\( M_t/M_t/n_t : \text{Strong Approximations} \)

Parameters: \( \lambda_t, \mu_t, n_t \).

\[
Q(t) = \begin{cases} Q(t) | t \geq 0 \end{cases} : \text{total number in system.}
\]

Model:

\[
Q(t) \equiv Q(0) + A_1 \left( \int_0^t \lambda_s ds \right) - A_2 \left( \int_0^t \mu_s \cdot (Q(s) \land n_s) ds \right),
\]

where \( A_1 \) and \( A_2 \) are two independent Poisson (1) processes.

Source: Predictable2_FINAL.tex
Uniform Acceleration of the $M_t/M_t/n_t$ Queue

$\lambda \leftrightarrow \eta \lambda, \ n \leftrightarrow \eta n$.

Take the limit as $\eta \to \infty$.

Physical interpretation: scaling up capacity in response to a similar scale up of the offered load.

Formally:
for any $\eta > 0$, consider

$$Q_\eta(t) = Q_\eta(0) + A_1 \left( \int_0^t \eta \lambda_s ds \right)$$

$$- A_2 \left( \int_0^t \mu_s \cdot (Q_\eta(s) \wedge \eta n_s) \, ds \right).$$
Assume \( \lambda \) and \( \mu \) are locally integrable functions. Then

\[
\lim_{\eta \to \infty} \frac{1}{\eta} Q^n(t) = Q^{(0)}(t) \quad u.o.c., a.s.
\]
given convergence at \( t = 0 \).

Here

\[
Q^{(0)}(t) = Q^{(0)}(0) + \int_0^t \left[ \lambda_s - \mu_s \cdot (Q^{(0)}(s) \wedge n_s) \right] ds,
\]
or more compactly

\[
\frac{d}{dt} Q^{(0)}(t) = \lambda_t - \mu_t \cdot (Q^{(0)}(t) \wedge n_t).
\]
Assume \( \lambda \) and \( \mu \) are locally integrable functions. Then

\[
\lim_{\eta \to \infty} \sqrt{\eta} \left[ \frac{1}{\eta} Q^{\eta}(t) - Q^{(0)}(t) \right] \overset{d}{=} Q^{(1)}(t),
\]
given convergence at \( t = 0 \).

The convergence is in \( D[0, \infty] \), and

\[
Q^{(1)}(t) = Q^{(1)}(0)
\]

\[
- \int_0^t \mu_s \mathbf{1}_{\{Q^{(0)}(s) < n_s\}} Q^{(1)}(s)^+ ds
\]

\[
+ \int_0^t \mu_s \mathbf{1}_{\{Q^{(0)}(s) \leq n_s\}} Q^{(1)}(s)^- ds
\]

\[
+ B \left( \int_0^t \lambda_s + \mu_s \cdot (Q^{(0)}(t) \wedge n_s) ds \right).
\]

Here \( B(\cdot) \) is a standard Brownian motion.
Strong Approximations: $A_i(t) \leftrightarrow t + B_i(t)$

$$
\frac{1}{\eta} Q^n(t) \approx \frac{1}{\eta} Q^n(0) + \\
+ \int_0^t \left[ \lambda_s - \mu_s \left( \frac{1}{\eta} Q^n(s) \wedge n_s \right) \right] ds \\
+ \frac{1}{\eta} B_1 \left( \eta \int_0^t \lambda_s ds \right) - \frac{1}{\eta} B_2 \left( \eta \int_0^t \mu_s \left( \frac{1}{\eta} Q^n(s) \wedge n_s \right) ds \right)
$$

1. **FSLLN:** As $\eta \uparrow \infty$, $\frac{1}{\eta} Q^n \rightarrow Q^{(0)}$ u.o.c., a.s., given convergence at $t = 0$.
   
   Here $Q^{(0)}$ is the unique solution to the ODE

   $$
   \frac{d}{dt} Q^{(0)}(t) = \lambda_t - \mu_t \left( Q^{(0)}(t) \wedge n_t \right), \quad t \geq 0.
   $$

   Proof: FSLLN for the $B_i$'s, combined with Gronwall.

2. **FCLT:** As $\eta \uparrow \infty$, $\sqrt{\eta} \left[ \frac{1}{\eta} Q^n - Q^{(0)} \right] \overset{d}{\rightarrow} Q^{(1)}$,
   
   given convergence at $t = 0$.

   Here $Q^{(1)}$ is the unique solution to the SDE...

   Proof:
Proof of FCLT

1. **Brownian Term**: as before, based on self-similarity & additivity & time-change, we get that it is distributed as:

\[ B \left( \int_0^t [\lambda_s + \mu_s(Q^{(0)}(s) \land n_s)] ds \right), \quad t \geq 0. \]

2. Drift \( = \sqrt{\eta} \int_0^t \left[ f_s \left( \frac{1}{\eta} Q^{\eta}(s) \right) - f_s (Q^{(0)}(s)) \right] ds \)

where \( f_t(x) = \lambda_t - \mu_t (x \land n_t), \quad x \in \mathbb{R}^1. \)

If indeed \( Q^{\eta}(t) \overset{d}{=} \eta Q^{(0)}(t) + \sqrt{\eta} Q^{(1)}(t) + o(\sqrt{\eta}), \)
then letting \( \epsilon = 1/\sqrt{\eta}, \)

Drift \( \overset{d}{\approx} \int_0^t \frac{1}{\epsilon} \left[ f_s(Q^{(0)}(s) + \epsilon Q^{(1)}(s)) - f_s(Q^{(0)}(s)) \right] ds \)

\[ \overset{\epsilon \downarrow \infty}{\longrightarrow} \int_0^t \land f_s(Q^{(0)}(s); Q^{(1)}(s)) ds \]

in which \( \land f_t(x; y) = \lim_{\epsilon \downarrow 0} \frac{1}{\epsilon} \left[ f_t(x + \epsilon y) - f_t(x) \right], \)

must be defined for continuous, but non-differentiable functions.
Let \( f : \mathbb{R}^1 \rightarrow \mathbb{R}^1 \) be continuous at \( x \), with left- and right-hand derivatives at \( x \). Then,

\[
\wedge f(x; y) = f'(x^+)y^+ - f'(x^-)y^- , \quad x, y \in \mathbb{R}^1.
\]

Example:

\[
f(x) = x \wedge a
\]

\[
\wedge f(x; y) = y \, 1_{x<a} + 1_{x=a}[0 \cdot y^+ - 1 \cdot y^-] = y \, 1_{x<a} - y^- \, 1_{x=a} = y^+ \, 1_{x<a} - y^- \, 1_{x\geq a}
\]

Example: Slutsky’s Theorem (extended)

Suppose \( X_n \rightarrow \mu, \sqrt{n} (X_n - \mu) \xrightarrow{d} Z \), as \( n \uparrow \infty \).

Then \( f(x_n) \rightarrow f(\mu) \) and

\[
\sqrt{n} [f(X_n) - f(\mu)] \xrightarrow{d} \wedge f(\mu; Z) = f'(\mu^+) Z^+ - f'(\mu^-) Z^-
\]

\[
(= f'(\mu) Z \quad \text{when} \quad \exists f'(\mu)).
\]
**Example:**  FCLT for $M_t/M_t/n_t$

Recall

$$Q^{(1)}(t) \overset{d}{\approx} \int_0^t \wedge f_s(Q^{(0)}(s); Q^{(1)}(s))ds + B(\cdots)$$

where $f_t(x) = \lambda_t - \mu_t(x \wedge n_t)$. 

Since $\wedge f_t(x, y) = -\mu_t[y^+1_{x < n_t} - y^-1_{x \leq n_t}]$, we conclude

**FCLT:** 

$$\lim_{\eta \to \infty} \sqrt{\eta} \left[ \frac{1}{\eta} Q^{\eta}(t) - Q^{(0)}(t) \right] \overset{d}{=} Q^{(1)} , \quad t \geq 0,$$

where $Q^{(1)}$ is the unique solution of the following SDE:

$$Q^{(1)}(t) = Q^{(1)}(0) - \int_0^t \mu_s \cdot 1_{\{Q^{(0)}(s) < n_s\}} Q^{(1)}(s)^+ ds + B \left( \int_0^t [\lambda_s + \mu_s(Q^{(0)}(s) \wedge n_s)] ds \right) , \quad t \geq 0.$$
Differential Equations for the Diffusion Moments of the $M_t/M_t/n_t$ Queue

If $\{ t \mid Q^{(0)}(t) = n_t \}$ has Lebesque measure zero, then $Q^{(1)}(\cdot)$ is a Gaussian process. Furthermore,

$$\frac{d}{dt} \mathbb{E} [Q^{(1)}(t)] = -\mu_t 1_{\{Q^{(0)}(t) \leq n_t\}} \mathbb{E} [Q^{(1)}(t)],$$

$$\frac{d}{dt} \text{Var} [Q^{(1)}(t)] = -2\mu_t 1_{\{Q^{(0)}(t) \leq n_t\}} \text{Var} [Q^{(1)}(t)]$$

$$+ \lambda_t + \mu_t \left( Q^{(0)}(t) \land n_t \right),$$

and

$$\frac{d}{dt} \text{Cov} \left[ Q^{(1)}(s), Q^{(1)}(t) \right]$$

$$= -\mu_t 1_{\{Q^{(0)}(t) \leq n_t\}} \text{Cov} \left[ Q^{(1)}(s), Q^{(1)}(t) \right].$$

The above is solvable numerically, in a spreadsheet, via forward increments.
Waiting Time
Virtual Waiting Times for the $M_t/M_t/n_t/\infty/\text{FCFS}$ Queue

Fix a time $\tau$.

Define $\{ \hat{Q}(t) \mid t \geq 0 \}$ to be the queue length process associated with an $M_t/M_t/n_t$ system, with parameters $\mu_t$ and $n_t$ as before, but with arrival rates $\hat{\lambda}_t$ that are modified as follows:

$$\hat{\lambda}_t = \begin{cases} \lambda_t & \text{if } t \leq \tau, \\ 0 & \text{if } t > \tau. \end{cases}$$

The virtual waiting time for a customer arriving at time $\tau$ is $W(\tau) - \tau$ where

$$W(\tau) = \inf \{ t \geq \tau \mid \hat{Q}(t) \leq n_t - 1 \}.$$ 

The uniformly-accelerated version is

$$W^\eta(\tau) = \inf \{ t \geq \tau \mid \hat{Q}^\eta(t) \leq \eta n_t - 1 \}.$$
Virtual Waiting Time: Fluid Limit

FSLLN \[ \lim_{\eta \to \infty} W^\eta = W^{(0)} \text{ a.s.} \]

where

\[ W^{(0)}(\tau) = \inf \left\{ t \geq \tau \mid \hat{Q}^{(0)}(t) \leq n_t \right\} \]

with

\[ \hat{Q}^{(0)}(t) = Q^{(0)}(\tau) - \int_{\tau}^{t} \mu_s n_s ds. \]

The analysis is extendable to the process \( \left\{ W^{(0)}(\tau) \mid \tau \geq 0 \right\} \)

(from merely the random variable \( W^{(0)}(\tau) \)).
Virtual Waiting Time: Diffusion Limit

FCLT

\[ \lim_{\eta \to \infty} \sqrt{\eta} \ (W^{\eta} - W^{(0)}) \overset{d}{=} W^{(1)} \]

where

\[ W^{(1)}(\tau) = \frac{\hat{Q}^{(1)}(W^{(0)}(\tau))}{\mu W^{(0)}(\tau) n W^{(0)}(\tau)} \]

If \( \hat{Q}^{(1)} \) is a Gaussian process, then \( \text{Var} \left[ W^{(1)}(\tau) \right] \) is calculated via

\[ \text{Var} \left[ \hat{Q}^{(1)}(W^{(0)}(\tau)) \right] = \text{Var} \left[ Q^{(1)}(\tau) \right] + \int_{0}^{W^{(0)}(\tau)} \mu s n_s ds. \]

The analysis is extendable to the

stochastic process \( \{ W^{(1)}(\tau) \mid \tau \geq 0 \} \)

(from merely the random variable \( W^{(1)}(\tau) \)).