(w^N)$ coincides with the weighted value $\varphi(w)$ whenever $w \in \mathcal{W}$ and $w_i^N > 0$ for all $i \in N$. Various properties of weighted values, including an axiomatic characterization can be found in [10], [4] and [1].

Let $v \in G$. In the main theorem of this paper we prove that any point in the core $C(v)$ of v is the weighted value $\varphi^v(w)$ of v with respect to some weight system w. That is,

$$C(v) \subseteq \varphi^v(\mathcal{W}). \tag{1}$$

Weighted values are special type of random order values (see [4] and [8] for a proof) which were defined by Weber [12] as follows:

Let $\mathcal{R}$ be the set of all $n!$ orders of N (where $n = \#N$), and let $\Delta(\mathcal{R})$ be the set of all probability measures on $\mathcal{R}$. For each $P \in \mathcal{R}$ the *random order value* $\psi(P) : G \longrightarrow R^N$ is defined in Weber [12] as follows: For all $v \in G$ and for all $i \in N$

$$\psi_i^v(P) = \sum_{S \subseteq N \setminus \{i\}} [v(S \cup \{i\}) - v(S)] P(Q^i = S),$$

where for all $r \in \mathcal{R}$, Q_r^i denotes the set of all players preceeding i in the order r.

The set of all weighted values is a closed, non-convex subset of the closed convex set of all random order values. It is the closure of the set of all weighted Shapley values. It is proved in this paper that the set of weighted values is a "thin" subset of the set of random order values. It is homeomorphic to an $n-1$ dimensional ball, while the set of all random order values has a dimension $2^{n-1}(n-2)+1$. For each $v \in G$, the set $\psi^v(\Delta(\mathcal{R})) = \{\psi^v(P) : P \in \Delta(\mathcal{R})\}$ of all random order values of v was discussed in [12], and it was proved there that:

$$C(v) \subseteq \psi^v(\Delta(\mathcal{R})), \tag{2}$$

(1) is a strict improvement over (2), not only because of the difference in dimensions mentioned above, but also because it can be shown that in general, for $v \in G$, the set $\varphi^v(\mathcal{W})$ is not convex and therefore, strictly contained in the set $\psi^v(\Delta(\mathcal{R}))$.

Stronger relations between cores and weighted values are obtained for convex games. For example, it is proved that v is convex iff $\varphi^v(\mathcal{W}) = C(v)$, and that v is strictly convex iff φ^v maps $\mathcal{W}$ homeomorphically onto $C(v)$.

The set $\mathcal{W}$ has been used (under different titles) in the theory of non–cooperative solutions. e.g., see [5], [6] and [7]. The main theorem of [6] states that $\mathcal{W}$ is homeomorphic to a ball. Here we supply an independent proof to this result by showing that $\mathcal{W}$ is homeomorphic to a core of any strictly convex game.

2. Preliminaries. Let N be a finite set with $n \geq 1$ elements which we call *players*. The set of all nonempty subsets of N will be denoted by $\mathcal{M}$. For $S \subseteq N$ we denote $N \backslash S$ by S^c and for $i \in N$ we write $S \cup i$ for $S \cup \{i\}$ and $S \backslash i$ for $S \backslash \{i\}$. For $S \in \mathcal{M}$ we write x^S, y^S, etc... for elements in R^S. If $S \subset T$ then x^T_S is the projection of x^T on R^S. For finite set X Δ^X is the unit simplex in R^X. We denote by $Int(\Delta^X)$ the relative interior of Δ^X. A *game* on N is a function $v : 2^N \longrightarrow R$ with $v(\emptyset) = 0$. The space of all games on N will be denoted by G. For $x \in R^N$ and $S \in \mathcal{M}$ we write $x(S)$ for $\sum_{i \in S} x_i$; for $S = \emptyset$, $x(S) = 0$. The *core* $C(v)$ of v is the set of all $x \in R^N$ for which $x(N) = v(N)$ and $x(S) \geq v(S)$ for all $S \subseteq N$.

$\mathcal{R}$ denote the set of all $n!$ (complete) orders on N; for $r \in \mathcal{R}$ we shall write the relation $<_r$. The set of all players preceding player i in r is denoted by Q^i_r. The *contribution vector* $c^v(r) \in R^N$ for the game v and the order r is defined by $c^v_i(r) = v(Q^i_r \cup i) - v(Q^i_r)$ for each $i \in N$. Let P be a probability measure over all $n!$ orders of N. i.e., $P \in \Delta^{\mathcal{R}}$. The *P–random order value* $\psi(P) : G \longrightarrow R^N$ is defined by $\psi^v(P) = E_P(c^v)$ for each $v \in G$, where E_P is the expectation with respect to P.

A game v is *convex* if for every $T \subset S$ and for every $i \notin S$, $v(T \cup i) - v(T) \leq v(S \cup i) - v(S)$. v is *strictly convex* if the last inequalities are always strict. The structure of the core of a convex game is described in [11] (see also [2]):

Let Σ be the set of all ordered partitions of N. Elements of Σ are of the form $\sigma = (S_1, \ldots, S_k)$, where $k \geq 1$, $\cup^k_{h=1} S_h = N$, $S_i \cap S_j = \emptyset$ and $S_i \neq \emptyset$.

For each $\sigma = (S_1, \ldots, S_k)$ in Σ let F^v_σ be the set of all $x \in C(v)$ such that $x(\cup_{j=1}^h S_j) = v(\cup_{j=1}^h S_j)$ for all $1 \leq h \leq k$. When v is convex, F^v_σ is a nonempty face of $C(v)$ of dimension $n - k$ at most. In particular for σ with $k = n$, F^v_σ consists of one point which is a contribution vector, and $C(v)$ is the convex hull of all distribution vectors. When v is strictly convex then F^v_σ is of dimension $n - k$ and all faces F^v_σ are distinct. Finally, if the core of v contains all the contribution vectors then v is convex; if in addition all the contribution vectors are distinct then the game is strictly convex.

3. Weighted values. A *weight vector* is an element of $Int(\Delta^N)$. For a weight vector w^N the it weighted Shaley value $\varphi(w^N) : G \to R^N)$ is defined in Shapley [9] as the unique linear operator satisfying for each unanimity game u_S:

$$(*) \qquad \varphi_i^{u_S}(w^N) = w_i^N / w^n(S) \qquad \text{for } i \in S,$$

and

$$\varphi_i^{u_S}(w^N) = 0 \qquad \text{for } i \notin S.$$

Since $\varphi(w^N)$ is linear, (*) defines $\varphi^v(w^N)$ for all $v \in G$.

Owen in [8] has shown that the wighted Shapley values are random order values. For a wieght vector w^N we define P_{w^N} in $\Delta^{\mathcal{R}}$ as follows. Let T_i for $i \in N$ be independent random variables distributed over $[0,1]$ such that for every $0 \leq t \leq 1$ and $i \in N$,

$$Prob(T_i \leq t) = t^{w_i^N}.$$

For $r \in \mathcal{R}$ define:

$$P_{w^N}(r) = P(T_i < T_j, \forall i <_r jinN).$$

For every $v \in G$, $\varphi^v(w^N) = \psi^v(P_{w^N})$. Note that higher pobability is assigned by $P_(w^N)$ to orders in which the 'heavy' players arrive later than the 'lighter' players.

Observe that if $w_i^N = 1\backslash n$ for all $i \in N$ then $\varphi(w^N)$ is the ordinary Shapley value.

We generalize now the notion of a weight vector to enable some players to have zero weight. The values corespondig to the genaralized weights will be called *weghited values.* These values were defined by Shapley in [10] and axiomatized by Kalai and Samet in [4]. The following consideration will lead us to this generalization. We can not use directly (*) to define a value in this case since for S which contains only zero weight players (*) is not defined. We need therefore to assign secondary weights to the zero weight players, which will be used only when no positive weight players are present. These new wights may be zero for some of the players and we have to assign also weights to these doubly zero weighted players, and so on. We are natually lead to the following definition. A *generalized weight vector* is a $2k$-tuple $(S_1, \dots, S_k, w^{S_1}, \dots, w^{S_k})$ such that $(S_1, \dots, S_k) \in \Sigma$ and $w^{S_h} \in Int(\Delta^{S_j})$ for $h = 1, \dots, k$. The interpretation of the generalized weights is as follows. The plyers in S_k are the 'heviest', they are the non-zero players with weights given by w^{S_k}, while the rest of the players are zero players. Among the zero players the 'heviest' are the members of S_{k-1} with weights $w^{S_{k-1}}$. All players in $\cup_{h \leq k-2} S_h$ are zero weight players relative to plyers in S_{k-1}, etc. Given the generalied weight vector $(S_1, \dots, S_k, w^{S_1}, \dots, w^{S_k})$ we can find the relative weights of player in each coalition S. For a given S let $m = \max\{h | S_j \cap S \neq \emptyset\}$. Define now w^S by: $w_i^S = w_i^{S_m} / w^{S_m}(S \cap S_m)$ for $i \in S \cap S_m$ and $w_i^S = 0$ for $i \in S \setminus S_m$. Thus $S \cap S_m$ consists of all the 'heaviest players in S. The relative weight of these plyers is determined by their weights in S_m. The rest of the players have zero weight in S.

Denote now $w = (w^S)_{S\in\mathcal{M}}$. w is a vector in $\prod_{S\in\mathcal{M}} \Delta^S$ and it is easy to verify that it satisfies:

$$w_i^S = w_i^T / w^T(S) \tag{(**)}$$

for each $i \in S \subseteq T$ for which $w^T(S) > 0$.

A vector $w \in \prod_{S\in\mathcal{M}} \Delta^S$ which satisfies (**) is called a *weight system.* The set of all weight systems is denoted by $\mathcal{W}$. we saw that each generalized weight vector coresponde to a weight system. It is easy to see that this correspondence is one to one. We show now that it is also onto $\mathcal{W}$. Let $w \in \mathcal{W}$ and define $\sigma(w) \in \Sigma$ as follows. Let $T_1 = \{i \in N : w_i^N > 0\}$, and for $h \geq 2$ we define T_h to be the set

$$\{i \in (\cup_{j=1}^{h-1} T_j)^c : w_i^{(\cup_{j=1}^{h-1} T_j)^c} > 0\}$$

when this set is not empty.

Clearly $(T_1, \ldots, T_k)$ is an ordered partition. Now for each $1 \leq h \leq k$, let $S_h = T_{k-h+1}$ and $\sigma(w) = (S_1, \ldots, S_k)$. It is easy to see now that w is the weight system that correspnds to $(S_1, \ldots, S_k, w^{S_1}, \ldots, w^{S_k})$.

For a given $w \in \mathcal{W}$ we define now the *weighted value* $\varphi(w)$ as the linear function $\varphi(w) : G \to R^N$ which is defined for each unanamity game u_S by:

$$\varphi^{u_S}(w) = w_i^S.$$

It can be easily verified that when $w^N > 0$ the weighted value $\varphi(w)$ coincides with the weighted Shapley value $\varphi(w^N)$.

Weighted values are also random order values. The probability distribution P_w in $Delta^{\mathcal{R}}$ which defines $\phi(w)$ is described as follows.

We say that an order r of N is *consistent* with $\sigma = (S_1, S_2, \ldots, S_k)$ in Σ if for each $1 \leq h \leq k-1$ each player in S_h precedes each player in S_{h+1}. For each $w \in \mathcal{W}$ we define a probability measure P_w over all orders of N, whose support is the set of all orders which are consistent with $\sigma(w)$.

Now let $(S_1, \ldots, S_k, w^{S_i}, \ldots, w^{S_k})$ be the generalized weight vector which corresponds to w. Since $w^{S_h} \in Int(\Delta^{S_h})$ we can define a probability distribution $P_{w^{S_j}}$ on the orders of S_j in the same way P_{w^N} was defined above. We define now

$$P_w(r) = \prod_{h=1}^{k} P_{w^{S_k}}(r_h),$$

where r_h is the order on S_h induced by r. P_w is the probabity distribution for which $\phi^v(w) = \psi^v(P_w)$ for each game v.

Notice that in all the orders which are consistent with $\sigma(w)$, the non–zero players–those in S_k–are preceeded by all other players, all the players in S_{k-1} preceede the players in $\cup_{h \leq k-2} S_h$ etc. This is in accordance with the interpretation of ‘heavy’ weight players which was given before.

Weight syatems can be used to define a different family of values which we call *dual weighted values.* For $w \in \mathcal{W}$ the dual weighted value $\varphi^*(w)$ is defined by $(\varphi^*)^v(w) = \varphi^{v^*}(w)$ where v^* is the dual game of v which satisfies for each S: $v^*(S) = v(N) - v(S)$. Dual weight values were used by Shapley in [?] for cost allocation problems. Axiomatization of $\varphi^*(w)$ for a fixed w was given in [?], while the axiomatization of the whole family of the dual weighted values is found in [4]. Dual weighted values are also random order values. The probability distribution

P^*_w which determines $\varphi^*(w)$ assigns to each order r the probability $P_w(r^*)$ where P_w is the probability distribution which defines $\varphi(w)$ and r^* is reverse of the order r.

3. Main Results.

The main theorem is the following:

Theorem A. *For every game v, each element in the core of v is the weighted value of v with respect to some weighted system on N. That is, $C(v) \subseteq \varphi^v(\mathcal{W})$.* $\square$

Stronger relations between the core and the weighted values exist when the game is convex.

The inclusion stated in Theorem A improves a simpler inclusion relationship that exists between the core and the set of random order values. For a given game $v \in G$, $\psi^v(\Delta(\mathcal{R})) = \{\psi^v(P) : P \in \Delta(\mathcal{R})\}$ is the set of all random order values of v. It was proved by Weber in [12] that:

$$(*) \qquad C(v) \subseteq \psi^v(\Delta(\mathcal{R})).$$

Since weighted values are also random order values $\varphi^v(\mathcal{R}) \subseteq \psi^v(\Delta(\mathcal{R}))$ and Theorem A ilplies (*). Moreover, $\varphi^v(\mathcal{R})$ is not always convex and therefore maybe strictly contained in the set $\psi^v(\Delta(\mathcal{R}))$ and thus Theoren A is a strict omprovment over (*).

Theorem B. *A game v is convex iff $C(v) = \varphi^v(\mathcal{W})$. Moreover, v is strictly convex iff φ^v is a homeomorphisim between $\mathcal{W}$ and $C(v)$; In this case for each $\sigma \in \Sigma$, φ^v maps homeomorphically $\mathcal{W}_\sigma$ onto the relative interior of the face F^v_σ of $C(v)$.* $\square$

A simple corollary of Theorem B is:

Corollary 1. *The set $\mathcal{W}$ of all weight systems on N is homeomorphic to a $n-1$ dimensional ball .* $\square$

In light of this corollary, Theorem A improves upon (*) in another way. The set of all weighted values has the dimension of $\mathcal{W}$ which according to this Corollary is $n-1$. The set of all random order values has a dimension of $2^{n-1}(n-2)+1$. Thus a much "thiner" set of values is required in order to cover the core of each game.

The set $\mathcal{W}$ has been used, under the title 'conditional system' in the theory of non–cooperative solutions. e.g., see [5], [6] and [7]. Condition (*) can be phrased as saying that for each S, w^S is the conditional probability on S which is derived from the probability distributions on supersets of S whenever such derviation is possible. The main theorem of [6] states that $\mathcal{W}$ is homeomorphic to a ball. Here we supply an independent proof to this result by showing that $\mathcal{W}$ is homeomorphic to a core of any strictly convex game.

Convex games can be also characterized by another property of the weighted values.

For every $i \in N$ and for every $w \neq u$ in $\mathcal{W}$ we write $w >_i u$ if $w_i^S \geq u_i^S$ for every S which contains i and $w_{N\setminus i} = u_{N\setminus i}$.

We say that φ^v is *monotonic* if for each i and each $w >_i u$, $\varphi_i^v(w) \geq \varphi_i^v(u)$. φ^v is *strictly monotonic* if the last inequality is strict.

Theorem C. *The game v is (strictly) convex iff φ^v is (strictly) monotonic.* $\square$

4. Proofs. We will need the following lemma:

Lemma 2.

Let $w \in \mathcal{W}$ and let $T \in \mathcal{M}$. Then

$$(P_w)_T = P_{(w_T)}\,,$$

where $(P_w)_T$ is the probability over all orders of T induced by P_w. That is,

$$(P_w)_T(\bar{r}) = P_w\{r : r_T = \bar{r}\}$$

for every order $\bar{r}$ of T, where r_T is the order on T induced by r.

Proof. The maps $w \longrightarrow (P_w)_T$ and $w \longrightarrow P_{(w_T)}$ are continuous, and $\mathcal{W}_{(N)}$ is dense in $\mathcal{W}$. Therefore it suffices to prove the lemma for $w \in \mathcal{W}_{(N)}$.

Let then $w^N > 0$ and let $\bar{r}$ be an order of T. It can be deduced from [4] that:

$$(P_w)_T(\bar{r}) = (\prod_{i \in T} w_i)/A(w),$$

where $w_i = w_i^N$ for all $i \in T$ and

$$A(w) = \prod_{i \in T}(\sum_{h <_{\bar{r}} i} w_i).$$

And also from [4] we get:

$$P_{(w_T)}(\bar{r}) = (\prod_{i \in T} \tilde{w}_i)/A(\tilde{w}),$$

where $\tilde{w}_i = w_i/w^N(T)$ for all $i \in T$. Therefore,

$$(P_w)_T(\bar{r}) = P_{(w_T)}(\bar{r}). \quad \square$$

Proof of Theorem C. For (strictly) convex games v, φ^v has a property stronger than (strict) monotonicity as we state in Lemma 3.

Lemma 3. *Let v be a convex game and let $w >_i u$. Then for each $j \neq i$, $\varphi_j^v(w) \leq \varphi_j^v(u)$. If v is strictly convex then at least for one player j, $\varphi_j^v(w) < \varphi_j^v(u)$.*

Since $\varphi(w)$ is efficient (that is, $\varphi^v(w)(N) = v(N)$), Lemma 3 implies that φ^v is (strictly) monotonic for (strictly) convex games v. To prove Lemma 3 we write $\varphi_j^v(w)$ by grouping orders which have the same set of players preceding j. Thus,

$$\varphi_j^v(w) = \sum_{S \subseteq N \setminus j} (v(S \cup j) - v(S)) P_w(Q^j = S) \tag{3}$$

Let $i \neq j$ then by rearranging (3) we have:

(4)

$$\varphi_j^v(w) = \sum_{S \subseteq N \setminus \{i,j\}} (v(S \cup j) - v(S)) P_w(Q^j = S) + \sum_{S \subseteq N \setminus \{i,j\}} (v(S \cup i \cup j) - v(S \cup i)) P_w(Q^j = S \cup i).$$

Therefore,

$$\varphi_j^v(w) = A + B,$$

Where

$$A = \sum_{S \subseteq N \setminus \{i,j\}} (v(S \cup j) - v(S))(P_w(Q^j = S) + P_w(Q^j = S \cup i)); \tag{5.1}$$

and

$$B = \sum_{S \subseteq N \setminus \{i,j\}} [(v(S \cup i \cup j) - v(S \cup i)) - (v(S \cup j) - v(S))] P_w(Q^j = S \cup i). \tag{5.2}$$

Obviously, Lemma 5 will now follow from the following two claims:

Claim 1. *Let $i \in N$. Then for each $j \neq i$ and for each $S \subseteq N \setminus \{i,j\}$,*

$$P_w(Q^j = S) + P_w(Q^j = S \cup i) = P_{\tilde{w}}(Q^j = S),$$

where $\tilde{w} = w_{N \setminus i}$.

Claim 2. *Let $i \in N$. If $w >_i u$ then for all $j \neq i$ and for all $S \subseteq N \backslash \{i, j\}$,*

$$P_w(Q^j = S \cup i) \leq P_u(Q^j = S \cup i),$$

and at least for one j and S strict inequality holds.

Proof of Claim 1. Obviously,

$$P_w(Q^j = S) + P_w(Q^j = S \cup i) = (P_w)_{N \backslash i}(Q^j = S).$$

Therefore the result follows from the first lemma of this section.

Proof of Claim 2. We separate the proof into two cases.

Case 1. $w^N > 0$ and $u^N > 0$.

Let $j \neq i$ and let $S \subseteq N \backslash \{i, j\}$. Then, it is easily verified that:

$$P_u(Q^j = S \cup i) = \int_0^1 [\prod_{k \in S \cup i} t^{u_k}][\prod_{k \in \bar{S}} (1 - t^{u_k})] u_j t^{u_j - 1} dt, \tag{6}$$

where $u_k = u_k^N$ and $\bar{S} = N \backslash (S \cup \{i, j\})$.

Let $\lambda = (1 - w_i)/(1 - u_i)$. Since $w >_i u$ then $0 < \lambda < 1$, $w_k = \lambda u_k$ for all $k \neq i$ and $w_i = \lambda u_i + 1 - \lambda$. Subtitute $t = s^\lambda$ in (6). Then,

$$P_u(Q^j = S \cup i) = \int_0^1 G(s) ds,$$

where

$$G(s) = [\prod_{k \in S \cup i} s^{\lambda u_k}][\prod_{k \in \bar{S}} (1 - s^{\lambda u_k})] \lambda u_j s^{\lambda u_j - 1}.$$

Hence,

$$P_u(Q^j = S \cup i) > \int_0^1 G(s) s^{1 - \lambda} ds = P_w(Q^j = S \cup i).$$

Case 2. $w^N \not> 0$ or $u^N \not> 0$.

In this case, weak inequalities follow from Case 1 and continuity arguments. It remains to show that there exists $j \neq i$ and $S \subseteq \{i,j\}^c$ for which

$$P_w(Q^j = S \cup i) < P_u(Q^j = S \cup i).$$

Indeed, since $w >_i u$ there exists $j \neq i$ such that $u_i^{\{i,j\}} < w_i^{\{i,j\}}$. Let $S = \{k \in \{i,j\}^c : P_u(k < j) = 1\}$. Then it is easy to verify that strict inequality holds for these j and S.

We have shown that (strict) convexity of v implies (strict) monotonicity of φ^v. Conversely, assume φ^v is monotonic and that v is not convex. Then there exist $i \neq j$ and $S \subseteq \{i,j\}^c$ such that

$$v(S \cup i) - v(S) > v(S \cup j \cup i) - v(S \cup j). \tag{7}$$

Consider w such that $\sigma(w) = (S, \{i,j\}, (S \cup i \cup j)^c)$. Then by (6)

$$\varphi_i^v(w) = w_i^{\{i,j\}}[v(S \cup j \cup i) - v(S \cup j)] + w_j^{\{i,j\}}[v(S \cup i) - v(S)]. \tag{8}$$

Consider u defined by $u^T = w^T$ for each $T \neq \{i,j\}$, $u_i^{\{i,j\}} = w_i^{\{i,j\}} - \varepsilon$ and $u_j^{\{i,j\}} = w_j^{\{i,j\}} + \varepsilon$ for small enough $\varepsilon > 0$. Clearly $w >_i u$ and $\sigma(u) = \sigma(w)$. Thus we can write for u an expression similar to (8). From (7) and (8)

$$\varphi_i^v(w) < \varphi_i^v(u). \tag{9}$$

This contradicts the monotonicity of φ^v.

If v is assumed not to be a strictly convex game then we can garantee only weak inequality in (7) and therefore weak inequality in (9). This is, however, sufficient to contradict strict monotonicity of φ^v. This complete the proof of Theorem C. $\square$

We prove Theorem B through Lemmas 4–6.

Lemma 4. *Let v be a convex game. Then for each $\sigma \in \Sigma$, $\varphi^v(\mathcal{W}_\sigma) \subseteq F^v_\sigma$. Moreover, if v is strictly convex then $\varphi^v(\mathcal{W}_\sigma) \subseteq Int(F^v_\sigma)$–the relative interior of F^v_σ.*

Proof. Let $\sigma = (S_1, \ldots, S_k)$. Define games $v_1, \ldots, v_k$ as follows. v_j is a game on S_j such that for each $S \subseteq S_j$, $v_j(S) = v(S \cup [\cup_{h<j} S_h]) - v(\cup_{h<j} S_h)$. Also for each $w \in \mathcal{W}$ and for each $1 \leq j \leq k$ denote $w_j = w_{S_j}$. It is easy to see that for each $w \in \mathcal{W}_\sigma$ and $1 \leq j \leq k$, $\varphi^v(w)_{S_j} = \varphi^{v_j}(w_j)$. Since $\varphi(w_j)$ is efficient we have:

$$\varphi^v(w)(S_j) = v_j(S_j) = v(\cup_{h\leq j} S_h) - v(\cup_{h<j} S_h) \tag{10}$$

for all $1 \leq j \leq k$, where $\cup_{h<1} S_h = \emptyset$.

Therefore,

$$\varphi^v(w)(\cup_{h\leq j} S_h) = v(\cup_{h\leq j} S_h) \tag{11}$$

for every $1 \leq j \leq k$.

This implies that $\varphi^v(w) \in F^v_\sigma$.

Let v be a strictly convex game. We have to show that for each $S \in \mathcal{M}$ which is different from each of the sets $\cup_{h\leq j} S_h$, $h = 1, \ldots, k$

$$\varphi^v(w)(S) > v(S).$$

Observe that

$$\varphi^v(w) = \sum_{r\in\mathcal{R}_\sigma} c^v(r) P_w(r), \tag{12}$$

where $\mathcal{R}_\sigma$ is the set of all orders which are consistent with σ, and that $P_w(r) > 0$ for all $r \in \mathcal{R}_\sigma$. By [11] $c^v(r)(S) > v(S)$ for every $S \in \mathcal{M}$ which is not an initial segment with respect to r (that is, $S \neq Q^i_r$ for all $i \in N$) and $c^v(r)(S) = v(S)$

for every initial segment S. Now, if S is not one of the sets $\cup_{h\le j}S_h$, $h = 1,\dots,k$ then there exists $r \in \mathcal{R}_\sigma$ such that S is not an initial segment with respect to r. Therefore,

$$\varphi^v(w)(S) = \sum_{r\in\mathcal{R}_\sigma} c^v(r)(S)P_w(r) > v(S)$$

by (12). $\square$

Lemma 5. *Let v be a strictly convex game. Then φ^v is one–to–one.*

Proof. Since relative interiors of different faces of $C(v)$ are disjoint it sufficies to prove that φ^v is 1–1 on each $\mathcal{W}_\sigma$. Suppose that $\varphi^v(w) = \varphi^v(u)$ for $w, u \in \mathcal{W}_\sigma$. Then by the construction of Lemma 4, $\varphi^{v_j}(w_j) = \varphi^{v_j}(u_j)$ for every $1 \le j \le k$. There exists at least one j for which $w_j \ne u_j$. For this j, v_j is strictly convex, $w_j^{S_j} > 0$ and $u_j^{S_j} > 0$. Therefore, it suffices to prove that for every strictly convex game v, φ^v is 1–1 on $\mathcal{W}_{(N)}$.

For each $a \in R^N_{++}$ we denote by $w(a)$ the weighted system defined by $w^S = a_S/a(S)$. We now show that if $a \ne b$, $a \le b$ and there exists j such that $a_j = b_j$ then $\varphi^v_j(w(a)) > \varphi^v_j(w(b))$. Suppose firstly that there exists $i \ne j$ such that $a_i < b_i$ and $a_k = b_k$ for each $k \ne i, j$. Then $w(b) >_i w(a)$ and thus by Lemma 3 $\varphi^v_j(w(a)) > \varphi^v_j(w(b))$. In the general case, one can find a sequence $a^1, a^2, \dots, a^l$ such that $a^1 = a$, $a^l = b$ and a^j and a^{j+1} satisfy the above assumption for a and b.

Now for $w \ne u$ in $\mathcal{W}_{(N)}$ define $\lambda_0 = \min\{\lambda > 0 : \lambda w^N \ge u^N\}$ and let $b = \lambda_0 w^N$. Then $b \ge u^N$, $b \ne u^N$ and there exists j such that $b_j = u^N_j$. Therefore,

$$\varphi^v(w) = \varphi^v(w(b)) \ne \varphi^v(w(u^N)) = \varphi^v(u). \quad \square$$

Lemma 6. *Let v be a strictly convex game. Then $\varphi^v(\mathcal{W}) = C(v)$.*

Proof. In what follows we denote the relative interior of a convex set X by $Int(X)$. Recall that $C(v) = F^v_{(N)}$. Therefore, by Lemma 4 φ^v maps $\mathcal{W}_{(N)}$ into the relative interior of $C(v)$. Since the closure of $\mathcal{W}_{(N)}$ is $\mathcal{W}$ and φ^v is continuous, it suffices to prove that $\varphi^v(\mathcal{W}_{(N)}) = Int(C(v)$. To see this note that $\mathcal{W}_{(N)}$ is homeomorphic to Δ^N and therefore, both $\mathcal{W}_{(N)}$ and $Int(C(v)$ are homeomorphic to R^{n-1}. Since φ^v is continuous and 1–1, we deduce from the Invariance of Domain Theorem (see [3]) that $\varphi^v(\mathcal{W}_{(N)})$ is open in $Int(C(v)$. Let $B = Int(C(v))\backslash\varphi^v(\mathcal{W}_{(N)})$. We now show that B is open in $Int(C(v))$ and therefore must be empty since $Int(C(v))$ is connected. Indeed, if B is not open then there exists $z \in B$ and a sequence (z^m), $m \geq 1$ in $\varphi^v(\mathcal{W}_{(N)})$ such that $z^m \longrightarrow z$. For each m there exists w^m in $\mathcal{W}_{(N)}$ such that $\varphi^v(w^m) = z^m$. Since $\mathcal{W}$ is compact we may assume without loss of generality that $w^m \longrightarrow w \in \mathcal{W}$. Since φ^v is continuous $\varphi^v(w) = z$ and hence $w \notin \mathcal{W}_{(N)}$ and $w \in \mathcal{W}_\sigma$ for some $\sigma \neq (N)$. But then, by Lemma 4, $\varphi^v(w) \in Int(F^v_\sigma)$ and $Int(F^v_\sigma) \cap Int(F^v_{(N)}) = \emptyset$. A contradiction. $\square$

Proof of Theorem B. By Lemmas 5 and 6, for a strictly convex game v, φ^v is a 1–1 continuous map from $\mathcal{W}$ onto $C(v)$ and since $\mathcal{W}$ is compact it is a homeomorphism. By Lemmas 4, 5 and 6, $\varphi^v : \mathcal{W}_\sigma \longrightarrow F^v_\sigma$ is a 1–1 continuous map onto $Int(F^v_\sigma)$ and therefore it is a homeomorphism.

Now let v be a convex game. There exists a sequence $(v^m)_{m=1}^\infty$ of strictly convex games on N such that $v^m(S) \longrightarrow v(S)$ for each $S \subseteq N$. For each m, $\varphi^{v^m}(\mathcal{W}) = Co\{c^{v^m}(r)\}$, were Co stands for "convex hull" and r ranges over all orders of N. Since $c^{v^m}(r) \longrightarrow c^v(r)$ and $\varphi^v(w)$ is continuous in both w and v,

$\varphi^v(\mathcal{W}) = Co\{c^v(r)\} = C(v)$.

Conversely, suppose $\varphi^v(\mathcal{W}) = C(v)$. For each order r of N let $\sigma_r = (\{i_1\}, \{i_2\}, \dots, \{i_n\})$, where $i_1, i_2, \dots, i_n$ is the order of the players according to r. $\mathcal{W}_{\sigma_r}$ consists of a single weighted system w_r and $\varphi^v(w_r) = c^v(r)$. Thus $c^v(r) \in C(v)$ for each order r and therefore v is convex. Moreover, if φ^v is a homeomorphism then all $n!$ vectors $c^v(r)$ are distinct and therefore v is strictly convex. $\square$

Proof of Theorem A. Let v be a game with a nonempty core and let $x \in C(v)$. We have to show that there exists $w \in \mathcal{W}$ for which $\varphi^v(w) = x$. We first transform (using Theorem B) the set of weighted system $\mathcal{W}$ to a core of a fixed strictly convex game U and then use a fixed point argument. Let U be a fixed Strictly convex game such that $U - v$ is also strictly convex. For each $z \in C(U)$ denote $G^v(z) = \varphi^v(w)$, where w is the unique element in $\mathcal{W}$ for which $\varphi^U(w) = z$. We now show that there exists $z \in C(U)$ for which $G^v(z) = x$. Define $g : C(U) \longrightarrow R^N$ as follows:

$$g(z) = x - G^v(z) + z.$$

We proceed to prove that g has a fixed point in $C(U)$. To show this we will show that g maps $C(U)$ into $C(U)$ and then we use the Brouwer fixed point theorem. Indeed, let $z \in C(U)$. We have to show that for each $S \subseteq N$

$$x(S) - G^v(z)(S) + z(S) \geq U(S)$$

or

$$x(S) - v(S) - [G^v(z)(S) - v(S)] + z(S) - U(S) \geq 0.$$

Since $x(S) - v(S) \geq 0$ it suffices to show

$$z(S) - U(S) \geq G^v(z)(S) - v(S). \tag{13}$$

There exists $w \in \mathcal{W}$ such that $\varphi^U(w) = z$ and $G^v(z) = \varphi^v(w)$. Substituting these in (13) gives:

$$\varphi^U(w)(S) - U(S) \geq \varphi^v(w)(S) - v(S)$$

or

$$\varphi^{U-v}(w)(S) \geq (U - v)(S),$$

which is implied by Theorem B since $U - v$ is convex. $\square$

References

1. Hart, S. and Mas–Colell, A., *Potential, Value and Consistancy*, Revised version of DP–1127 of Harvard University (1987).
2. Ichiishi, T., *Game Theory for Economic Analysis*, Academic Press, New York, 1983.
3. Istratescu, I. V., *Fixed Point Theory. Math and its Applications; 7.*, D. Reidel Publishing Company, Holland, 1981.
4. Kalai, E. and Samet, D., *On Weighted Shapley Values*, I. J. Game Theory **16** (1987), 205–222.
5. Mclennan, A., *Consistant Conditional Systems in Noncooperative Game Theory*, Mathematical Sciences Research Institute Working Paper 05217–86 (1987).
6. Mclennan, A., *The Space of Conditional Systems is a Ball*, Preprint, Dept. of Economics University of Minneapolis (1987).
7. Myerson, R. B., *Multistage Games with Comunication*, Econometrica **54** (1986), 323–358.
8. Owen, G., *Multilinear Extensions of Games*, J. Manag. Sci. **18** (1972), 64–79.

9. Shapley, L. S., *In: Kuhn HW, Tucker AW (eds) Contributions to the theory of games II (Annals of Mathematical Studies)*, Princeton University Press, Princeton, 1953, pp. 307–317.
10. Shapley, L. S., *Additive and Non–Additive Set Functions*, Ph. D thesis, Princeton University. (1953).
11. Shapley, L. S., *Cores of Convex Games*, I. J. Game Theory **1** (1971), 11–26.
12. Weber, R. J., *In: Roth, A. E.(ed.) The Shapley Value*, Cambridge University Press, New York, 1988, pp. 101–119.